Cosmological magnetic field observations
Lecture 1
Magnetic field as a cosmological observable

Notation conventions and units

Introduction and motivations
- Cosmological epochs
- Cosmological observables
- Existing cosmological probes
- Magnetic field as a cosmological observable?

Cosmological magnetic field description
- Maxwell equations in curved space-time
- “locally measured” and “comoving” electric and magnetic fields
- Stochastic magnetic fields: power spectrum, strength, correlation length

Magnetic field across cosmological epochs
- Magneto-hydrodynamics in expanding Universe
- Magnetic field evolution laws (example helical field)
- Relic magnetic field in the present-day Universe

Homogeneous isotropic Universe; Notation conventions

Expanding homogeneous and isotropic Universe with Friedman-Robertson-Walker-Lemaitre metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x}^2 \qquad (1)$$

Two alternative time coodinate choices: "proper time" t, "conformal time" $\tilde{t} = \int dt/a(t)$:

$$ds^2 = a^2(\tilde{t})(-d\tilde{t}^2 + d\vec{x}^2)$$

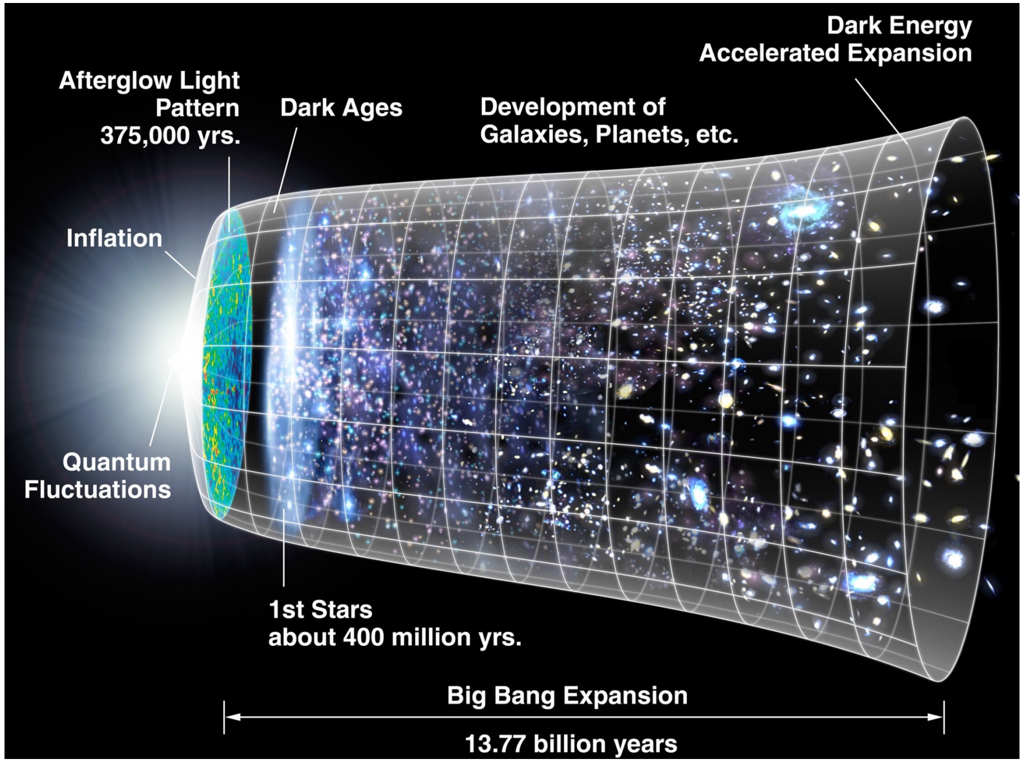
Covariant (V_μ) and contravariant (V^μ) vectors:

$$V_\mu = g_{\mu\nu}V^\nu: \quad V_0 = -V^0; \quad V_i = a^2V^i$$

Greek subscripts ($\mu, \nu, \ldots$) run through (0,1,2,3), $x^0 = t$ or $x^0 = \tilde{t}$. Latin subscripts ($i, j, \ldots$) run through (1,2,3). Repeated indices imply sum over all values.

We will only consider flat Eucledian spatial geometry with metric $d\,\vec{x}^2 = \delta_{ij}dx^i dx^j$. The scale factor is $a(t_0) = 1$, there t_0 is "today". Redshift:

$$1 + z = \frac{a(t_0)}{a(t)} = \frac{1}{a}$$



Christoffel symbols

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\alpha}\left(g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda,\alpha}\right)$$

Comma stands for : $_{,\mu} \equiv \partial_\mu \equiv \partial/\partial x^\mu$. Semocolon is for covariant derivative

$$V^\mu_{;\nu} = V^\mu_{,\nu} + \Gamma^\mu_{\nu\alpha}V^\alpha$$

Non-zero Christoffel symbols for the metric (1):

$$\Gamma^i_{j0} = \frac{\dot{a}}{a}\delta_{ij}; \quad \Gamma^0_{ij} = a\dot{a}\delta_{ij}$$

Dot stands for d/dt.

Natural system of units

Natural units:

$$\hbar = 6.6\times10^{-16}\ \text{eV s} \equiv 1;$$
$$c = 3\times10^{10}\frac{\text{cm}}{\text{s}} \equiv 1;$$
$$\hbar c = 2\times10^{-5}\ \text{eV cm} \equiv 1;$$
$$k_B = 8.6\times10^{-5}\ \text{eV K} \equiv 1$$

Gaussian electromagnetic units (Jackson, Electrodynamics, Appendix). Maxwell equations in Minowski space-time:

$$\partial_i E_i = \mathbf{4\pi}\rho_e$$
$$\epsilon_{ijk}\partial_j B_k = \mathbf{4\pi}J_i + \partial_t E_i$$
$$\epsilon_{ijk}\partial_j E_k + \partial_t B_i = 0$$
$$\partial_i B_i = 0$$

where ϵ_{ijk} is Levi-Civita symbol, $\epsilon_{123} = 1$. Electromagnetic field energy density

$$\rho_{em} = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right)$$

Energy density of magnetic field of the strength 1 G:

$$\rho_B = \frac{(1\ \text{G})^2}{8\pi} \equiv \frac{1}{8\pi}\frac{\text{erg}}{\text{cm}^3};$$

$$1\ \text{G} \equiv \sqrt{\frac{\text{erg}}{\text{cm}^3}} = \sqrt{\frac{1\ \text{eV}}{1.6\times10^{-12}\ \cdot \text{cm}^3}} = \sqrt{\frac{1\ \text{eV}\cdot(2\times10^{-5}\ \text{eV})^3}{1.6\times10^{-12}}} = 0.069\ \text{eV}^2$$

Electron charge

$$e = \sqrt{\alpha_{EM}} \simeq \frac{1}{\sqrt{137}} \simeq 0.085$$

```
[1]: import astropy.units as u
     import astropy.constants as const
     from numpy import sqrt

[2]: const.hbar
```

[2]: 1.0545718×10^{-34} J s

```
[3]: hbar=const.hbar.to(u.eV*u.s)
     hbar
```

[3]: 6.5821196×10^{-16} eV s

```
[4]: c=const.c.to(u.cm/u.s)
     c
```

[4]: $2.9979246\times10^{10}\ \frac{\text{cm}}{\text{s}}$

```
[5]: hbarc=(hbar*c).to(u.eV*u.cm)
     hbarc
```

[5]: 1.9732698×10^{-5} cm eV

```
[6]: Gauss_in_eV2=sqrt((u.erg.to(u.eV)*hbarc**3)).value
     Gauss_in_eV2

[6]: 0.06925075434870955
```

Exercise 1. Find the value of Newton constant

$$G_N = 6.67\times10^{-8}\frac{\mathrm{cm}^3}{\mathrm{g\,s}^2}$$

in Natural system of units. Express in powers of eV, s, or cm.

