Basics of cosmological structure formation

Franco Vazza

franco.vazza2@unibo.it

MENU FOR THE WEEK

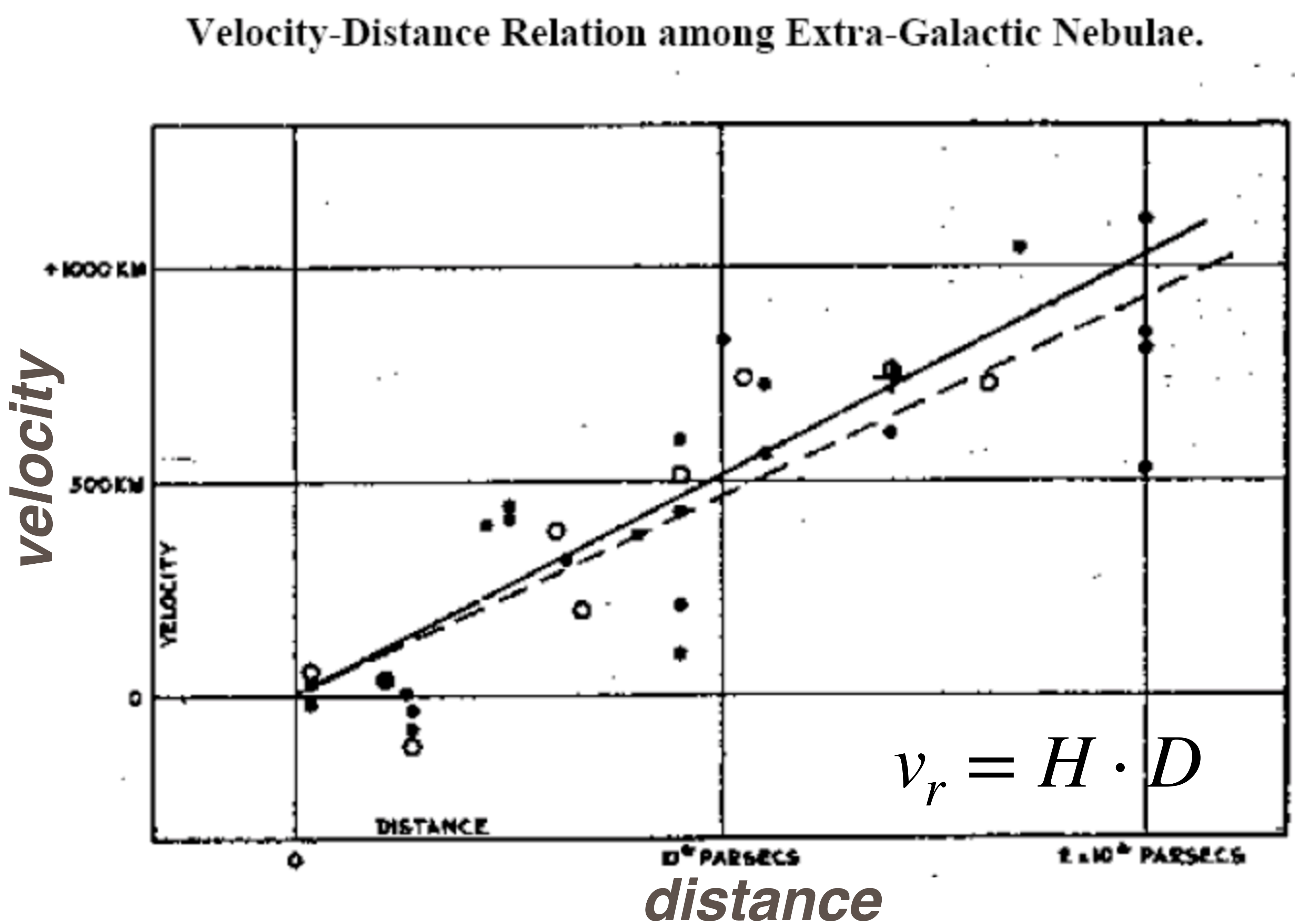
- Today: how to describe the onset of structure formation in cosmology, how to start incorporating the effect of magnetic fields
- Tuesday: non-linear structure formation, cosmological simulations
- Wednesday: clusters of galaxies and their magnetic signatures
- Thursday: simulating magnetic fields evolution and dynamo in cosmology
- Friday: constraining cosmic magnetism combining observations and numerical simulations

OUTLINE

In this lesson we will (briefly) see:

- how to describe the onset of structure formation in cosmology, starting from simple perturbations
- how to roughy predict the typical overdensity of the building blocks of the cosmic web
- how to start incorporating the presence of magnetic fields in the process of structure formation

THE UNIVERSE IS EXPANDING



Lemaitre 1927

Hubble 1929

Hubble-Lemaitre law: recession velocity increases with distance

HUBBLE-LEMAITRE LAW

$$v_r = H_0 \cdot D$$

H0 is the "constant" (actually not constant!) derived by Hubble.

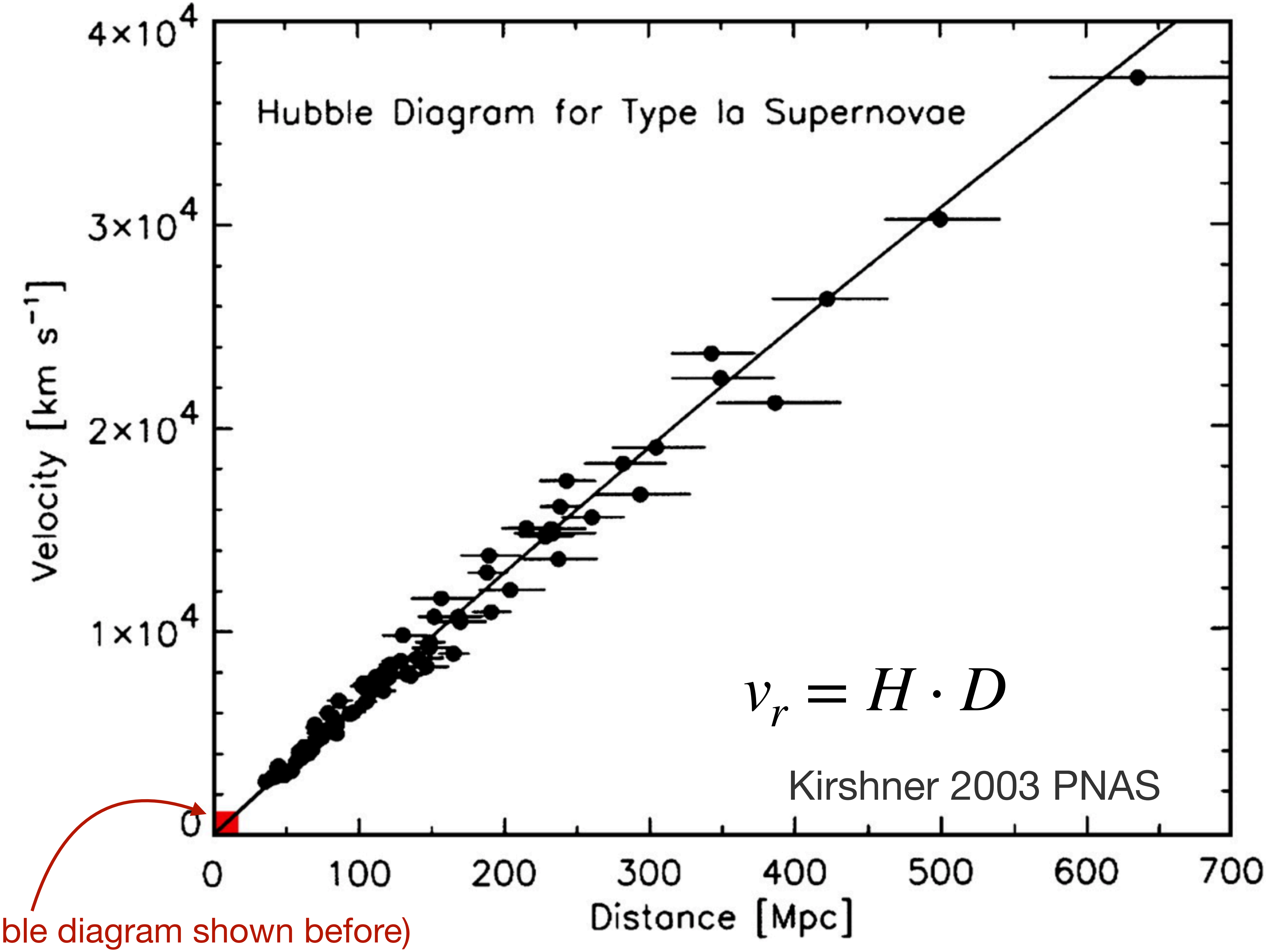
$$H_0 \approx 70 \frac{\mathrm{km/s}}{\mathrm{Mpc}}$$

Historically, the estimate has oscillated between 10 and 100 km/s Mpc.

Example: on average a galaxy distant from 20Mpc, moves away from us at the speed of $v_r = (20\mathrm{Mpc}) \cdot \mathrm{H}_0 \approx 1400\ \mathrm{km/s}$

Example: on average a galaxy with a distance rate of 21,000km/s is at a distance of $D = (21000\ \mathrm{km/s})/\mathrm{H}_0 \approx 300\ \mathrm{Mpc}$

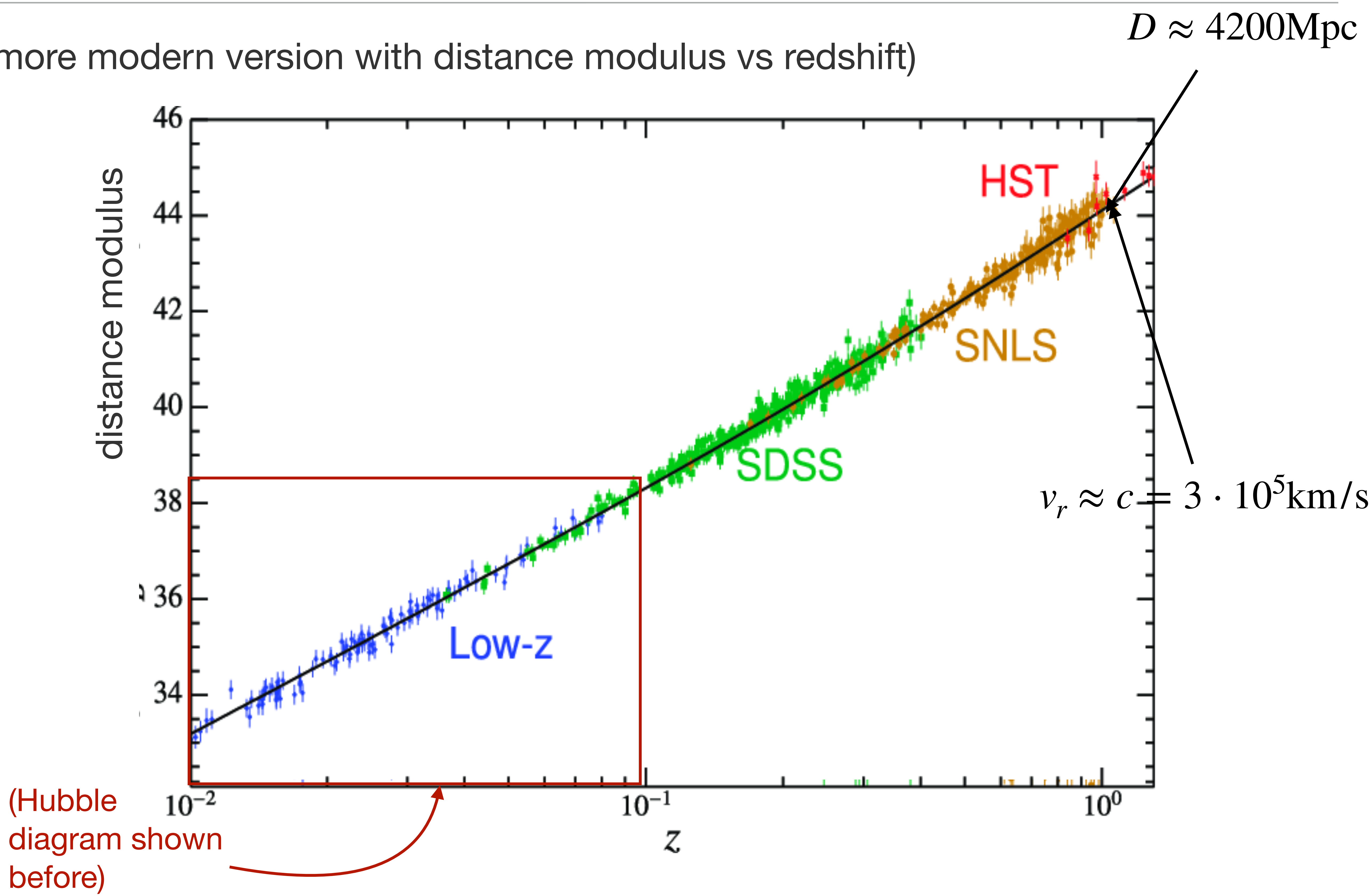
THE UNIVERSE IS EXPANDING



(Hubble diagram shown before)

THE UNIVERSE IS EXPANDING

(more modern version with distance modulus vs redshift)



Bachall 2015

NEWTONIAN COSMOLOGY

Many key aspects of structure formation in cosmology can be capture already combining Newtonian physics with Hubble-Lemaitre law

Key assumptions:

- homogeneity
- isotropy

Such assumptions remained for several decades only "working hypotheses", before the observations confirmed them.

Example: the distribution of the radio galaxies observed around the north celestial pole is very homogeneous and isotropic.