Exercise 2. Consider a black hole of mass M with gravitational radius $R_g = G_N M/c^2$. Find the mass of the black hole at which R_g becomes equal to its Compton wavelength $\lambda = h/mc$. Compare with Planck scale(s).

Exercise 3. Consider an electron moving with velocity $\vec{v}$ in homogeneous magnetic field $\vec{B} \perp \vec{v}$. Find the magnetic field strength at which the gyroradius becomes comparable to the de Broglie wavelength of the electron. Compare with Schwinger limit of Quantum Electrodynamics.

Exercise 4. Go through the "Units and Quantities" tutorial of astropy Python package and define a set of functions expressing measurements derived from observational data (typically in cgs units) into Natural units (in powers of eV, s or cm) and back.

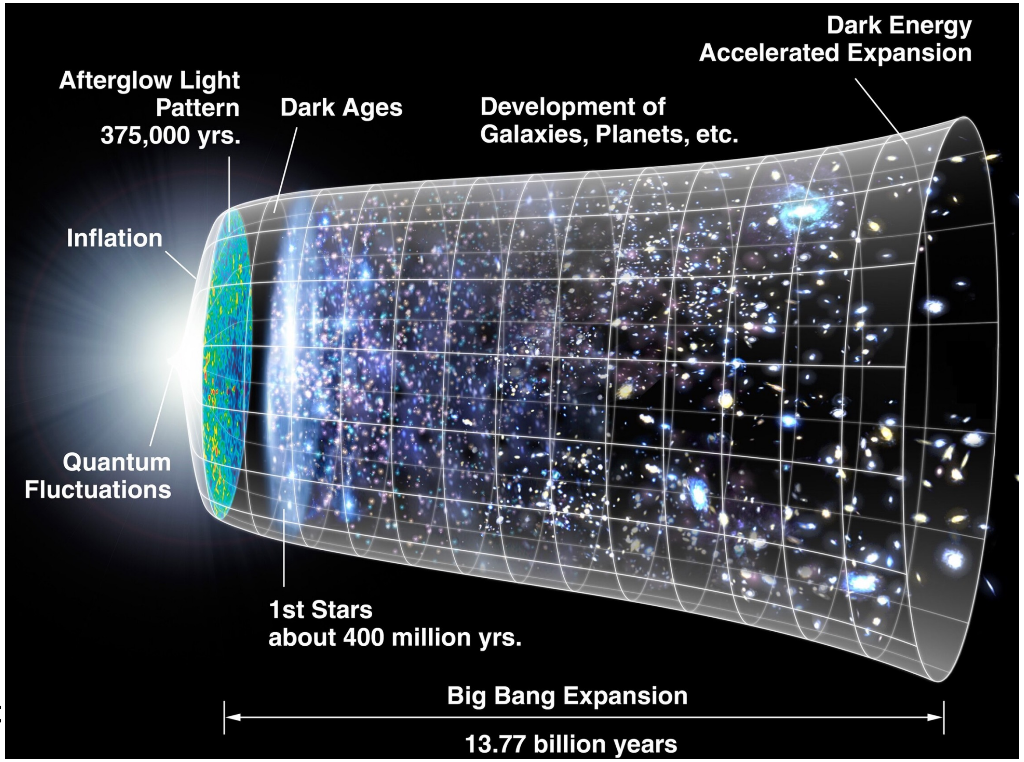
Cosmological epochs

Dynamical equations:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho = \frac{8\pi}{3M_{Pl}^2}\rho \quad \text{(Friedman)}$$

$$\dot{\rho} = -3H(p+\rho) \qquad \text{(energy conservation)}$$

$$p = w\rho \qquad \text{(equation of state)}$$



Radiation-dominated Universe:

$w = 1/3 \qquad \rho \propto a^{-4}; \qquad a \propto t^{1/2}$

Matter-dominated Universe:

$w = 0 \qquad \rho \propto a^{-3}; \qquad a \propto t^{2/3}$

Cosmological constant dominated Universe:

$w = -1 \qquad \rho \propto a^{0}; \qquad a \propto e^{Ht}$

Hubble time:

$$t_H = H^{-1} = \left(\frac{3M_{Pl}^2}{8\pi\rho}\right)^{1/2}$$

Hubble time today ($H_0 \simeq 70$ km/(s Mpc), 1 Mpc $= 3\times10^{24}$ cm):

$t_{H0} = H_0^{-1} \simeq 4\times10^{17}$ s $\simeq 1.4\times10^{10}$ yr.

Energy density in radiation-dominated Universe:

$$\rho = \frac{\pi^2}{30} g_* T^4$$

where g_* is the effective number of relativistic degrees of freedom ($g_* = 2$ for black body photon gas). Reference Hubble time at $T \sim 1$ MeV:

$$t_H = \left(\frac{90}{8\pi^3 g_*}\right)^{\frac{1}{2}} \frac{M_{Pl}}{T^2} \simeq 1 \left[\frac{T}{1\ \text{MeV}}\right]^{-2} \left[\frac{g_*}{20}\right]^{-1/2} \text{ s}$$

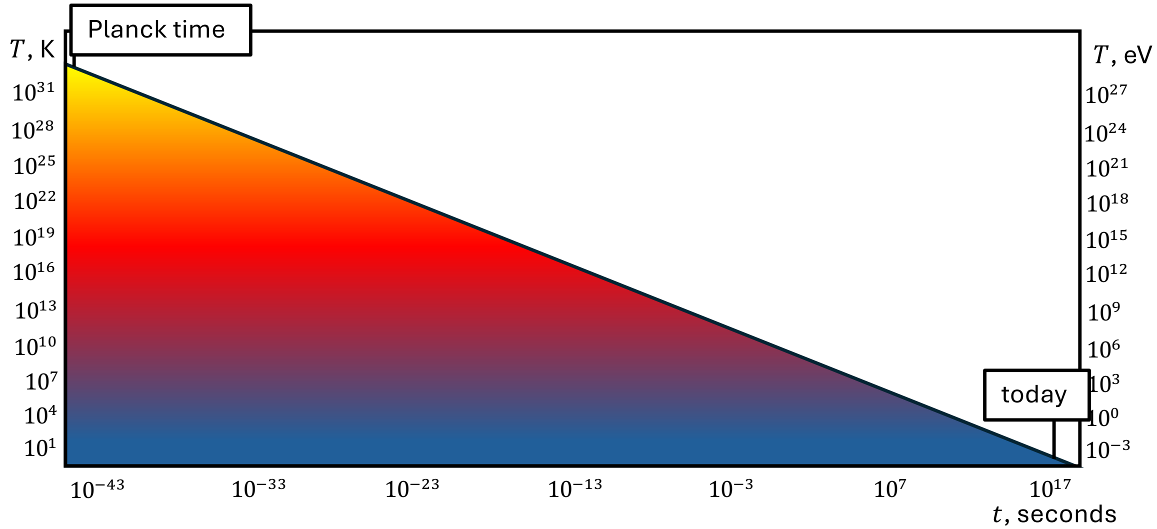
Cosmological epochs

Dynamical equations:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho = \frac{8\pi}{3M_{Pl}^2}\rho \quad \text{(Friedman)}$$

$$\dot{\rho} = -3H(p+\rho) \qquad \text{(energy conservation)}$$

$$p = w\rho \qquad \text{(equation of state)}$$



Radiation-dominated Universe:

$w = 1/3 \quad \rho \propto a^{-4}; \quad a \propto t^{1/2}$

Matter-dominated Universe:

$w = 0 \quad \rho \propto a^{-3}; \quad a \propto t^{2/3}$

Cosmological constant dominated Universe:

$w = -1 \quad \rho \propto a^{0}; \quad a \propto e^{Ht}$

Hubble time:

$$t_H = H^{-1} = \left(\frac{3M_{Pl}^2}{8\pi\rho}\right)^{1/2}$$

Hubble time today ($H_0 \simeq 70$ km/(s Mpc), 1 Mpc $= 3\times10^{24}$ cm):

$t_{H0} = H_0^{-1} \simeq 4\times10^{17}$ s $\simeq 1.4\times10^{10}$ yr.

Energy density in radiation-dominated Universe:

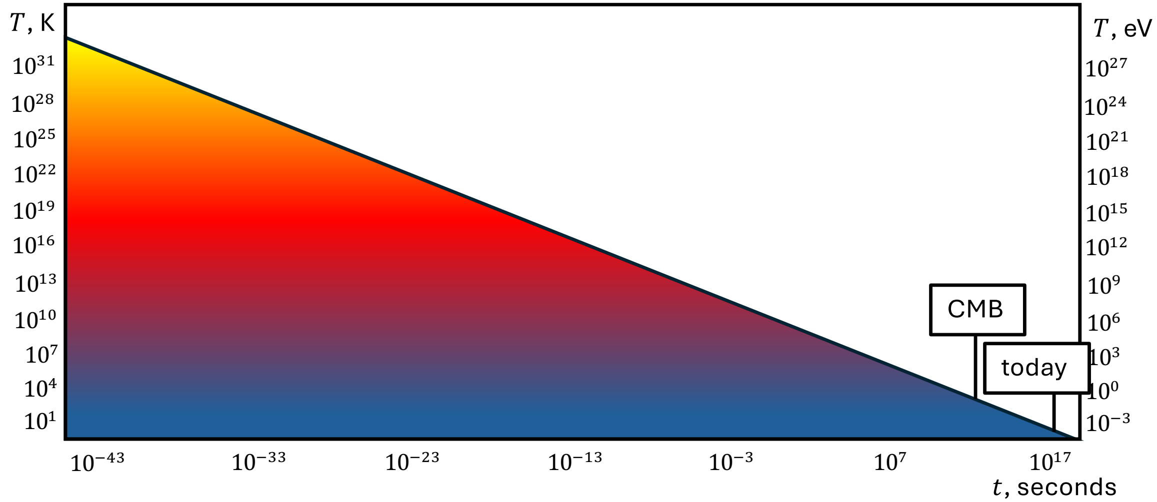
$$\rho = \frac{\pi^2}{30} g_* T^4$$

where g_* is the effective number of relativistic degrees of freedom ($g_* = 2$ for black body photon gas). Reference Hubble time at $T \sim 1$ MeV:

$$t_H = \left(\frac{90}{8\pi^3 g_*}\right)^{\frac{1}{2}} \frac{M_{Pl}}{T^2} \simeq 1 \left[\frac{T}{1\ \text{MeV}}\right]^{-2} \left[\frac{g_*}{20}\right]^{-1/2} \text{ s}$$