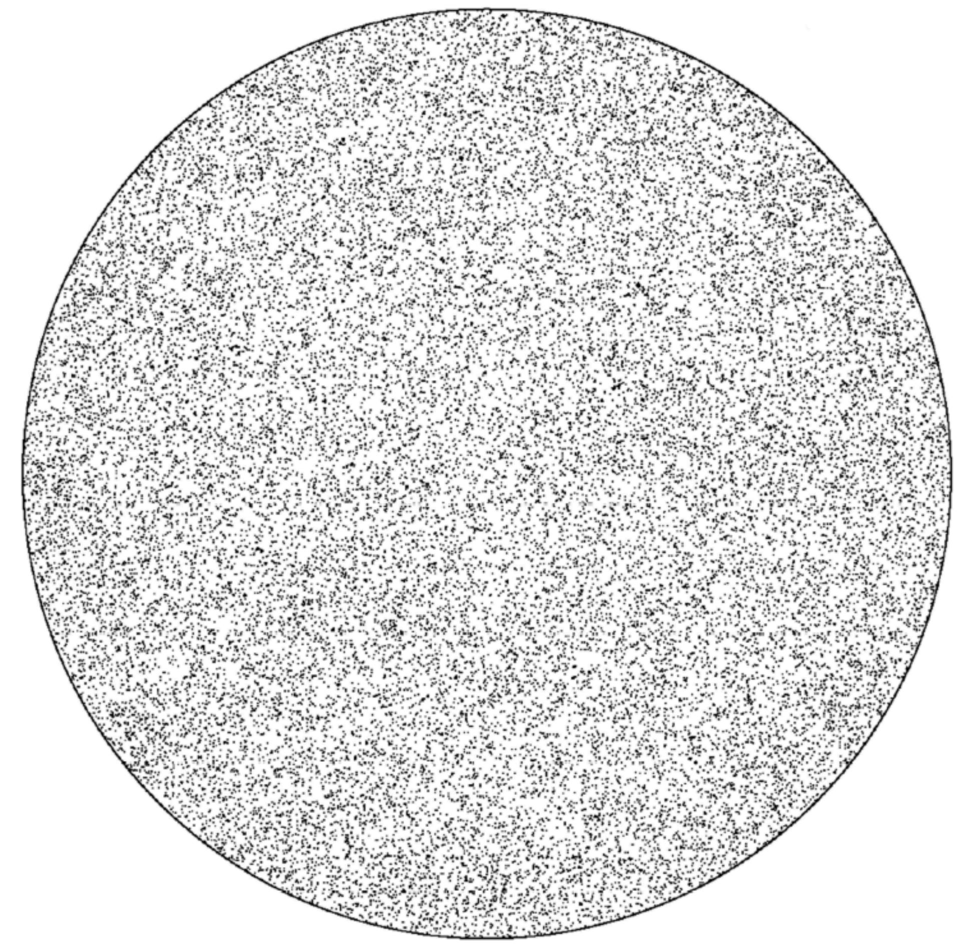


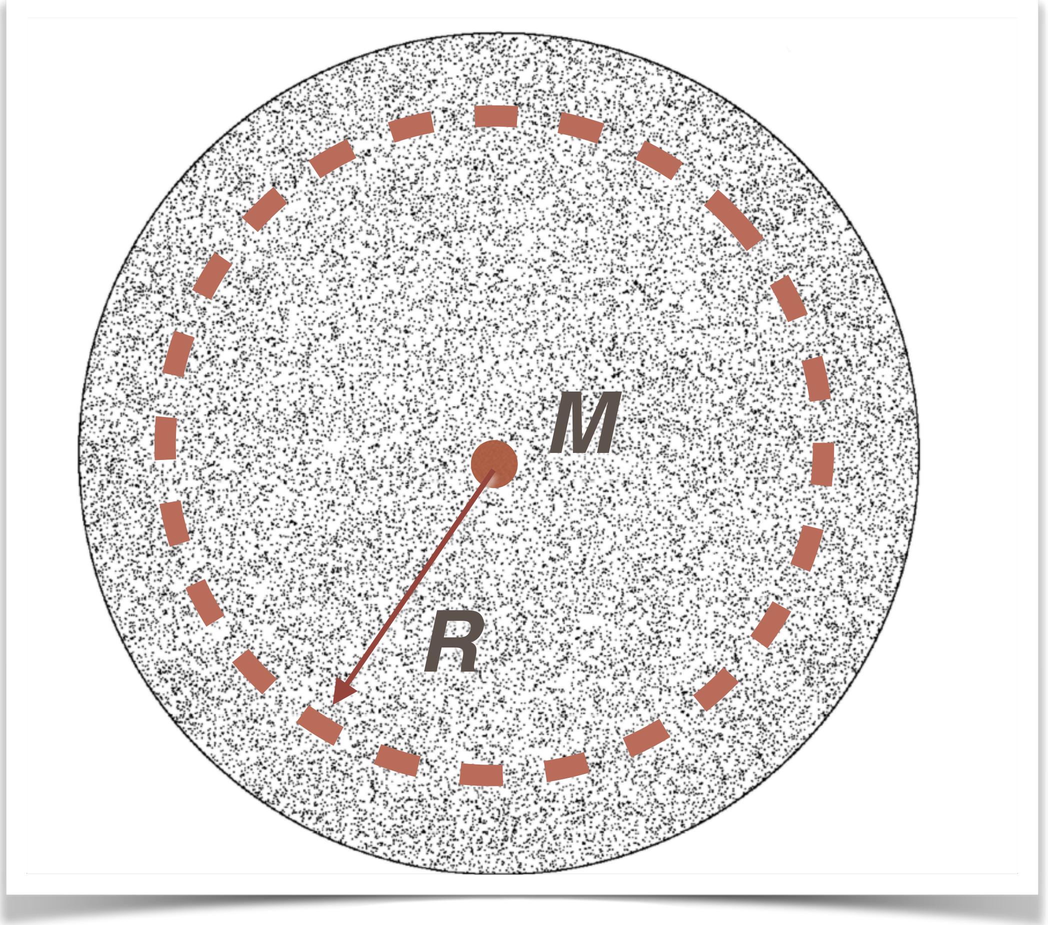
Figure 1. Positions of the $N \sim 4 \times 10^4$ radio sources stronger than $S = 2.5$ mJy at 1.4 GHz are indicated by points on this equal-area plot covering the sky within 15° of the north celestial pole. Nearly all of these sources are extragalactic and so distant (median redshift $\langle z \rangle \sim 1$) that their distribution is quite isotropic.

Condon 2018

NEWTONIAN DYNAMICS OF THE UNIVERSE

If the Universe expand as a whole, its "scale radius" R expands too. It was R=0 at t=0 and we normalise it so that R=1 today, so 0<R<1 in the past. [note: in modern cosmology R is referred as "a"]

We try to see, using simple Newtonian dynamics, what is the dynamics of the Universe, by considering a very large portion of it, with M is the total mass contained in the sphere of radius R.

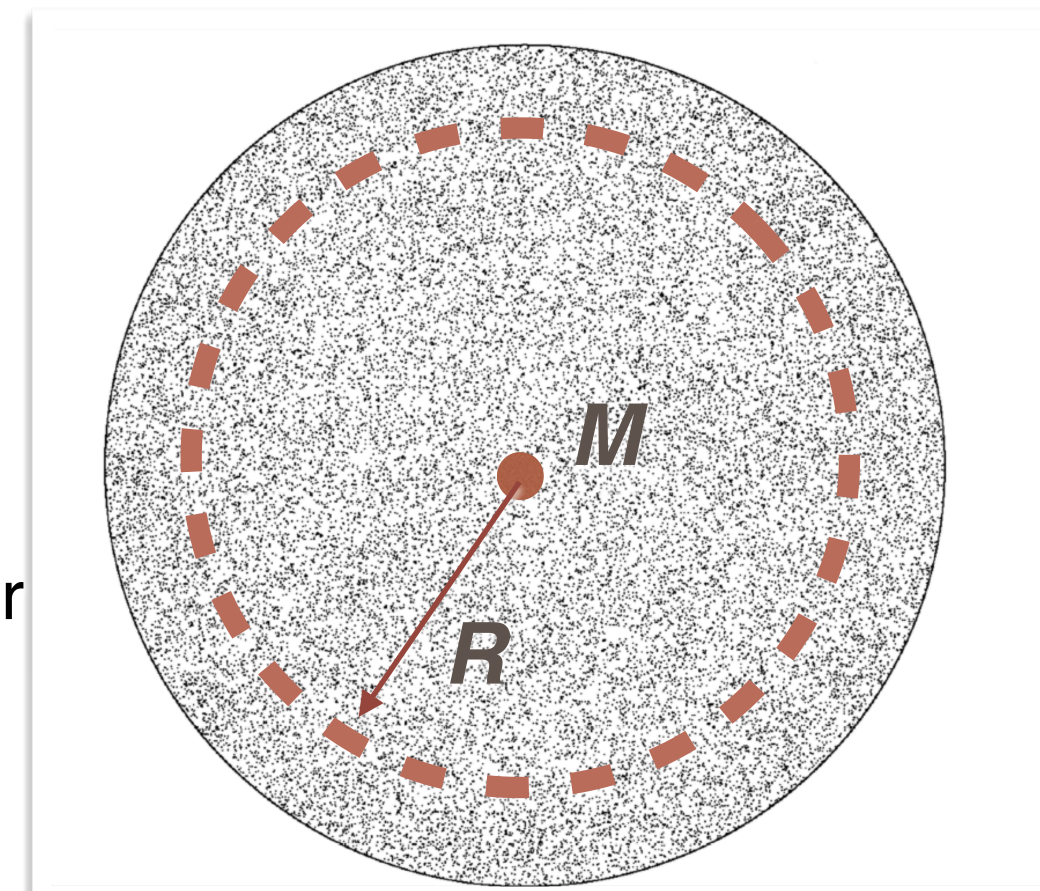


BIRKHOFF'S THEOREM

We first need Birkhoff's theorem for this: it is the General Relativity analogous to Gauss's theorem for electric field:

Field (gravitational) with spherical symmetry in an isotropic field is equivalent to a field generated by a distribution of point mass contained by the sphere.

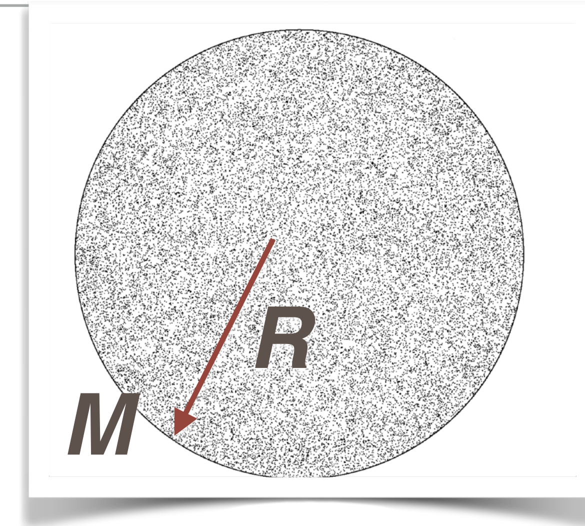
M is the total mass contained in the sphere of radius R.



We can use Netwonian mechanics instead of General Relativity if $\frac{GM}{Rc^2} \ll 1$, which means when:

- the typical velocity of particles is much lower than c: $\frac{v^2}{2} = \frac{GM}{R} \ll c^2 \rightarrow \frac{2GM}{c^2} \ll R$
- the system is much larger than its Schwarzschild radius: $R \gg R_S = \frac{2GM}{c^2}$

NEWTONIAN COSMOLOGY



The scale factor of an expanding Universe is
$R(t) = 0$ at $t = 0$ and $R_0 = 1$ at $t = t_0$

We apply Newton's law to a portion of the Universe with radius R, mass M and density $\rho = M/[(4\pi/3)R^3]$

From $\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi G}{3}\frac{\rho R^3}{R^2}$ (with $\rho(t)R^3 = \rho(t_0)R_0^3(t)$)

we get $R^2\ddot{R} + \frac{4\pi}{3}G\rho(t_0) = 0$

and after multiplying for $\dot{R}$ we have $\dot{R}[R^2\ddot{R} + \frac{4\pi}{3}G\rho(t_0)] = 0$

which after integration gives

$$\dot{R}^2 = \frac{8\pi}{3}\frac{G\rho(t_0)}{R} - k = \frac{\text{const}}{R} - k$$

with k = (U-T) being the total (potential+kinetic) energy

NEWTONIAN COSMOLOGY

Let's notice that if these two relations are true

$$R^2\ddot{R} + \frac{4\pi}{3}G\rho(t_0) = 0 \qquad\qquad \dot{R}^2 = \frac{8\pi}{3}\frac{G\rho(t_0)}{R} - k$$

in order to have a static Universe it should be:

$$\ddot{R} = 0 \qquad \textit{and} \qquad \dot{R} = 0$$

…which can only be possibile in a Universe without matter at all

$$\rho = 0$$

Instead if $\rho > 0$, only dynamical solutions for R(t) exist

NEWTONIAN COSMOLOGY

$$\dot{R}^2 = \frac{8\pi}{3}\frac{G\rho(t_0)}{R} - k$$

Three possible scenarios:

$k = 0$ **Einstein-DeSitter model "Flat universe"** $R \propto t^{2/3}$

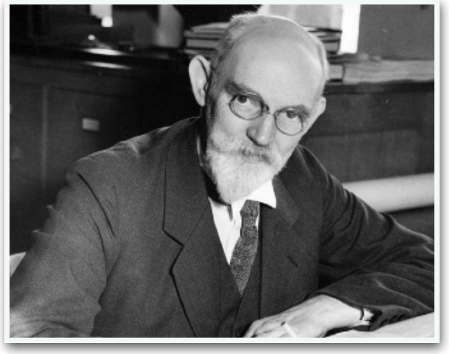
$k > 0$ **"Closed universe"**

$k < 0$ **“Open universe"** $R \propto t^{2/3}$ (for small t)
$R \propto t$ (for large t)

Notice: the connection between the total energy *k* and geometry makes sense only in Einstein’s GR theory

FLAT UNIVERSE

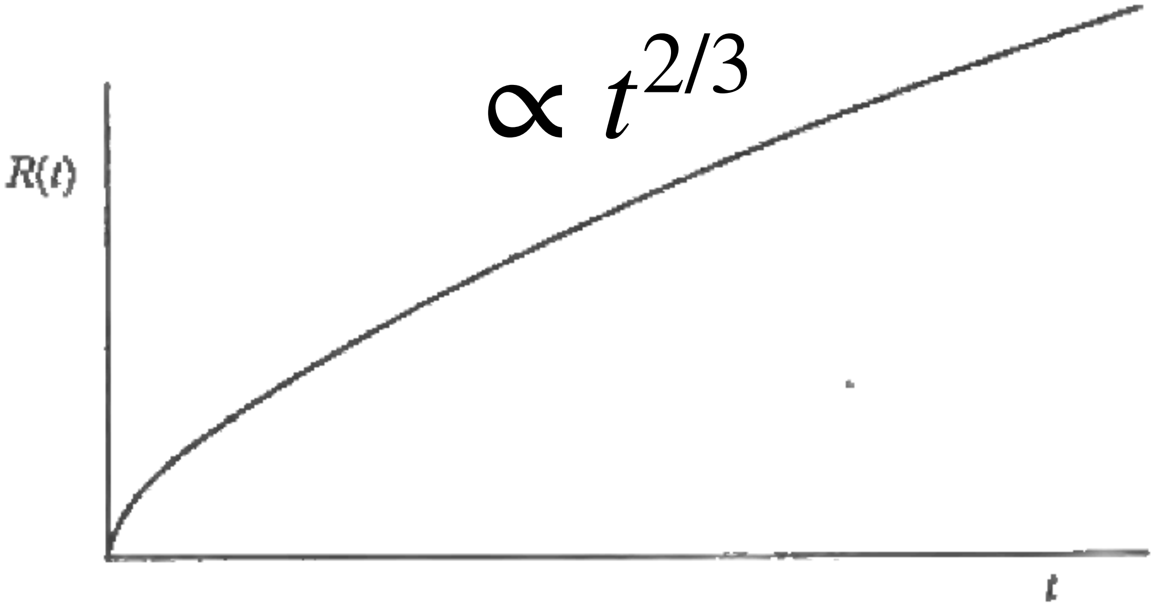
$k = 0$ $$\dot{R}^2 = \frac{8\pi}{3}\frac{G\rho(t_0)}{R}$$ (Einstein- DeSitter)



by solving we get $R \propto t^{2/3}$

The expansion velocity goes to 0 at infinite time

This Universe has Eucledian geometry. This is the most simple model that approaches, under many respects, the flat ΛCDM cosmology we think best describes our Universe.



OPEN UNIVERSE

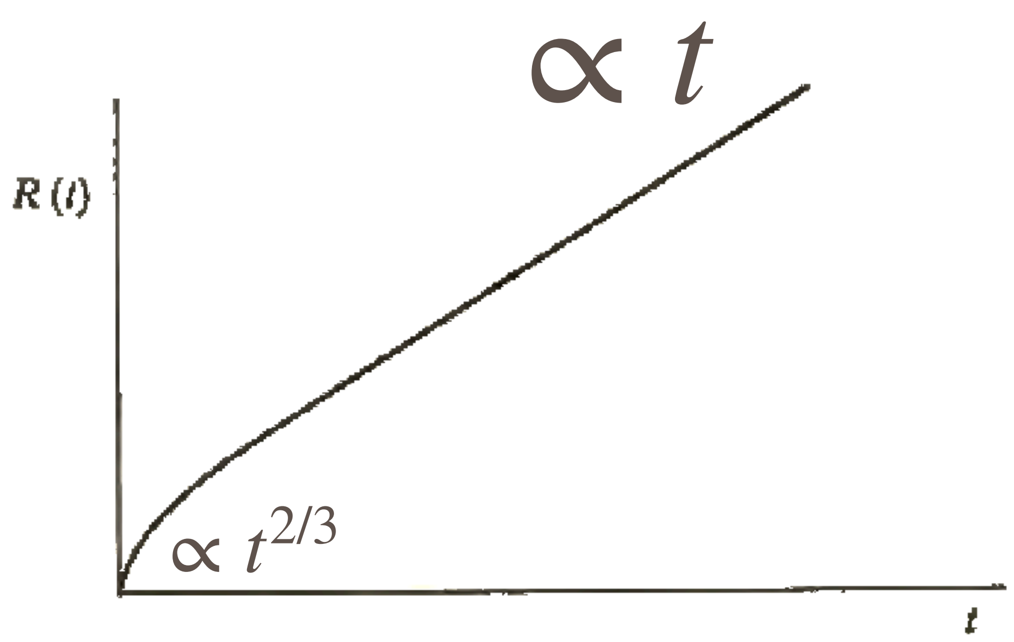
$k < 0$

$$\dot{R}^2 = \frac{8\pi}{3}\frac{G\rho(t_0)}{R} + |k|$$

Two different trends for small or large times

$t \to 0$ $\quad \dot{R}^2 \approx \frac{8\pi}{3}\frac{G\rho(t_0)}{R} \to R \propto t^{2/3}$

$t \to \infty$ $\quad \dot{R}^2 \approx k \to R \propto t$



This Universe has infinite volume.

The sum of the angles

of the triangle <180°

CLOSED UNIVERSE

$$k > 0 \qquad \dot{R}^2 = \frac{8\pi}{3}\frac{G\rho(t_0)}{R} - k$$

with maximum radius $R_{max} = \frac{8\pi}{3}\frac{G\rho_0}{k}$ for $\dot{R} = 0$

The analytical solution is parametrized by the "cycloid"

$$R(\theta) = R_0 \frac{\Omega}{2(\Omega - 1)}(1 - \cos\theta)$$

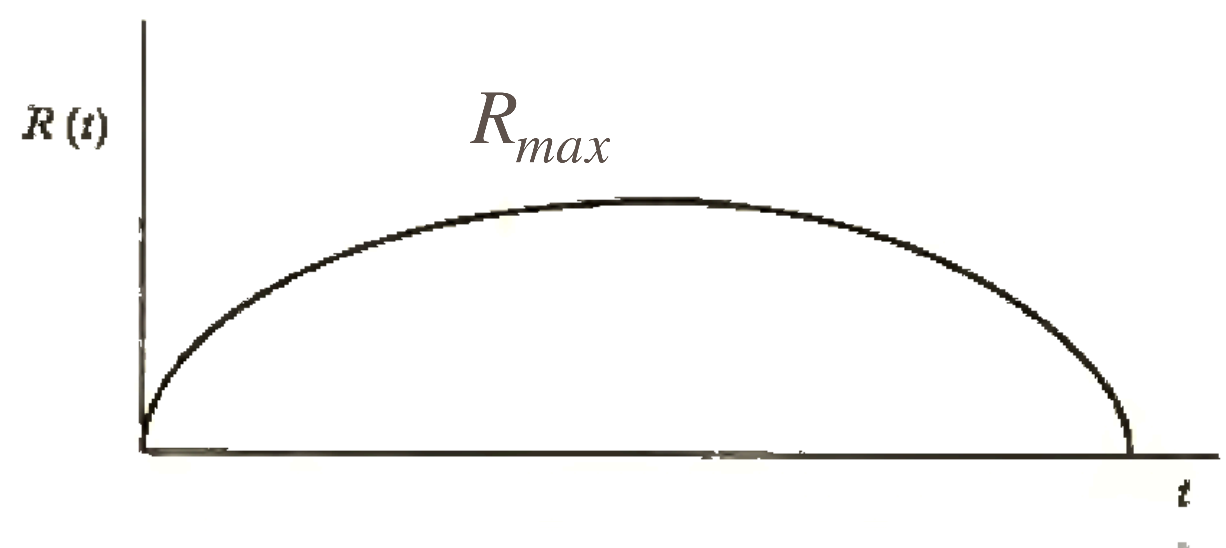
$$t(\theta) = \frac{1}{H_0}\frac{\Omega}{2(\Omega - 1)^{1/2}}(\theta - \sin\theta)$$

for $\theta = \pi$ we have the maximum radius:

$$R_{\max} = R_0 \frac{\Omega}{(\Omega - 1)}$$

$$t_{\max} = \frac{\pi}{2}\frac{\Omega}{(\Omega - 1)^{3/2}} H_0^{-1}$$

Such universe has a finite volumebut no boundaries. The final contraction velocity is infinite. The sum of the angle of the triangle >180°



THE COSMOLOGICAL CRITICAL DENSITY

What is the exact density making the Universe flat? if we set k=0:

$$\dot{R}^2 = H_0^2 R^2 = \frac{8\pi}{3}\frac{G\rho(t_0)}{R} = \frac{8\pi G}{3}\rho R^2$$

$$\rightarrow \rho_{cr} = \frac{3H_0^2}{8\pi G}$$

This is the density required to make the Universe "flat"

From the most recent estimates of H0, we get : $\rho_{cr} = 9.20 \cdot 10^{-27}\ \mathrm{kg/m^3}$

In cosmology, density is often referred to the critical one: $\Omega = \rho/\rho_{cr}$

From modern observations, we know that the Universe is flat, hence $\Omega = 1$.
However, matter (dark+ordinary) only makes $\Omega_M \approx 0.3$
The rest appears to be contributed by dark energy, providing $\Omega_\Lambda \approx 0.7$ today

RELATIVISTIC COSMOLOGY

Key differences wrt Newtonian cosmology:

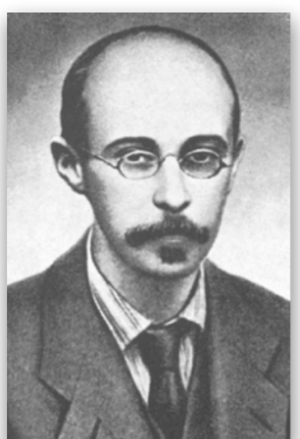
- 10 potentials, not one!
- Link between geometry and total energy k is evident through the metric
- mass-less components (e.g. photons) affects the metric through $E = mc^2$
- Pressure (like density) exerts gravitational attraction
- Non-linear theory
- The problem of an infinite dimension of the Universe is eliminated (delayed potentials)
- it allows including components with *negative* pressure (e.g. dark energy)

RELATIVISTIC COSMOLOGY

Einstein's General Relativity enters the room. If we assume

- the Robertson-Walker metric and
- a perfect fluid (of photons, matter or dark energy, but without viscosity or thermal conduction)

Einstein's cosmological eq. are simplified in:

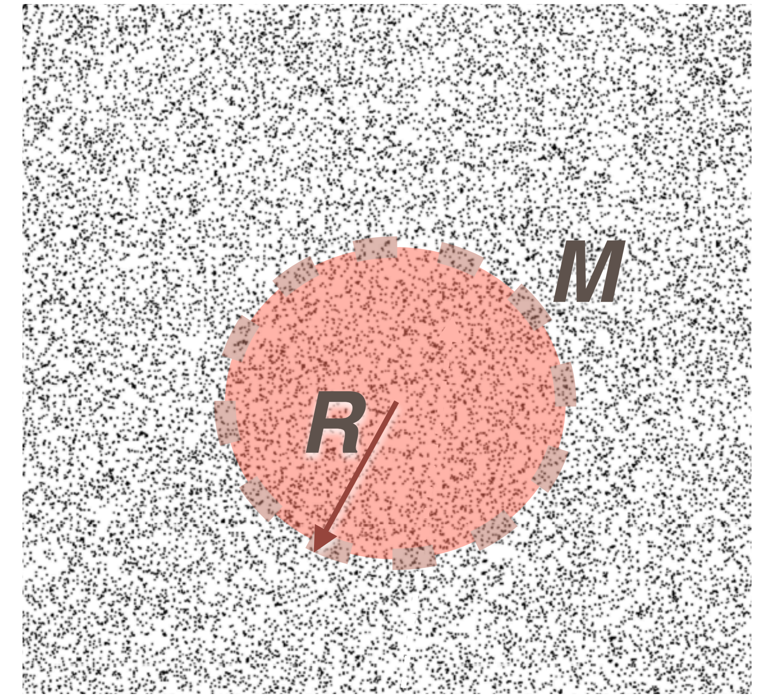


$$\begin{cases} \dot{R}^2 = \dfrac{8\pi}{3} G\rho R^2 - k \\ 2\dfrac{\ddot{R}}{R} + \dfrac{\dot{R}^2}{R^2} = -\dfrac{8\pi G p}{c^2} - \dfrac{k}{R^2} \end{cases}$$

known as "Friedman equations"

which admit solutions identical to the one we derived for the Newtonian case.

COLLAPSE OF ONE TOP HAT PERTURBATION



The simplest type of perturbation is the spherical top-hat.

We assume flat Einstein-de-Sitter Universe ($\rho = \rho_{cr}$) with

one overdense ($\rho > \rho_{cr}$) spherical region of radius R.

Birkhoff's theorem says we can ignore everything outside

and that this behaves like a closed Universe!

We can then use the same parametric solution for a closed universe:

$$R = A(1 - \cos\theta) \quad \text{and} \quad t = B(\theta - \sin\theta)$$

with $R_{max} = 2A$ and $t_{max} = \pi B$ and $A^3 = GMB^2$

Let's expand cos and sin at early times ($t \rightarrow 0$) up to the 5th order:

$$\sin\theta = \theta - \theta^3/6 + \theta^5/120 - \ldots \quad \text{and} \quad \cos\theta = 1 - \theta^2/2 + \theta^4/24 - \ldots$$

so we finally get (only valid for early times!):

$R = (A\theta^2/2)(1 - \theta^2/12)$ and

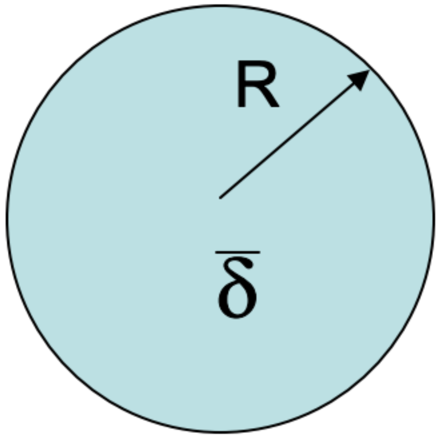
$t = (B\theta^3/6)(1 - \theta^2/20)$.

COLLAPSE OF ONE TOP HAT PERTURBATION

From this we derive the linear growth of the scale of the top-hat perturbation

$$R(t) = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left(1+\frac{\theta^2}{30}\right)\left(1-\frac{\theta^2}{12}\right) =$$

$$\boxed{R(t) = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]}$$



- The lowest order behaviour for $t \to 0$ is $R(t) \propto t^{2/3}$ as in EdS.
- The first correction to this behavious yields a fractional change in radius of order $\frac{\delta R}{R} = -\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}$ giving a density perturbation

$$\frac{\delta\rho}{\rho} = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3} \ll 1$$

- Next, let's see the overdensity at the maximum expansion of the sphere ("turnaround")

COLLAPSE OF ONE TOP HAT PERTURBATION

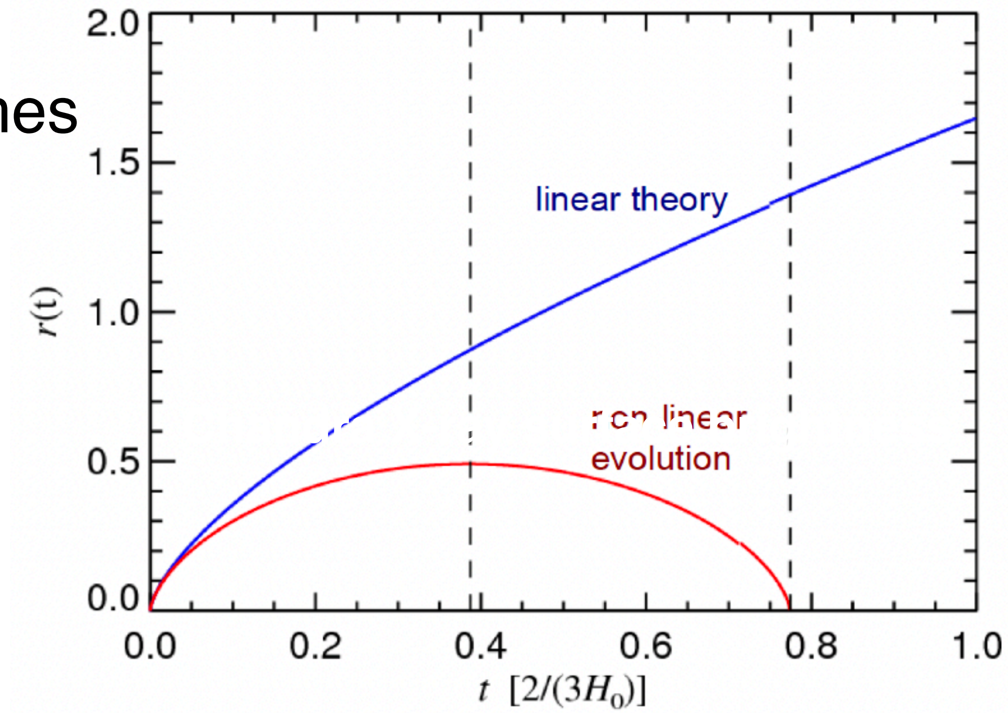
At the "turnaround" it is $\theta = \pi$ and radius $R = R_{\rm ta} = 2A$, at a time $t_{ta} = \pi B$.