Cosmological epochs

Atoms get unbound when temperature is above $T \sim 1$ eV

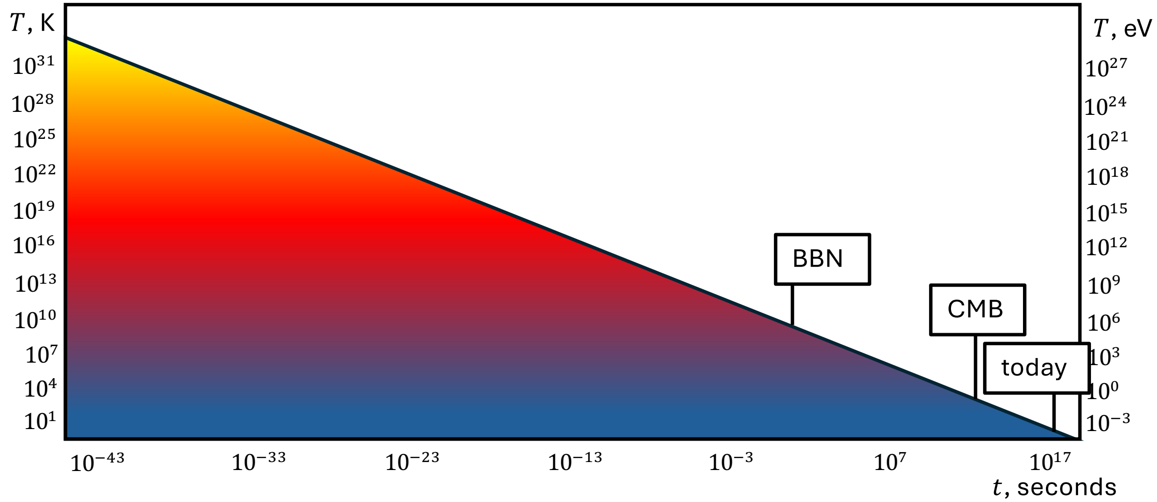


CMB - Cosmic Microwave Background radiation formation epoch

Cosmological epochs

Atoms get unbound when temperature is above $T \sim 1$ eV

Atomic nuclei get unbound when temperature is $T \sim 1$ MeV



CMB - Cosmic Microwave Background radiation formation epoch

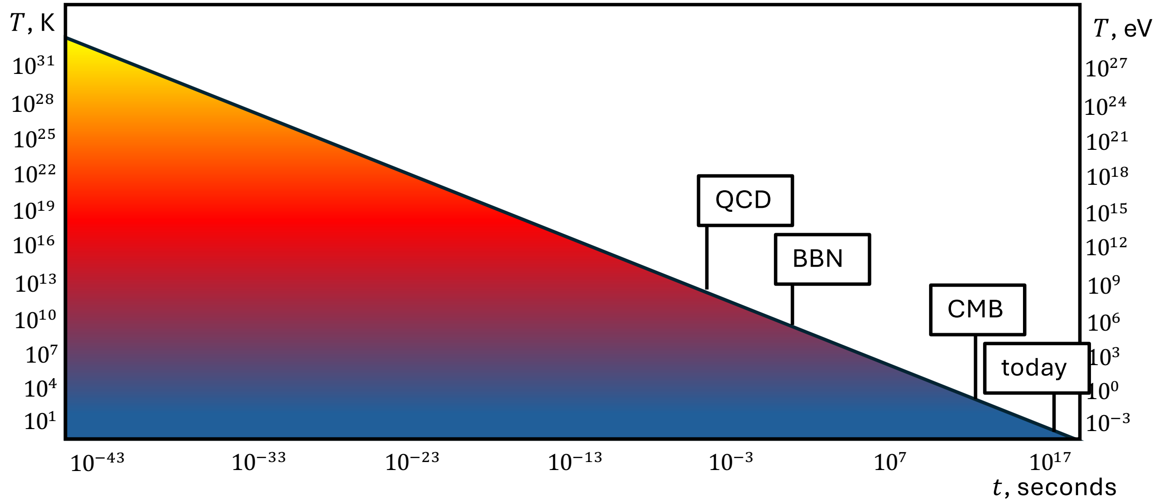
BBN - Big Bang Nucleosynthesis epoch

Cosmological epochs

Atoms get unbound when temperature is above $T \sim 1$ eV

Atomic nuclei get unbound when temperature is $T \sim 1$ MeV

Nucleons get unbound when temperature is $T \sim 100$ MeV



CMB - Cosmic Microwave Background radiation formation epoch

BBN - Big Bang Nucleosynthesis epoch

QCD - Quantum Chromodynamics phase transition epoch

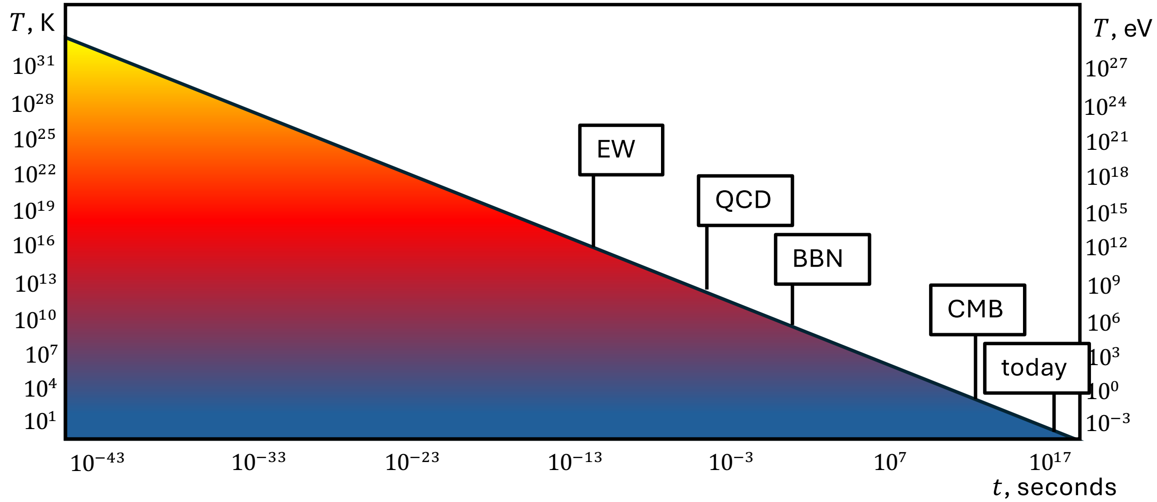
Cosmological epochs

Atoms get unbound when temperature is above $T \sim 1$ eV

Atomic nuclei get unbound when temperature is $T \sim 1$ MeV

Nucleons get unbound when temperature is $T \sim 100$ MeV

Higgs field acquires vacuum expectation value when temperature is $T \sim 100$ GeV



CMB - Cosmic Microwave Background radiation formation epoch

BBN - Big Bang Nucleosynthesis epoch

QCD - Quantum Chromodynamics phase transition epoch

EW - Electroweak phase transition phase transition epoch

Cosmological epochs

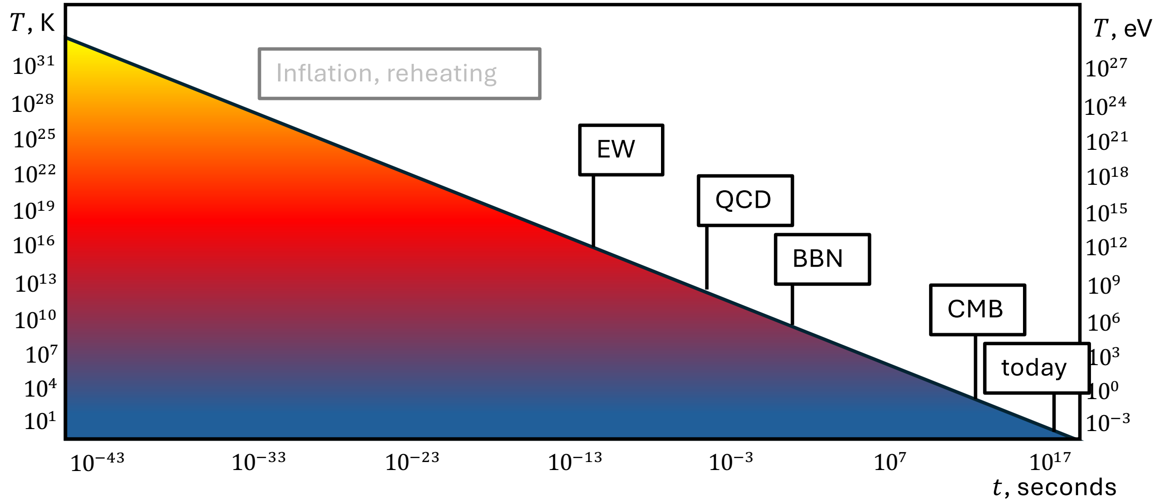
Atoms get unbound when temperature is above $T \sim 1\ \mathrm{eV}$

Atomic nuclei get unbound when temperature is $T \sim 1\ \mathrm{MeV}$

Nucleons get unbound when temperature is $T \sim 100\ \mathrm{MeV}$

Higgs field acquires vacuum expectation value when temperature is $T \sim 100\ \mathrm{GeV}$

Density perturbations seeding future galaxies should have been generated from quantum fields fluctuations at Inflation. Universe should have entered thermal equilibrium through a period of reheating after Inflation.



CMB - Cosmic Microwave Background radiation formation epoch

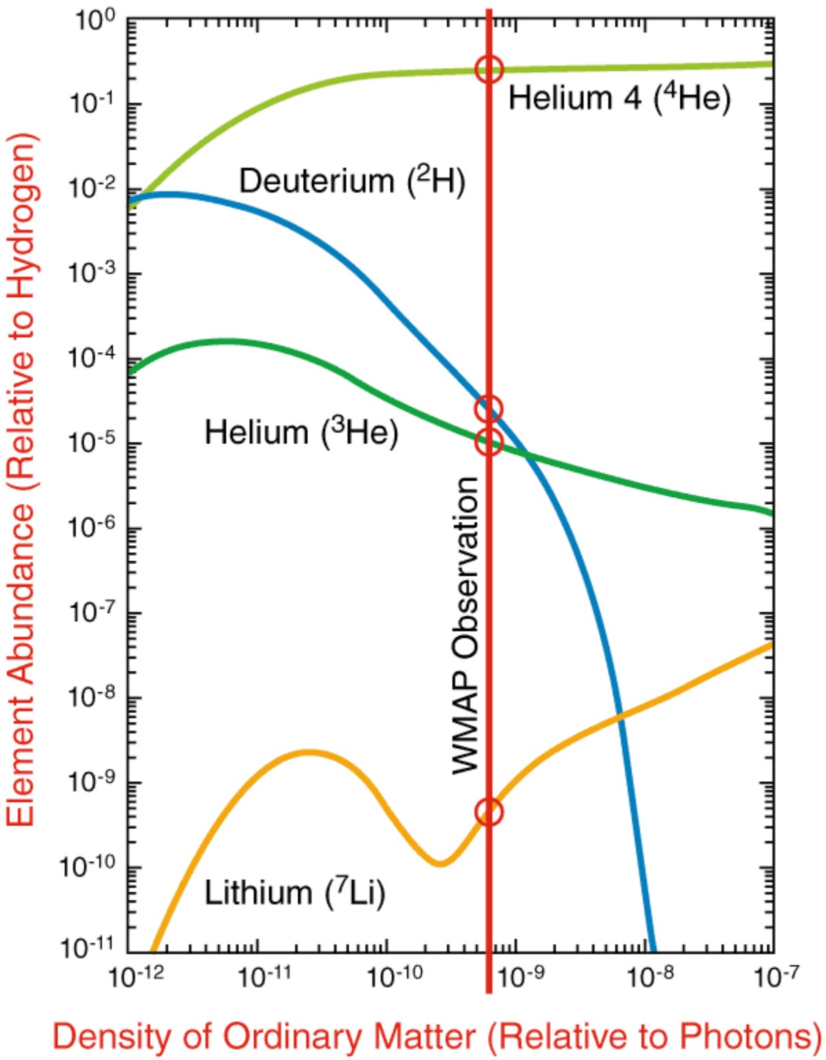
BBN - Big Bang Nucleosynthesis epoch

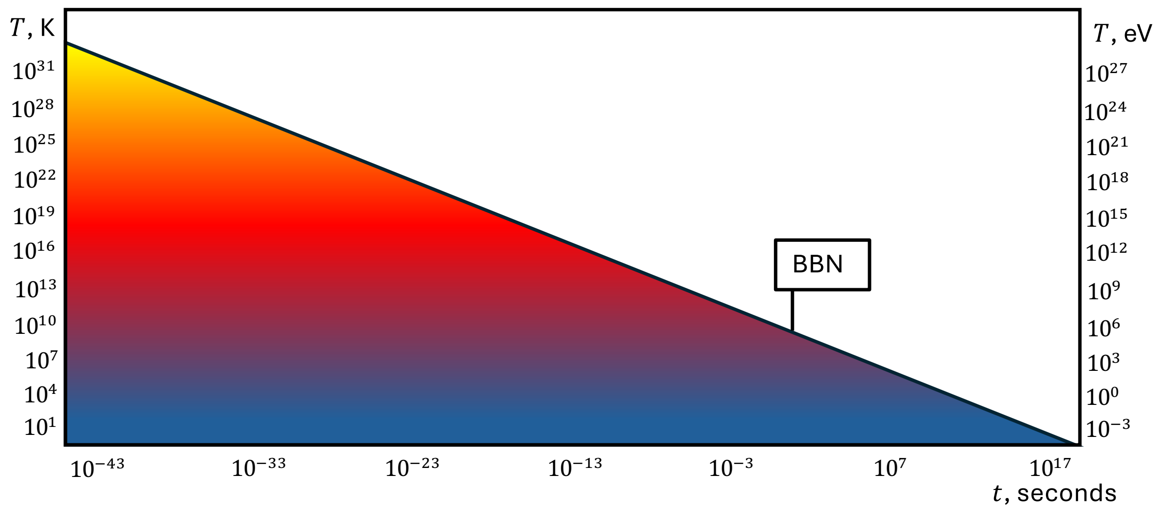
QCD - Quantum Chromodynamics phase transition epoch