At this time, the radius the sphere would have if it had the critical density is

$R_0 = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}$, so the

density contrast at turnaround is the inverse ratio of volumes

$$\frac{\rho}{\rho_{cr}} = \frac{R_0^3}{R_{\rm ta}^3} = \left[\frac{(2A)^3}{\left[\frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\right]^3}\right]^{-1} = \frac{9\pi^2}{16} \approx 5.5$$

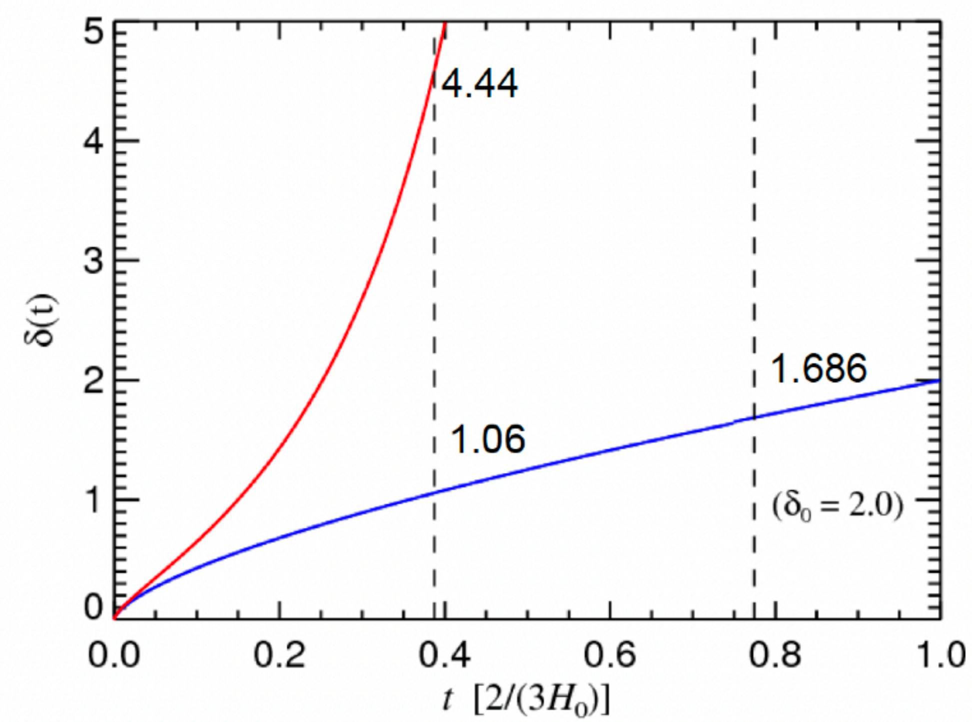


After turnaround, the sphere undergoes gravitational collapse until $\theta = 2\pi$ and $t = 2\pi B$.

The linear theory would (wrongly) predict an overdensity

$$\frac{\rho}{\rho_{cr}} = \frac{3}{20} 12\pi \approx 1.686,$$ which is a is reference value used

in analytical treatments of the growth of structure, such as the Press-Schechter formalism (see later).



COLLAPSE OF ONE TOP HAT PERTURBATION

The linear theory breaks down as the perturbation evolves. There are three interesting epochs:

(1) **TURNAROUND** The sphere breaks away from the general expansion and it reaches a maximum radius at $\theta = \pi$ and $t_{ta} = \pi B$. As we just saw, at turnaround the true over density with respect to the background is just $\rho(t_{ta})/\rho_{cr} \approx 5.5$

(2) **COLLAPSE** If only gravity operates, the sphere will collapse to a singularity at $\theta = \pi$.

(3) **VIRIALISATION** In reality the gas in the spherical overdensity will shock and thus be heated, and the dark matter will undergo "violent relaxation" in which particles get redistributed in phase space. The end point of this violent process is a gravitationally-bound dark-matter halo with gravitational binding energy $U = -GM/R_{ta}$ and kinetic energy K, linked by the virial theorem, $U = -2K$. This process is assumed to take place when the sphere has collapsed by a factor 2 from turnaround (so $\theta = 3\pi/2$ and $R_{vir} = R_{ta}/2$).

(4) Conventionally, it is assumed that this stable virialized radius is eventually achieved only at the collapse time ($\theta = 2\pi$), at which point the density contrast between the virialised sphere and the background universe is again obtained by the inverse ratio of volumes:

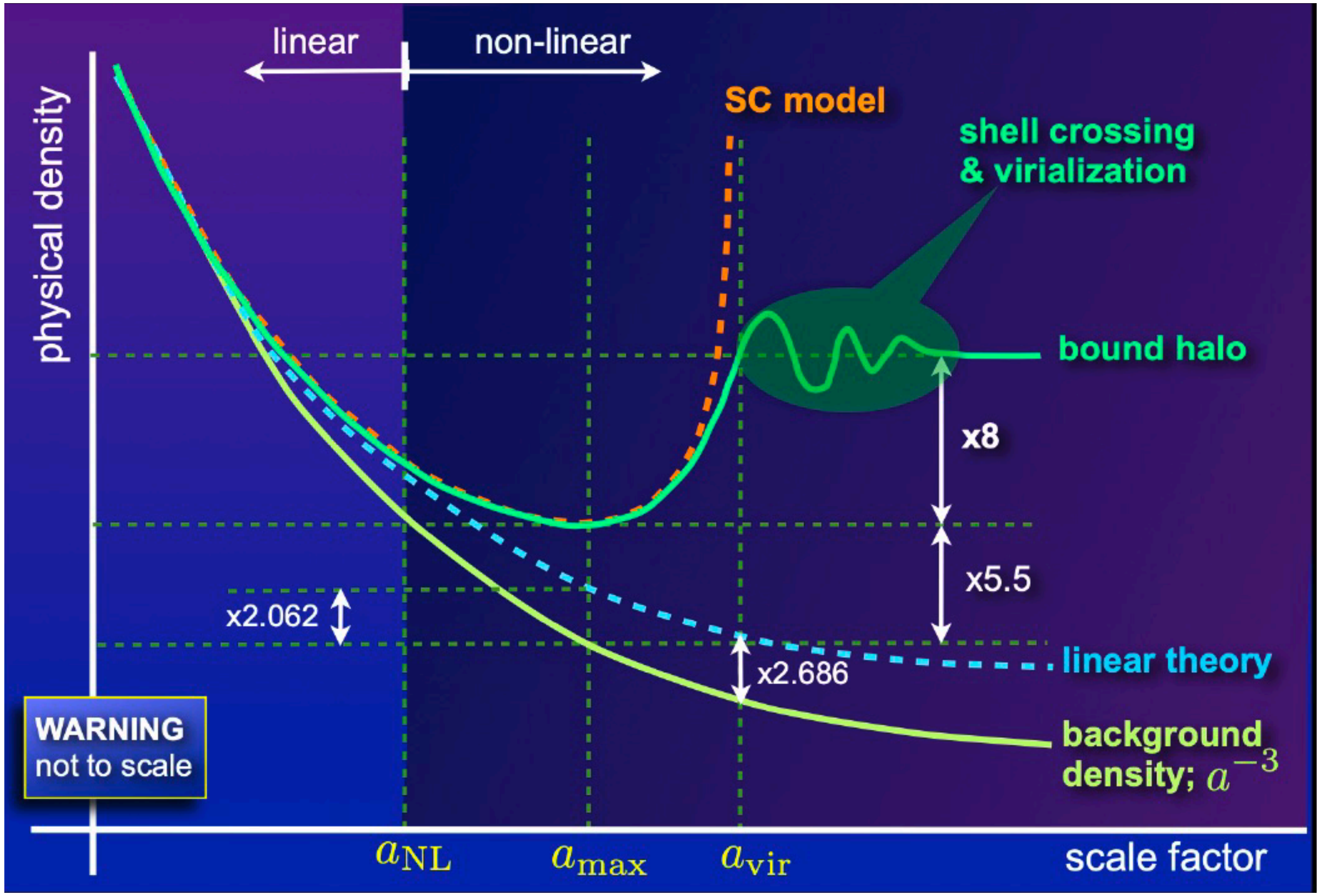
$$\delta_{vir} = \frac{\rho_{vir}}{\rho_{cr}} - 1 = \frac{R_0^3}{R_{vir}^3} - 1 = \frac{(6\pi)^2}{2} - 1 \approx 177$$

COLLAPSE OF ONE TOP HAT PERTURBATION

- This is strictly valid for a critical EdS Universe. $(\Omega_m = 1)$
- If $\Omega_m < 1$ because the Universe is open or because there is dark energy (ΛCDM) then the curvature or cosmological-constant terms in the Friedmann equation increase the expansion rate of the ambient Universe and so the ambient density is thus smaller at virialization.
  - Approximated relations are

$\delta_{vir} \approx (18\pi^2 + 60x - 32x^2)/\Omega_m(t_{vir}) - 1$ for $(\Omega_\Lambda = 0)$ and

$\delta_{vir} \approx (18\pi^2 + 82x - 39x^2)/\Omega_m(t_{vir}) - 1$ for $(\Omega_\Lambda \neq 0)$

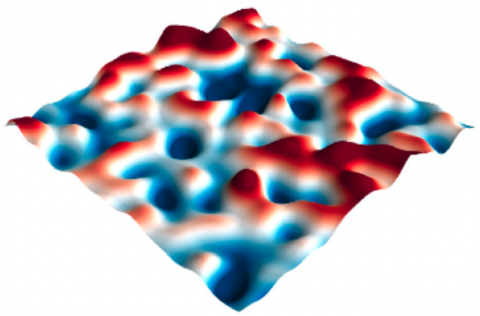


(credits: F. Van den Bosch)

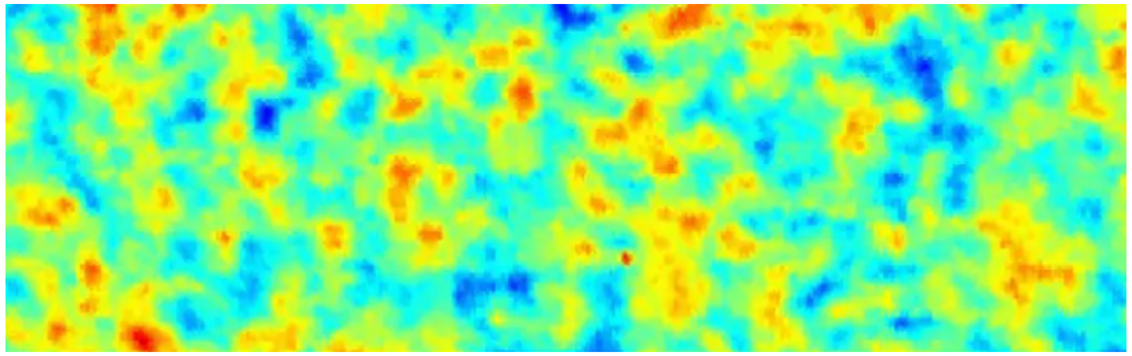
COLLAPSE OF PERTURBATIONS - FULL CASE

We consider an initial density perturbation field characterized by a dimensionless density fluctuation (or density contrast) as function of space ($\vec{x}$)

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}}$$



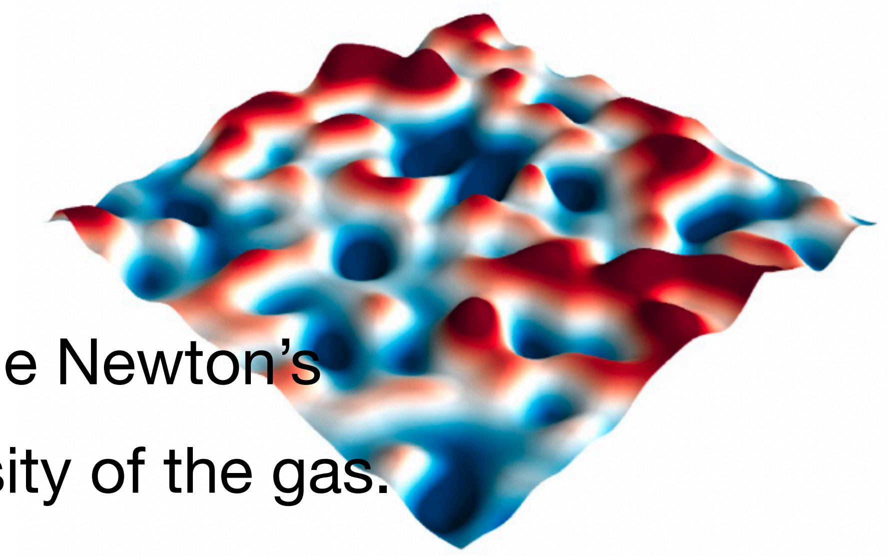
where $\rho(\vec{x})$is the matter density field at the position $\vec{x}$, and $\bar{\rho}$ is the mean mass density of the background Universe. The primordial properties of this field are set during inflation. Most inflationary models predict a homogeneous and isotropic Gaussian random fluctuation field (Guth and Pi 1982), which is confirmed by analysis the Cosmic Microwave Background.



The perturbed cosmological Friedmann's equations need to be solved here.
But in its linear evolutary stage, the problem can be simplified: for scales large enough, we treat a self-gravitating fluid with zero pressure, which collapses while the Universe expands.
All solutions we are going to see applies for a MATTER DOMINATED epoch ($z \leq 3000$)

COLLAPSE OF PERTURBATIONS - FULL CASE

The spatial scale above which we can consider a collapse only driven by gravity and with negligible pressure is the Jeans scale: $\lambda_j = \sqrt{\dfrac{15 k_B T}{4\pi G \mu \rho}}$,



with k_B the Boltzmann constant, T the gas temperature, G the Newton's constant, μ the mean molecular weight and ρ the mass density of the gas.