EW - Electroweak phase transition phase transition epoch

Inflation – seeding of density perturbations

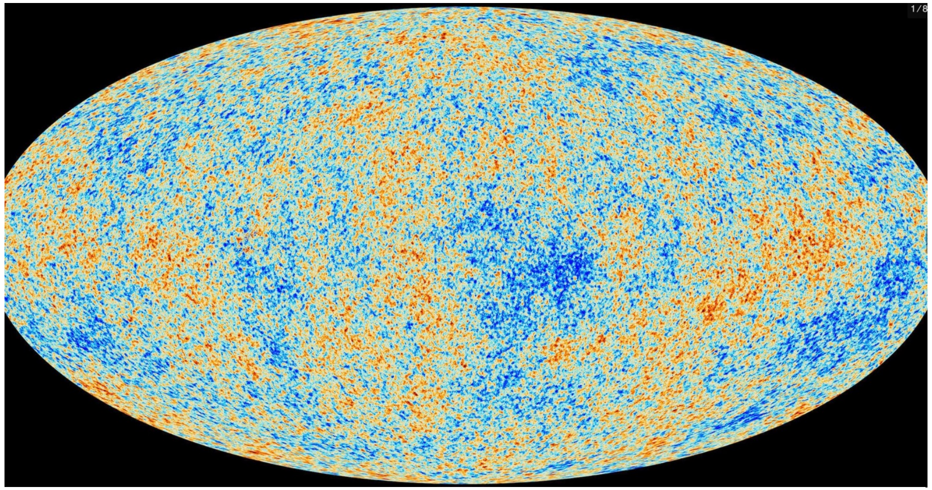
Cosmological probes

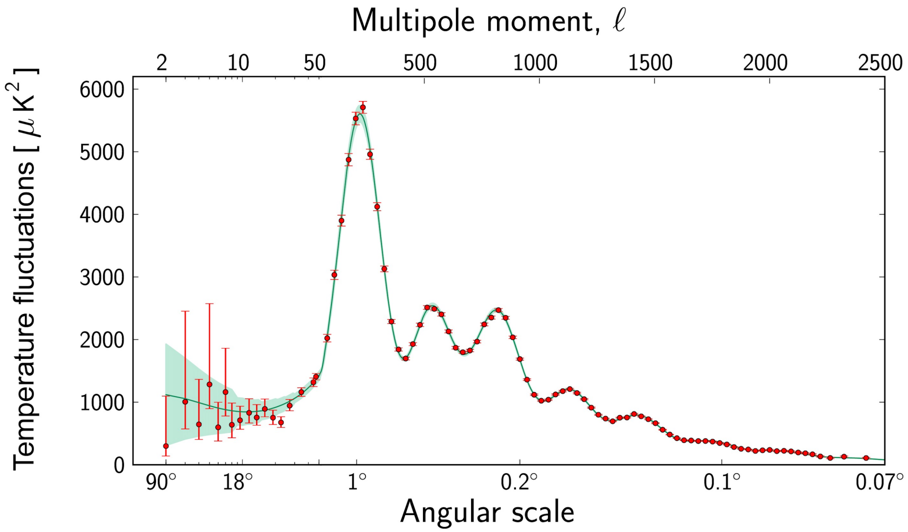


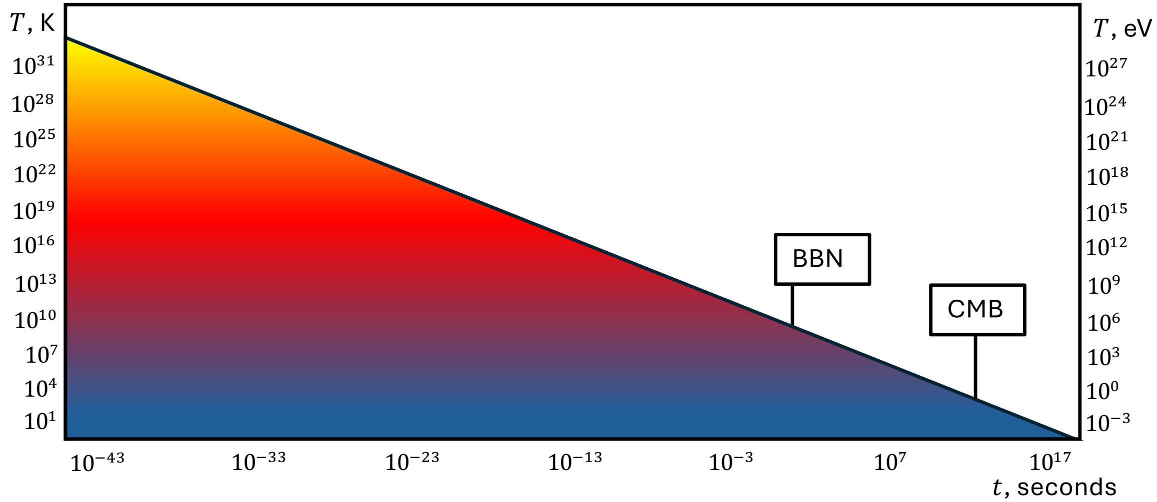


BBN - Big Bang Nucleosynthesis

Cosmological probes

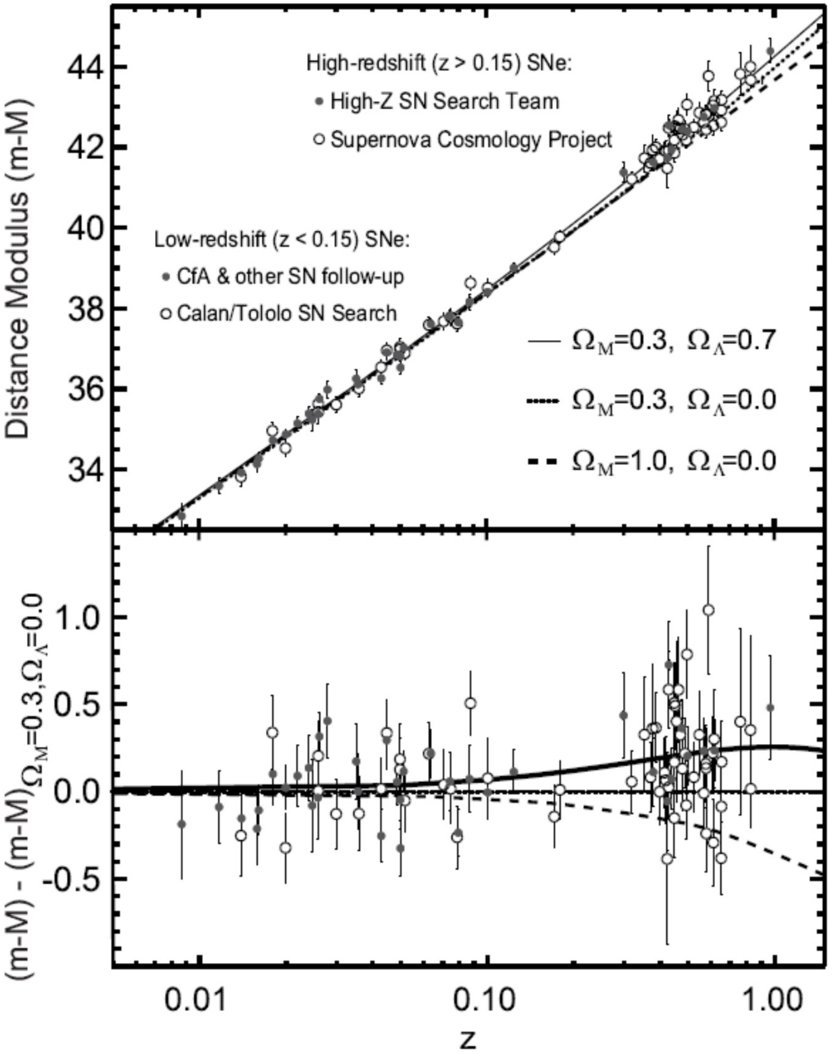


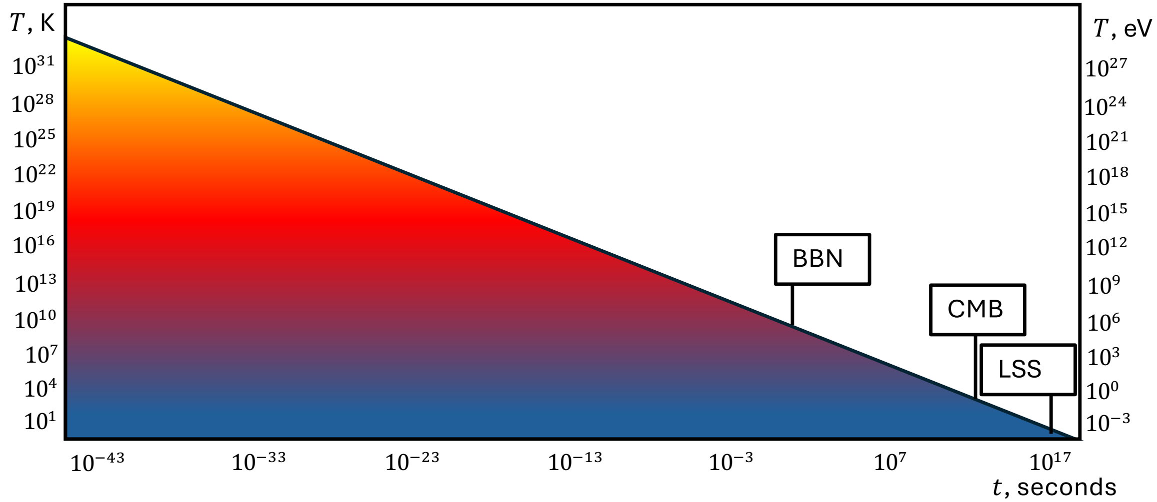




BBN - Big Bang Nucleosynthesis
CMB - Cosmic Microwave Background

Cosmological probes



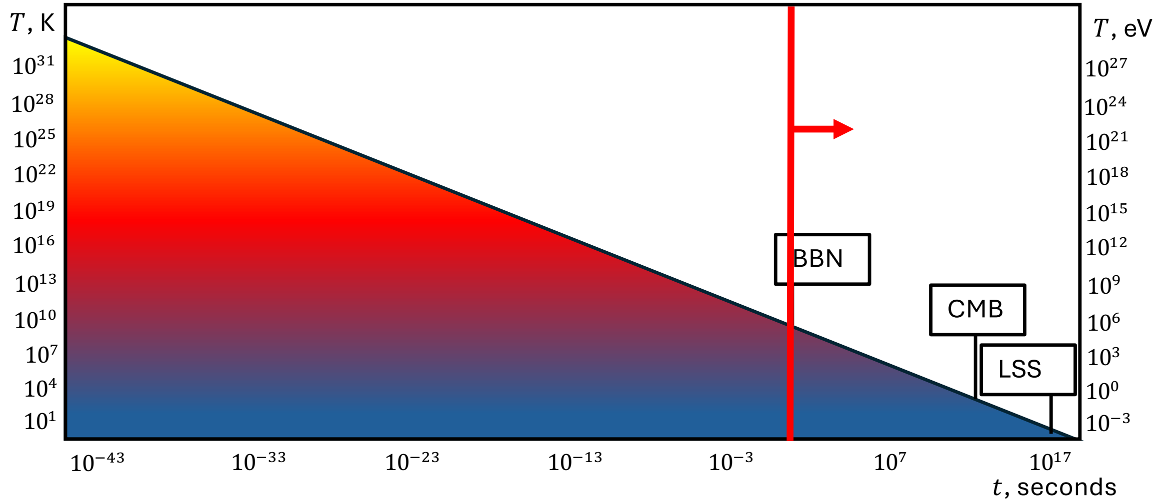


BBN - Big Bang Nucleosynthesis
CMB - Cosmic Microwave Background
LSS – Large Scale Structure