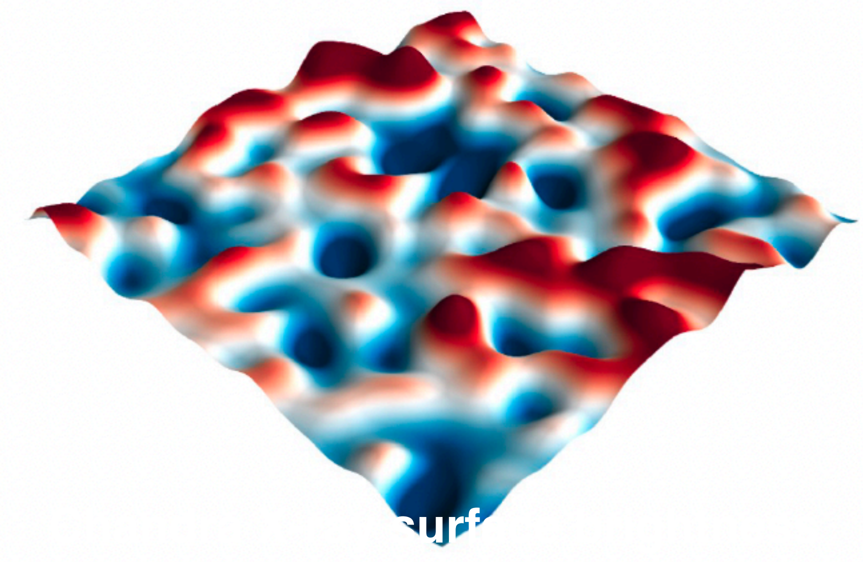
$L \geq \lambda_j$ perturbations undergo gravitational instability in a timescale: $t_J \sim 1/\sqrt{\rho G}$.

In EdS cosmology the expansion timescale of the Universe is $\tau_H = 1/H_0 \sim \sqrt{3/(8\pi G \rho)}$ (H_0 is the Hubble constant). Since $t_j \sim \tau$, *global* collapse is impossible.

However, density flucutations ($\delta \geq 0$) can produce patches of the Universe where $t_j \leq \tau_H$ and they undergo local collapse by "detaching from the Hubble flow" and produce patchy collapsing regions - similar to a set of top-hat perturbations.

COLLAPSE OF PERTURBATIONS - FULL CASE

The evolution of a self-gravitating, pressureless and non-relativistic **density contrast** δ on $L \geq \lambda_J$ scales is described by the continuity, the Euler, and the Poisson equations combined:



$$\frac{\partial \delta}{\partial t} + \nabla \cdot [(1+\delta)\mathbf{u}] = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + 2H(t)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla \phi}{a^2}$$

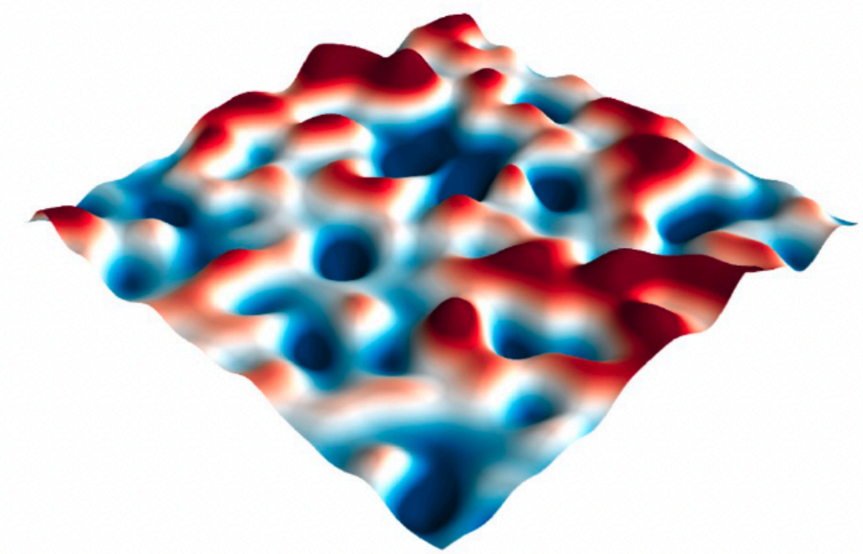
$$\nabla^2 \phi = 4\pi G \bar{\rho} a^2 \delta,$$

Note that patial derivatives are with respect to the **comoving** coordinate x, while $a(t)$ is the cosmic expansion factor such that $r = xa(t)$ is the **proper** coordinate, $v = \dot{r} = \dot{a}x + u$ is the total velocity of a fluid element (with $\dot{a}x$ giving the Hubble flow and $u = a\dot{x}$ giving the peculiar velocity),$\phi(x)$ is the gravitational potential and $H(t) = \dot{a}/a = E(t)H_0$ is the time-dependent Hubble parameter, which is in ΛCDM cosmology:

$$E(z) \equiv \frac{H(t)}{H_0} = \left[(1+z)^3 \Omega_m + (1+z)^2(1-\Omega_m-\Omega_\Lambda) + \Omega_\Lambda\right]^{1/2}.$$

COLLAPSE OF PERTURBATIONS - FULL CASE

When small density fluctuations ($\delta \ll 1$) are considered,
all the non-linear terms with respect to δ and u can be ignored,
the above equations can be written as



$$\frac{\partial^2 \delta}{\partial t^2} + 2H(t)\frac{\partial \delta}{\partial t} = 4\pi G\bar{\rho}\delta.$$

which delineates the Jeans instability of a fluid with no pressure under the counter-effect of the cosmic expansion.

This is a second order differential equation in time t, its solution can be written as

$$\delta(\mathbf{x}, t) = \delta_+(\mathbf{x}, t_i)D_+(t) + \delta_-(\mathbf{x}, t_i)D_-(t),$$

where $D_+(t)$ and $D_-(t)$ the growing and decaying modes of $\delta(x, t)$ and $\delta_+(x, t)$, $\delta_-(x, t)$ are the corresponding spatial distribution of the primordial matter field.
The specific evolution with time depends on the detail of the Cosmological model.

COLLAPSE OF PERTURBATIONS - FULL CASE

General solutions for equation $$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} + (c_s k^2 - 4\pi G\bar{\rho})\delta = 0$$

$\lambda \gg \lambda_J = c_s(\pi/\bar{\rho}G)^{1/2}$ we get growing/decaying solutions

$\lambda \ll \lambda_J = c_s(\pi/\bar{\rho}G)^{1/2}$ we get oscillating solutions (pressure)

In Einstein de Sitter $\bar{\rho} = 1/(6\pi G t^2)$ $a = a_0(t/t_0)^{2/3}$ and $\dot{a}/a = 2/(3t)$

hence $$\frac{d^2\delta}{dt^2} + (4/3t)\frac{\delta}{dt} - \frac{2}{3t^2}(1 - \frac{c_s^2 k^2}{4\pi G\bar{\rho}})\delta = 0$$

if we look for $\delta \propto t^n$ solutions we get $$\frac{\delta\rho}{\rho} = \exp(i\vec{k}\cdot\vec{r})t^{[1\pm 5(1-4v_s^2k^2/(25\pi G\rho)^{1/2}]/6}$$

when $\lambda \geq \lambda_J$ we thus have gravitational instability, and get

$\frac{\delta\rho}{\rho} \sim t^{[1\pm 5(1-(\lambda_J/\lambda)^2)^{1/2}]/6}$ with solutions $\delta_+ \sim t^{2/3}$ (growing) and $\delta_- \sim t^{-1}$ (decaying)

when $\lambda \ll \lambda_J$ we get instead oscillations $\frac{\delta\rho}{\rho} \sim \exp[i(\vec{k}\cdot\vec{r} \pm kc_s \ln t)]$

Solutions can be get for closed and open models, by using the appropriate $H(t)$,$a(t)$, $\rho(t)$

COLLAPSE OF PERTURBATIONS - FULL CASE

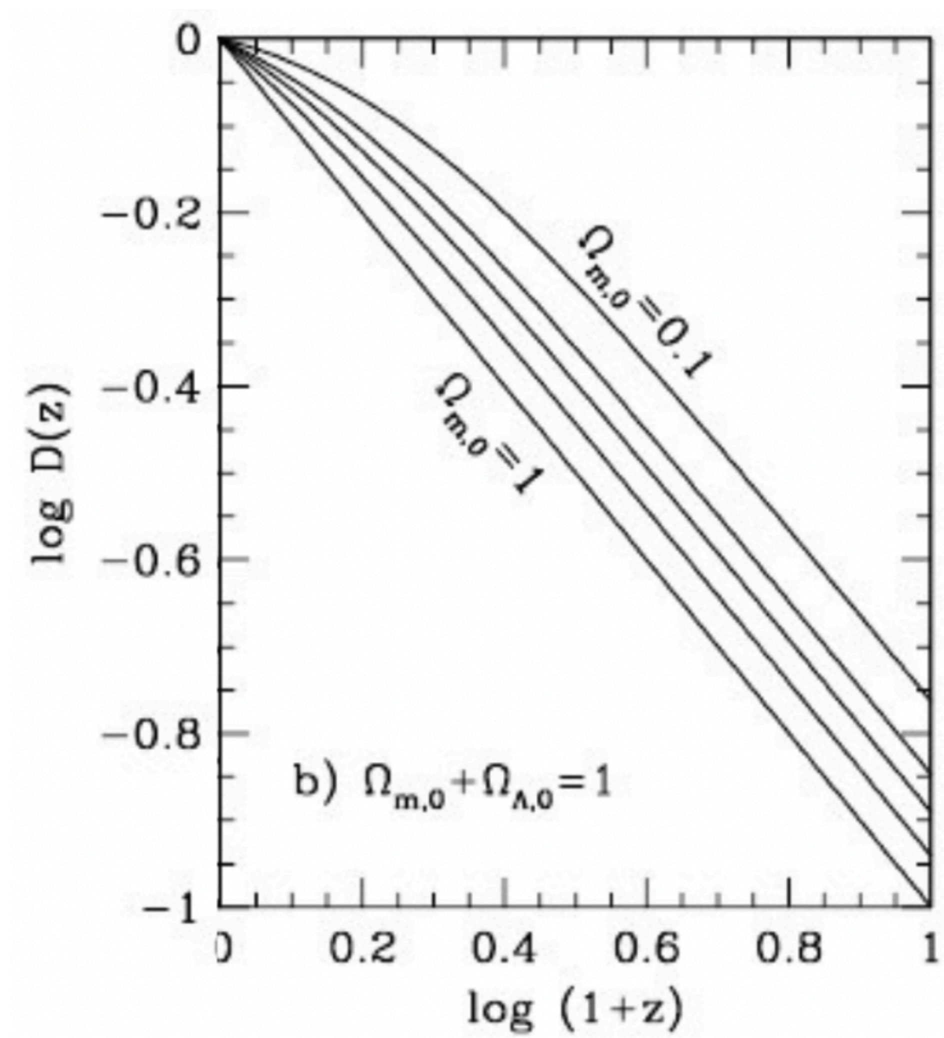
In the Einstein-de-Sitter model (EdS, $\Omega_M = 1$, $\Omega_\Lambda = 0$) the Hubble constant evolves as

$H(t) = \dfrac{2}{3t}$ In this case the solution for the growing and th decreasing modes are

$$D_+(t) = (t/t_i)^{2/3} \propto a(t) \quad \text{and } D\_(t) = (t/t_i)^{-1}$$

So in this particular case, cosmic expansion and gravitational instability proceed at the same rate.

Instead in models with $\Omega_M < 1$ (like ΛCDM, where Ω_M ~ 0.3) there is an epoch, when the cosmological constant begins to be significant, at which the characteristic time-scale of expansion turns out to be shorter than in the EdS case

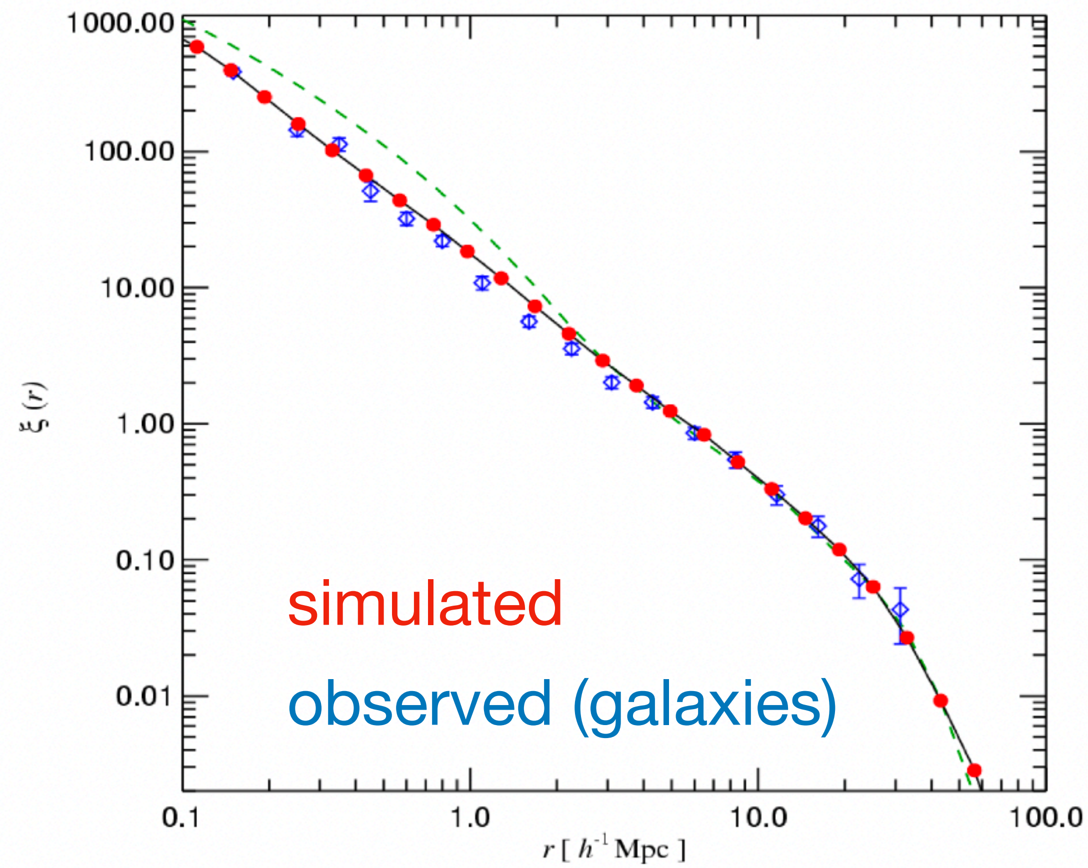


STATISTICAL VIEW: POWER SPECTRUM

We can study the statistical evolution of the $\delta(\vec{x}, t)$ density field. If the primordial density field is a realization of a random field, its statistics can be describe with a 2 point correlation function:

$$\xi(\vec{r}) = \langle \delta(\vec{x} + \vec{r})\delta(\vec{x}) \rangle,$$

where the brackets denote an average over all realizations over the entire Universe.