Cosmological probes

BBN currently provides the earliest cosmological data. No observational data on QCD or EW epoch is available.

Inflation framework can be indirectly probed with CMB and LSS data (properties of the matter perturbation power spectrum).



BBN - Big Bang Nucleosynthesis
CMB - Cosmic Microwave Background
LSS – Large Scale Structure

Magnetic field in primordial plasma

The universe is filled with freely moving charged particles that may generate and support electromagnetic fields.

Radiation filling the early Universe is a form of electromagnetic field.

The energy density of radiation is

$$\rho = \frac{\pi^2}{30} g_* T^4 \sim 1 \left[\frac{g_*}{3}\right] \left[\frac{T}{1\ \mathrm{eV}}\right]^4 \ \mathrm{eV}^4$$

Magnetic field with comparable energy density

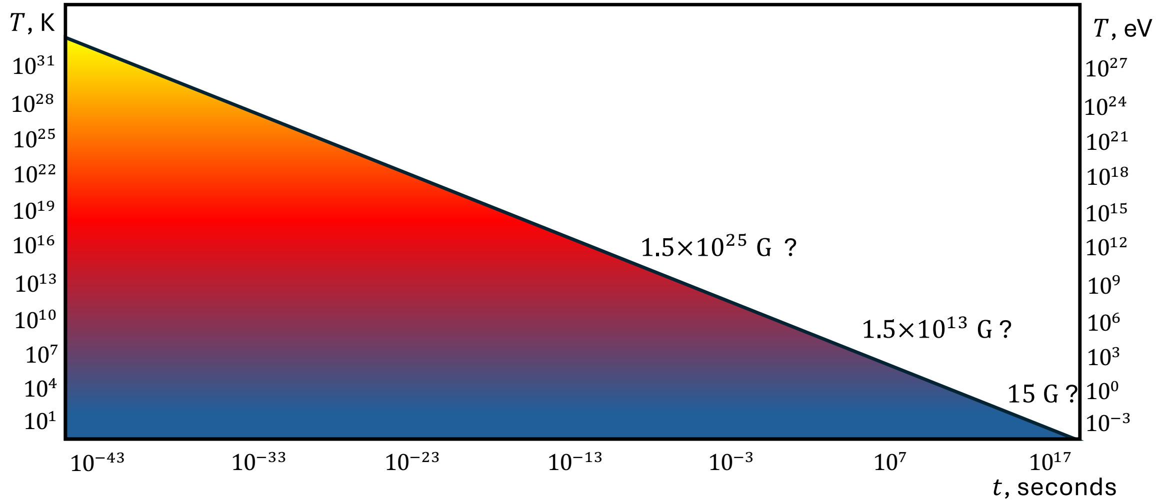
$$\rho_B = \frac{B^2}{8\pi} \sim 1\ \mathrm{eV}^4$$

has the strength

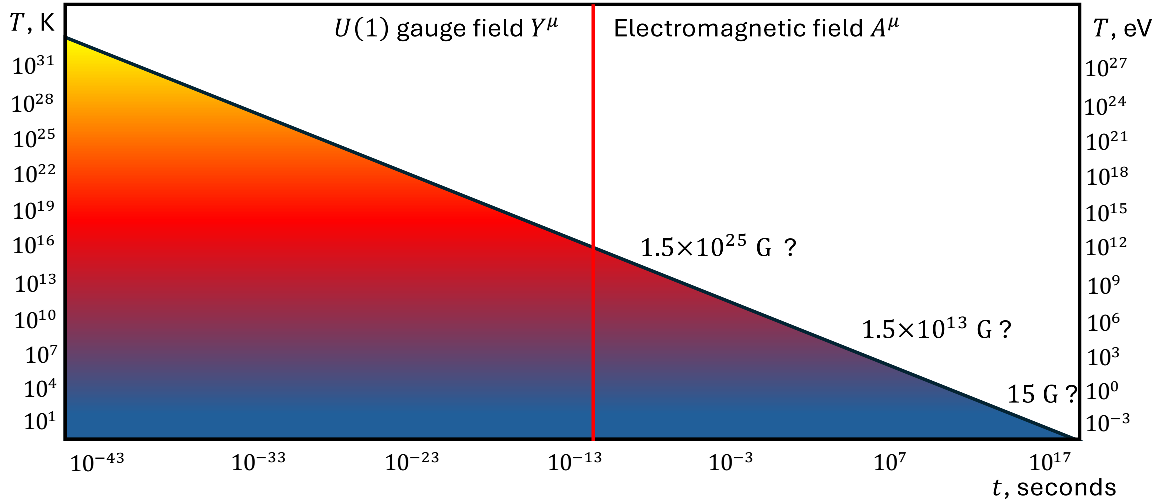
$$B \sim 15 \left[\frac{g_*}{3}\right]^{1/2} \left[\frac{T}{1\ \mathrm{eV}}\right]^2 \mathrm{G}$$

Kinetic energy of particles forming primordial plasma is large enough to support strong magnetic fields. Magnetic fields present before BBN can be much stronger than the strongest fields found in present-day Universe.

Similar to radiation (CMB), magnetic fields can evolve and survive over cosmological time scales. Relic cosmological magnetic fields can be present in the Universe today.



(Hyper)magnetic field in primordial plasma



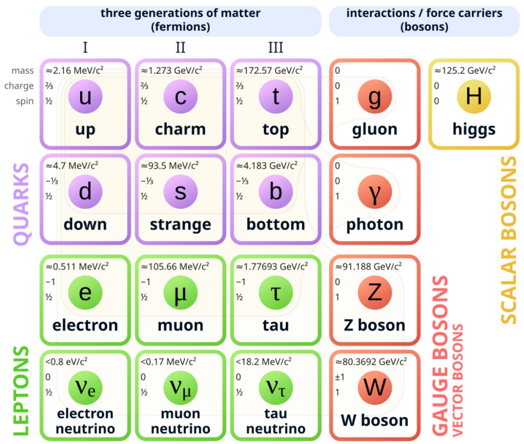
The electromagnetic field in its present form did not exist before the Electroweak phase transition.