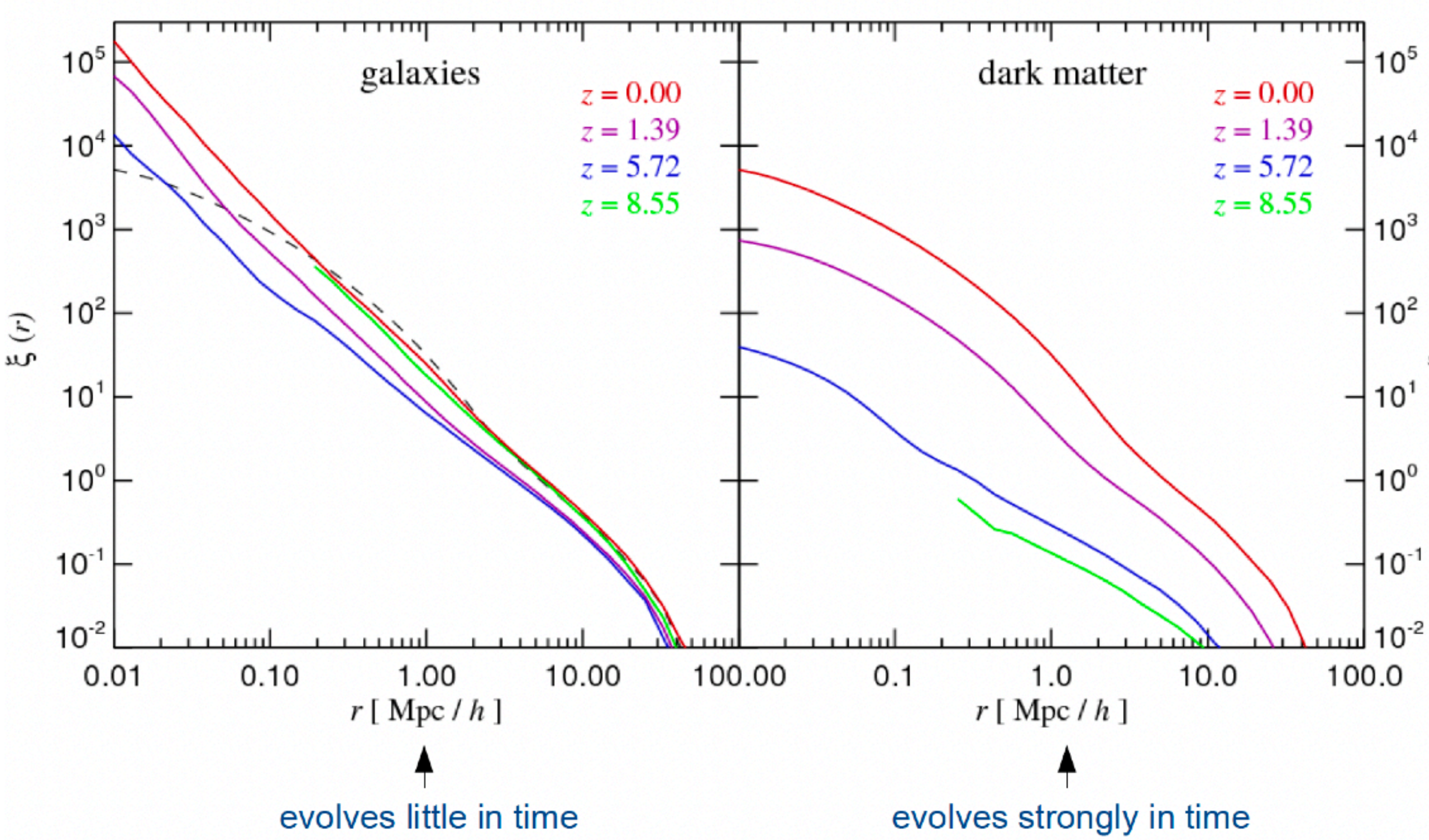


This tells us the average number of galaxies within a distance r

$$\langle N_r \rangle = \frac{4\pi r^3}{3}\langle n \rangle + 4\pi\langle n \rangle \int_0^r \xi(r'_{12}) r'_{12} dr'_{12}$$

average excess

The two-point correlation function of galaxies is to a very good approximation a power law. Galaxies and dark matter have slightly different $\xi(r)$

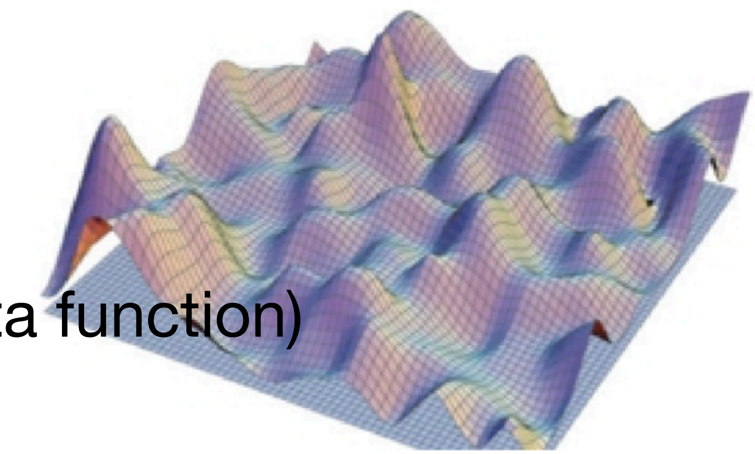


STATISTICAL VIEW: POWER SPECTRUM

In a complementary view (which leads to easier numerical computation) we can deal with the Fourier transform of the density field,

$$\tilde{\delta}(\vec{k}) = \int d^3x \mathrm{e}^{\mathrm{i}\vec{\mathrm{k}}\cdot\vec{\mathrm{x}}}\delta(\vec{\mathrm{x}})$$

This allows us to define the power spectrum $P(k)$, from



$$\langle\tilde{\delta}(\vec{k})\tilde{\delta}(\vec{k}')\rangle = (2\pi)^3\delta_D(\vec{k}+\vec{k}')P(k)$$

(where δ_D is the Dirac delta function)

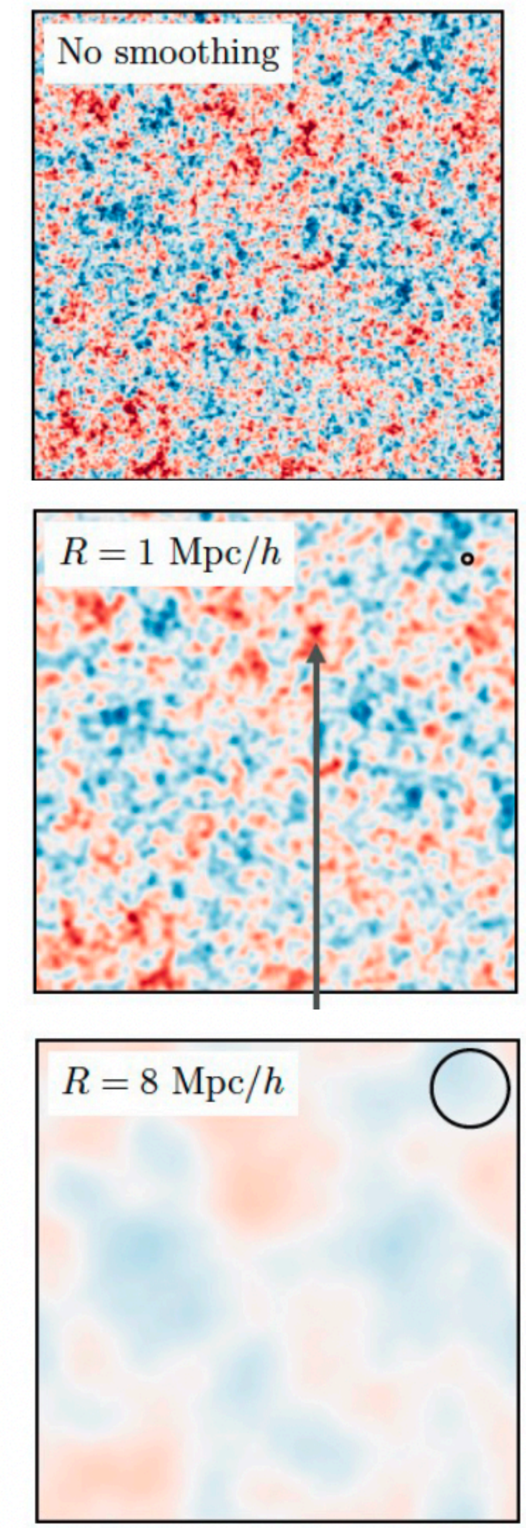
The correlation funciton and the power spectrum are related via: $$P(k) = \int d^3x\xi(\vec{x})e^{i\vec{k}\cdot\vec{x}}$$

If the power spectrum is a power law, $P(k) \propto k^n$, then the correlation function is a power law too $\xi(r) \propto r^{-\gamma}$ with $\gamma = n + 3$.

The spectral index n = 0 is a white-noise (i.e., no correlations) spectrum, and the n = 1 spectrum is a Peebles-Harrison-Zeldovich or "flat" scale-invariant spectrum.

STATISTICAL VIEW: POWER SPECTRUM

In practice, it is impossible to measure the density perturbation $\delta(\vec{x}, t)$ at a particular point. Instead, it is only possible to measure the density perturbation smoothed over some volume:



$$\delta_R(\vec{x}, t) = \int d^3r \; W(\vec{r}) \; \delta(\vec{x} + \vec{r})$$

where $W(\vec{r})$ is a "window function". The most simple is again the spherical top-hat: $W_R(r) = 3/(4\pi R^3)$ for $r \leq R$ or $W_R(r) = 0$ otherwise.

The mass variance smoothed within the radius R is thus:

$$\sigma^2(R) = \frac{1}{2\pi} \int P(k) W^2(k) k^2 dk$$

Notice: this quantity, if $R = 8\mathrm{Mpc/h}$ comoving, is used to normalize the power spectrum of cosmological model in CMB studies ($\sigma_8 \approx 0.7 - 0.8$ in ΛCDM).

STATISTICAL VIEW: POWER SPECTRUM

The comoving scale that entered the horizon at the epoch of matter-radiation equality is $L_H = 16(\Omega_m h^2)^{-1}$Mpc.

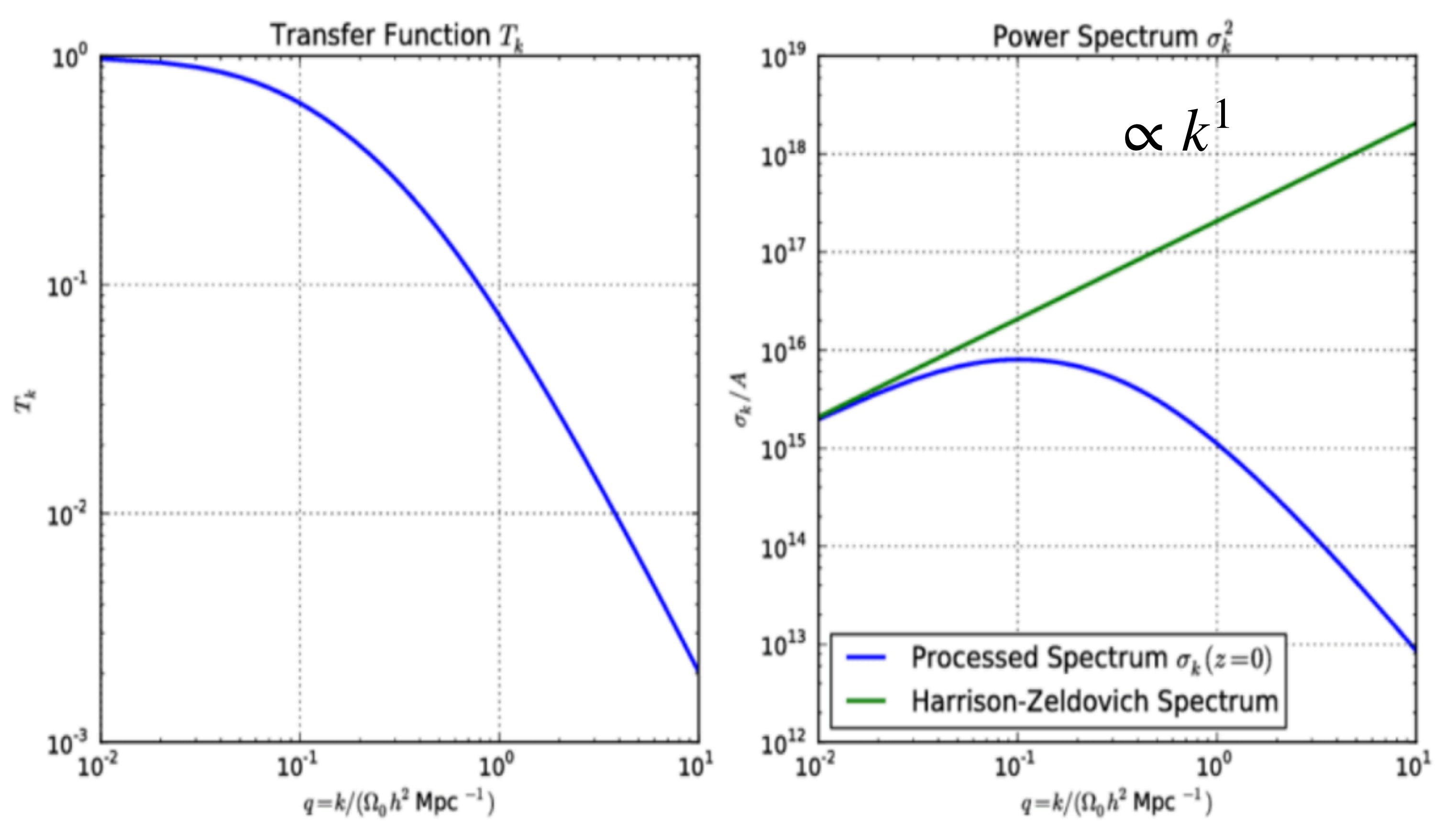
Modes with $k \geq 2\pi/L_H$ entered the horizon at earlier times, during radiation domination, while $k < 2\pi/L_H$ modes entered the horizon at later times, during matter domination.

The growth of density perturbations is different during radiation domination than it is during matter domination, and so $L_H = 2\pi/k_H$ has an important role in determining $P(k)$.

The <u>primordial power spectrum</u> predicted by inflation is the Harrison-Zeldovich one $P_{prim}(k) \propto k^1$. This gets "processed" by the growth of density perturbations once they enter the horizon, resulting in the power spectrum of the Universe today:

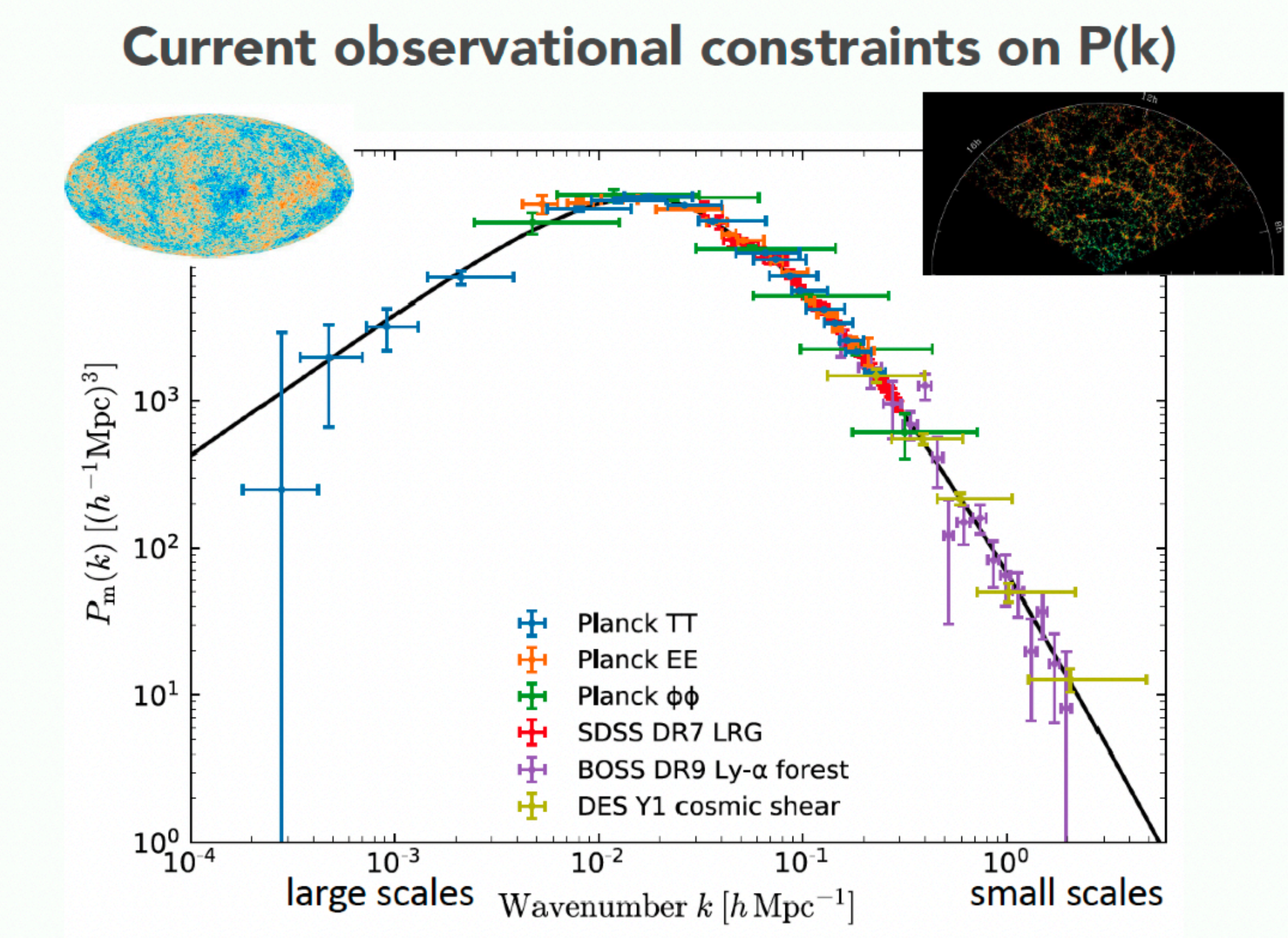
$P(k) = T(k)P_{prim}(k)$

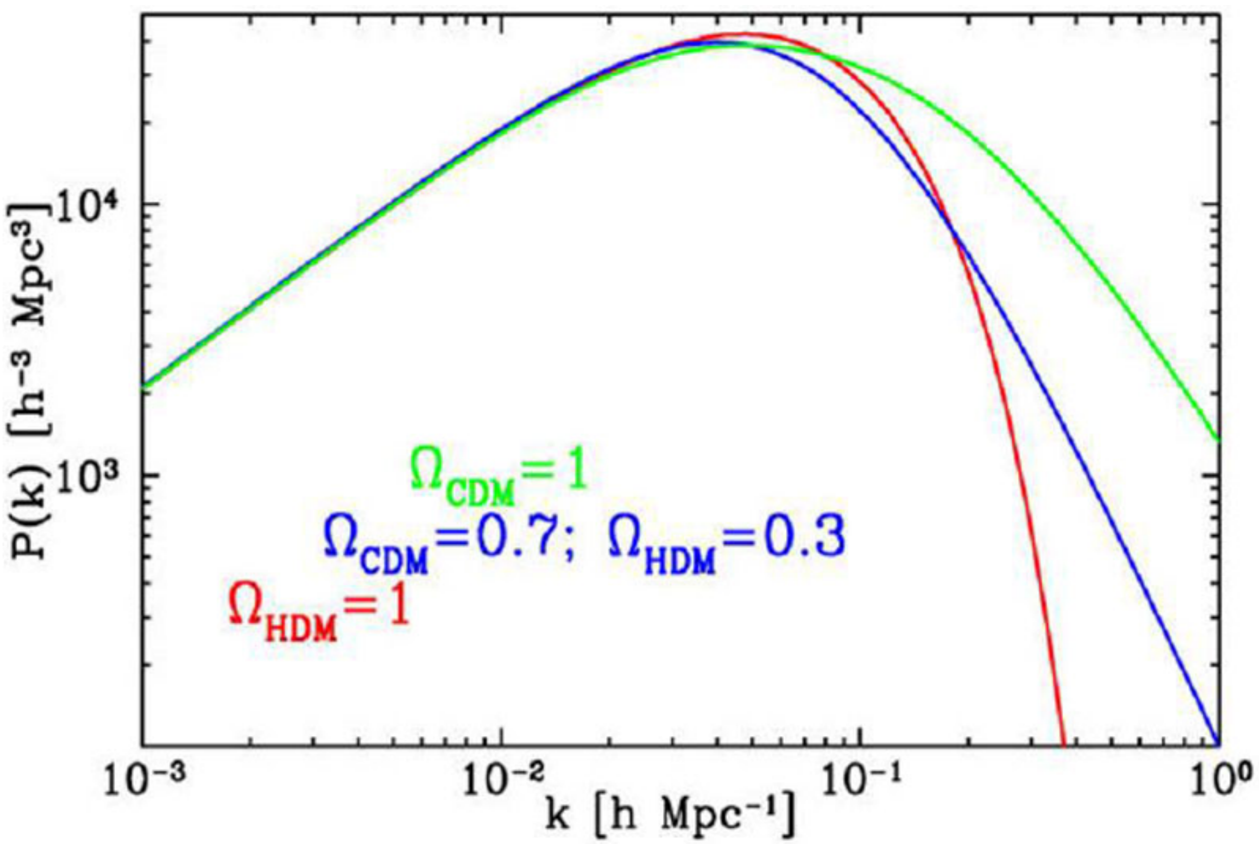
T(k) is a "transfer function" that accounts for the effects of gravitational amplification of a density-perturbation as a function of k. The calculation of T(k) is based on the solutions for $\delta(t)$ given before, and it requires sophiticated calculations/numerical integration.



STATISTICAL VIEW: POWER SPECTRUM

To a first order, the agreement between the theoretical and the observed power spectrum (from different astronomical probes) is extremely good.

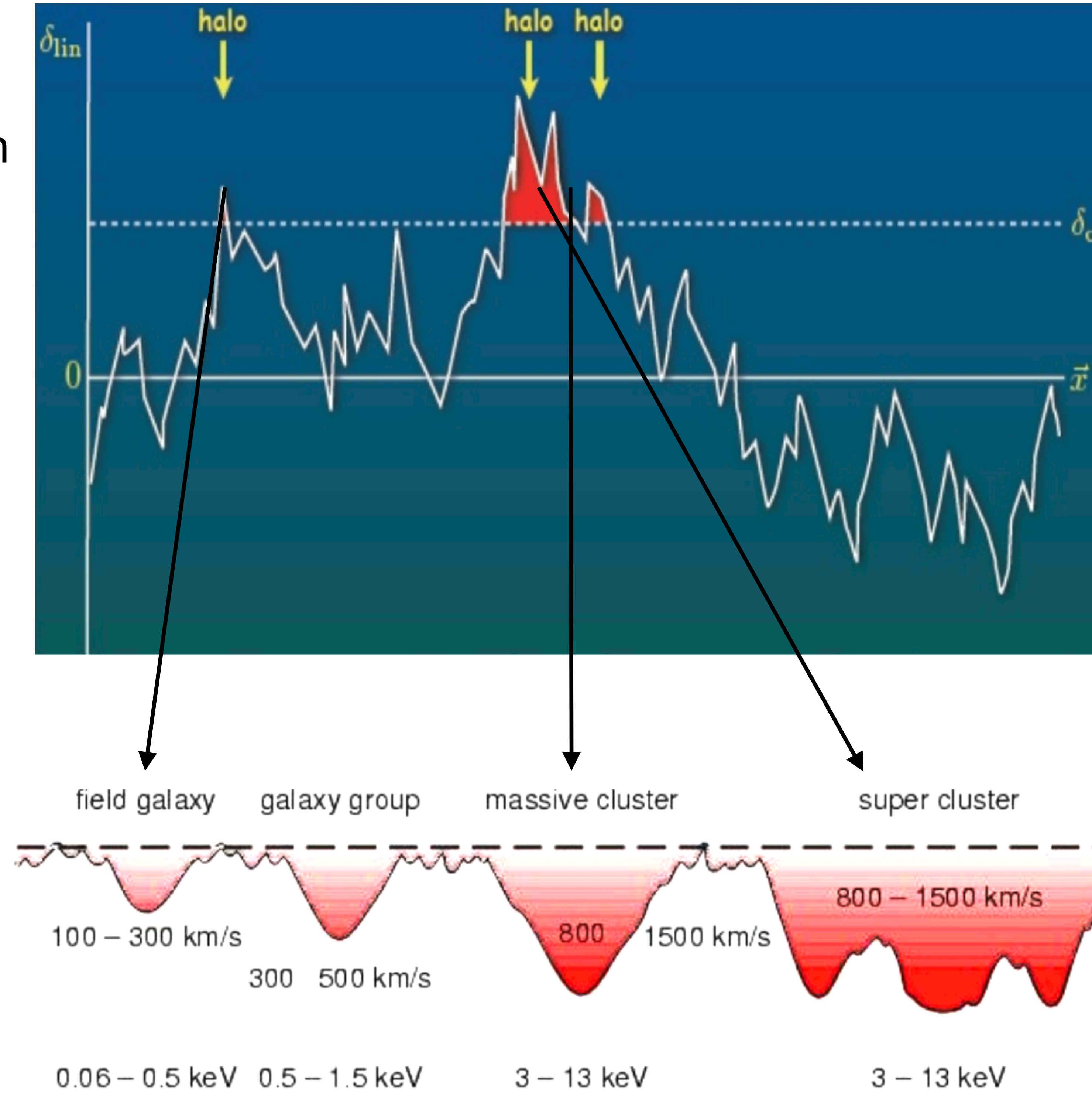


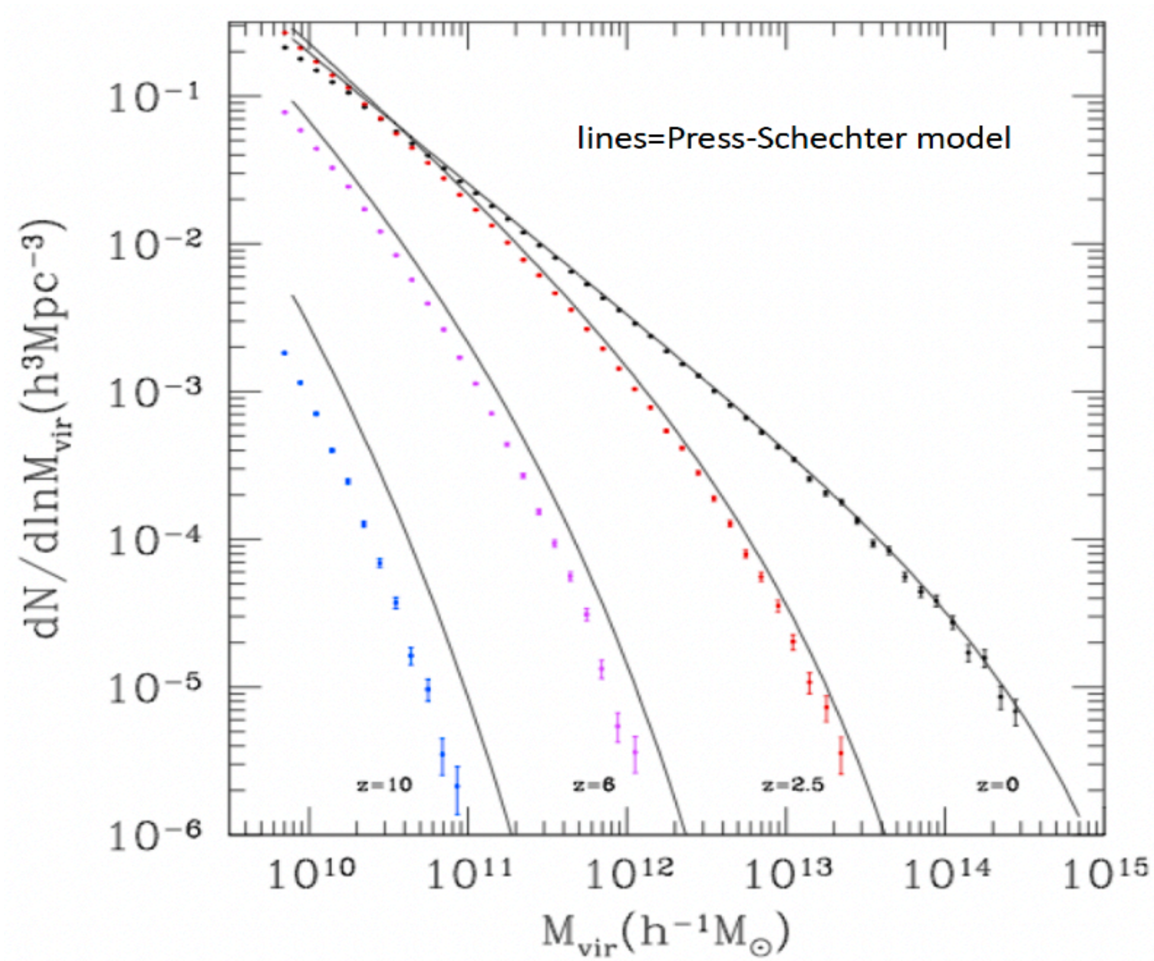


The power spectrum fully describes the statistics of a Gaussian field, and it is used to constrain cosmology, as the growth of perturbations in different cosmologies changes the $T(k)$ function

QUANTIFYING PRESENT DAY COSMIC STRUCTURES

From the statistical distribution of σ^2 fluctuations as a function of scale on can derive the expected [mass distribution of collapsing halos,] considering that all fluctuations with $\delta > 1.686$ will collapse (a few important models: Press and Schechter'74, Sheth & Tormen'98, Jenskins 2001, Tinker+10)





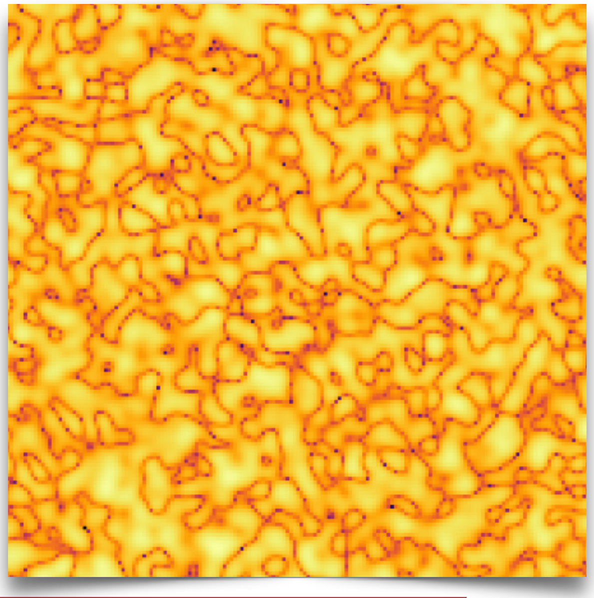
$$\frac{kT}{7\ \mathrm{keV}} = \left(\frac{M}{10^{15} h^{-1} M_\odot}\right)^{2/3} (1 + z_{\mathrm{vir}})$$

WHAT IF THERE IS A MAGNETIC FIELD?

In analogy with the ordinary Jeans length, $\lambda_j = c_s\sqrt{\frac{\pi}{\rho G}}$ (where c_s = sound speed)

we must introdyce the **magnetic** Jeans length to be $\lambda_B = v_A\sqrt{\frac{\pi}{\rho G}}$

Hence the <u>total Jeans length is larger</u> ($\sim \lambda_B + \lambda_j$) than the unmagnetised case.



For $L \gg \lambda_B$ magnetic effects can be neglected,

while for $L < \lambda_B$ magentic field affects the evolution of ordinary matter and can change the halo mass function.

It is $\lambda_B \sim 0.1\mathrm{Mpc}\frac{\mathrm{B_{rms}}}{[\mathrm{nG}]}$ (in comoving units) hence the evolution on scales tyical of galactic halos can be affected by a range of primordial magnetic fields.

WHAT IF THERE IS A MAGNETIC FIELD?

The evolution of the baryonic density fluctuations δ now becomes:

$$\frac{\partial \delta}{\partial t} + \frac{\vec{\nabla} \cdot \vec{v}}{a} + \frac{\vec{\nabla} \cdot (\delta \vec{v})}{a} = 0.$$

$$\frac{\partial \vec{v}_{\mathrm{b}}}{\partial t} + H \vec{v}_{\mathrm{b}} + \frac{(\vec{v}_{\mathrm{b}} \cdot \nabla) \vec{v}_{\mathrm{b}}}{a} + \frac{c_{\mathrm{b}}^2}{a} \nabla \delta_{\mathrm{b}} = \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi a^5 \rho_{\mathrm{b}}} - \frac{\nabla \phi}{a},$$

+ Poisson equations for gravity, continuity & momentum equation for DM etc..

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{a} \nabla \times (\vec{v}_{\mathrm{b}} \times \vec{B})$$ with $$\vec{B} = a^2 \vec{B}_{\mathrm{phys}}$$

The above equations are ordinary differential equations and their solution can be expressed as a linear combination of the homogeneous solution determined by the standard cosmological solution seen above, $\delta^{lin}_{\Lambda CDM}$ and the inhomogeneous solution source by magnetic fields, δ^{lin}_{B}.

As long as flucutations are in the linear regime: $\delta^{lin} \approx \delta^{lin}_{\Lambda CDM} + \delta^{lin}_{B}$

WHAT IF THERE IS A MAGNETIC FIELD?

By opportunely computing the transfer function T(k) to evolve the primordial spectrum under the effect of PMFs we can estimate the modifications ot the matter spectrum modified by magnetic fields.

The modifications depend on the amplitude and on the spectral shape of the input magnetic power spectrum $P_B(k) \propto k^{n_B}$

Modifications to standard cosmological perturbations are confined to $1/L \geq 1/\lambda_B$

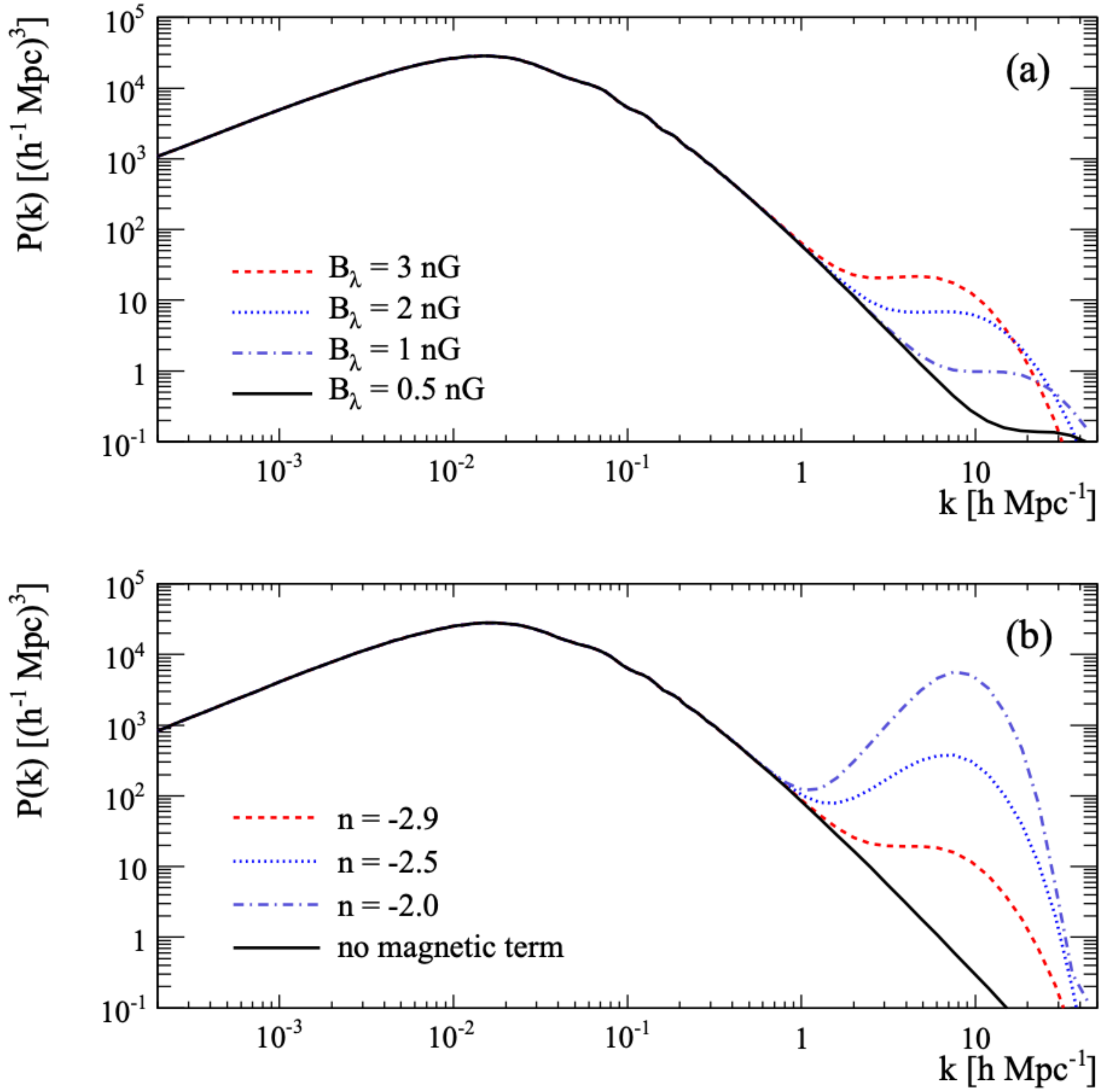


FIG. 1.— The magnetic field matter power spectra for $n_B = -2.9$ and for different values of B_λ (a) and for B_λ = 3 nG and for different values of n_B (b).

Kahniashvili et al. 2013, https://arxiv.org/pdf/1211.2769

WHAT IF THERE IS A MAGNETIC FIELD?

The modified spectrum determines a different distribution of $\sigma^2(R)$.

This might change a distorsion of the mass distribution of halos with respect to ΛCDM.

For $B_{rms} \sim 1 - 10\text{nG}$:

- suppression of $M \leq 10^9 M_\odot$ halos
- henancement of $M \geq 10^{10} - 10^{11} M_\odot$ halos

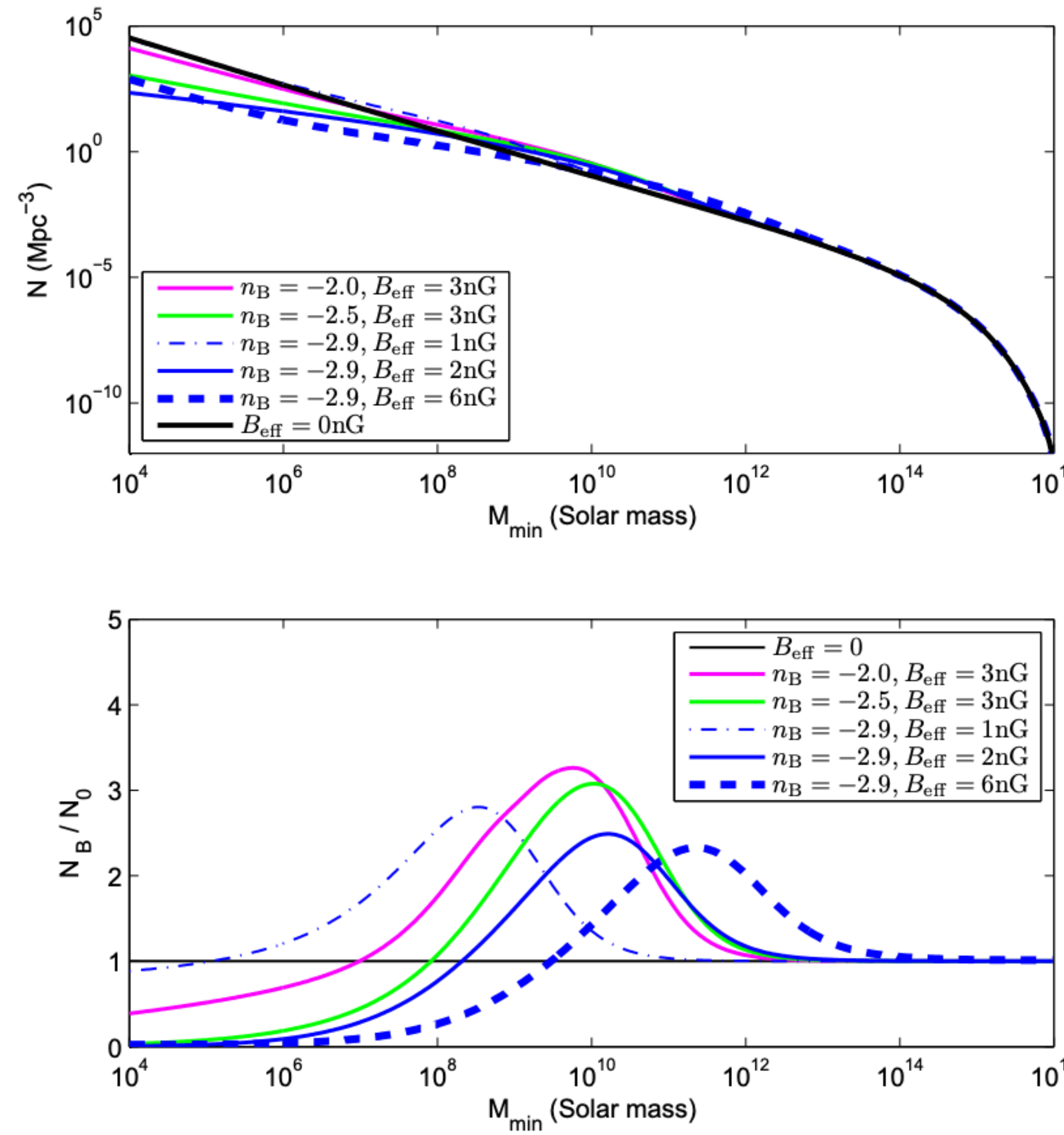


FIG. 4.— Halo number density $N(M > M_0)$ (top panel) and ratio of number density for magnetic and non-magnetic simulations $N_B)/N_0$ (bottom panel) for different effective magnetic field values B_{eff} and spectral index n_B, and $z = 0$. Number of small mass objects ($M \sim 10^4 M_\odot$) in magnetized case can be reduced down by factor of 100 compared to the non-magnetic number, object number count excess occurs for objects with mass around ($M \sim 10^{10} M_\odot$).

Kahniashvili et al. 2013, https://arxiv.org/pdf/1211.2769

WHAT IF THERE IS A MAGNETIC FIELD?

Notice however: the full incorporation of primordial magnetic fields in cosmological structure formation is still subject of active research.

The full range of physical effects of PMFs can be properly taken into account only with full numerical simulations:

- PMFs can source velocity fields which adds to primordial velocity fields, and can source additional overdensities;
- PMFs can get amplified via dynamo and inhibit collapse in halos;
- PMFs can dissipate energy and heat up baryons and change the thermal & ionisation history of the Universe

(see e.g. Ralegankar et al. 2025 for a recent work)

SOME SUGGESTED READING

- *"Introduction to Cosmology"* by *John Peacock (2010)* [*https://indico.ictp.it/event/a09159/session/2/contribution/1/material/0/0.pdf*](https://indico.ictp.it/event/a09159/session/2/contribution/1/material/0/0.pdf)

- *"Large-Scale Structure Formation: From the First Non-linear Objects to Massive Galaxy Clusters"* *Planelles, Schelicher & Bykov, 2014*

- *"Matter power spectrum induced by primordial magnetic fields: from the linear to the non-linear regime" JCAP08(2025)011* *Ralegankar, Garaldi & Viel 2025*