Instead, other massless gauge fields existed ($SU(2), U(1)$). During the Electroweak phase transition, the electromagnetic field was formed as a linear combination of the "hypercharge" $U(1)$ gauge field Y and one of the components, W_3, of the $SU(2)$ gauge field

$$\gamma = W_3 \sin\theta_W + Y \cos\theta_W$$
$$Z = W_3 \cos\theta_W - Y \sin\theta_W$$

$SU(2)$ $U(1)$

θ_W is called Weinberg angle.



Cosmological magnetic field observations
Lecture 1
Magnetic field as a cosmological observable

Notation conventions and units

Introduction and motivations

- Cosmological epochs
- Cosmological observables
- Existing cosmological probes
- Magnetic field as a cosmological observable?

Cosmological magnetic field description

- Maxwell equations in curved space-time
- “locally measured” and “comoving” electric and magnetic fields
- Stochastic magnetic fields: power spectrum, strength, correlation length

Magnetic field across cosmological epochs

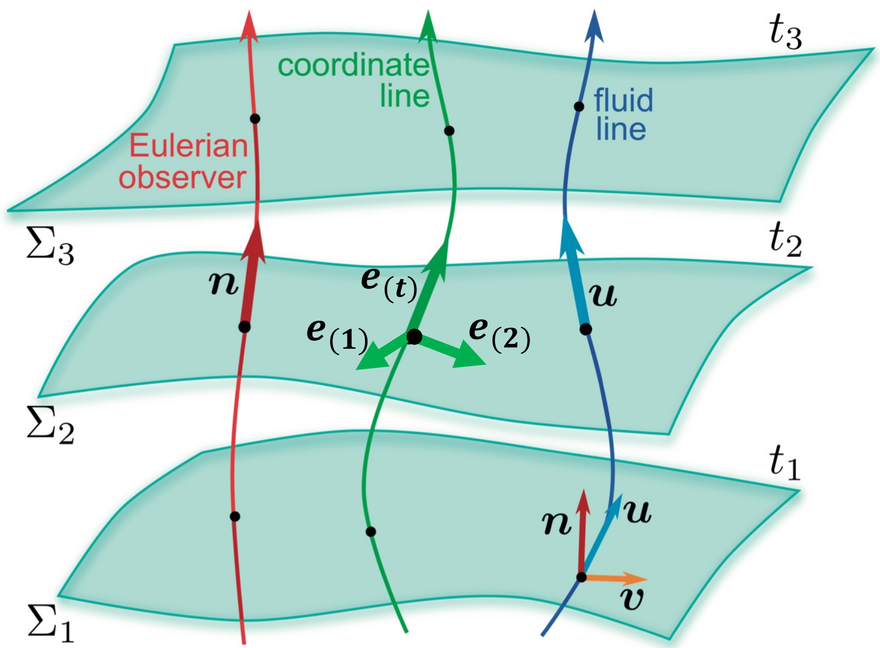
- Magneto-hydrodynamics in expanding Universe
- Magnetic field evolution laws (example helical field)
- Relic magnetic field in the present-day Universe

Electric and magnetic field in expanding universe

In flat Minkowski space-time, electric and magnetic fields are defined as components of electromagnetic field tensor:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$F_{i0} = E_i,\ F_{ij} = \epsilon_{ijk} B_k.$



In Friedman-Robertson-Walker space time with metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

one can introduce locally Minkowski frame

$$e^{\mu}_{(t)} = (1,0,0,0)$$

$$e^{\mu}_{(1)} = \left(0,\frac{1}{a},0,0\right)$$

$$e^{\mu}_{(2)} = \left(0,0,\frac{1}{a},0,\right)$$

$$e^{\mu}_{(3)} = \left(0,0,0,\frac{1}{a}\right)$$

and define "locally measured" electric and magnetic field via components of $F_{\mu\nu}$ in this frame:

$$F_{(i)(0)} = F_{\mu\nu} e^{\mu}_{(i)} e^{\nu}_{(0)} = \frac{1}{a} F_{i0} = E_i$$

$$F_{(i)(j)} = F_{\mu\nu} e^{\mu}_{(i)} e^{\nu}_{(j)} = \frac{1}{a^2} F_{ij} = \epsilon_{ijk} B_k$$

The "locally measured" $\vec{E}, \vec{B}$ can thus be defined in the $(t, \vec{x})$ coordinates through an Ansatz for $F_{\mu\nu}$:

$$F_{i0} = aE_i; \quad F_{ij} = a^2 \epsilon_{ijk} B_k$$

Comoving electric and magnetic fields

Maxwell equaitons in Minowski space-time in terms of $\vec{E}, \vec{B}$ had the form

$$\partial_i E_i = 4\pi\rho_e$$
$$\epsilon_{ijk}\partial_j B_k = 4\pi J_i + \partial_t E_i$$
$$\epsilon_{ijk}\partial_j E_k + \partial_t B_i = 0$$
$$\partial_i B_i = 0$$

In curved space time, Maxwell equations in terms of the field tensor:

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$
$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu$$

Consider e.g. $\mu = 0, \nu = i, \lambda = j$ homogeneous equation:

$$\partial_t F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} =$$
$$= \partial_t\left(\epsilon_{ijk} a^2 B_k\right) + a(\partial_i E_j - \partial_j E_i) = 0$$

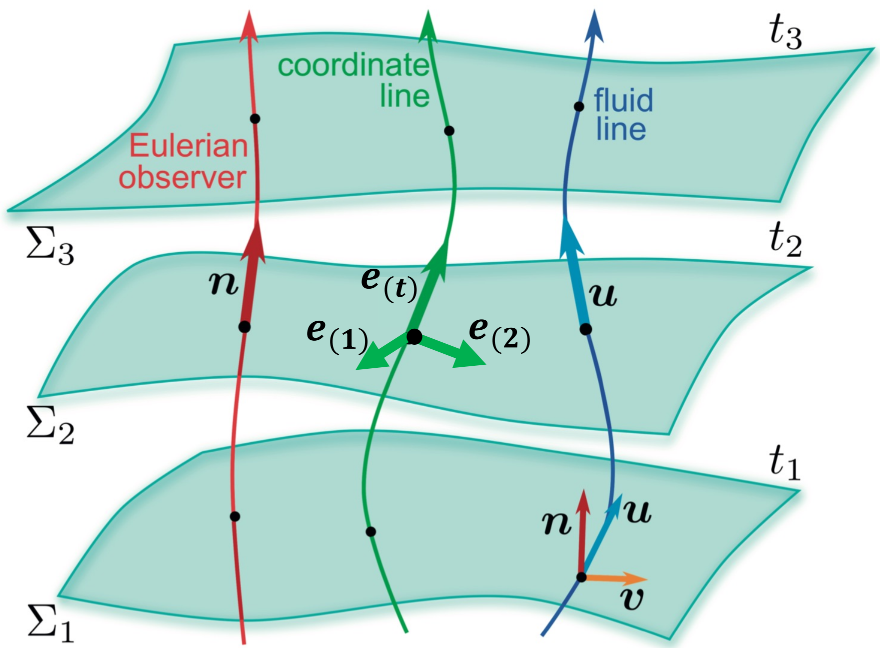
It relates in a non-trivial way time evolution of magnetic field with the evolution of the scale factor of the Universe. Introducing **comoving** elect and magnetic field strengths:

$$\boldsymbol{\tilde{B}_i = a^2 B_i}$$
$$\boldsymbol{\tilde{E}_i = a^2 E_i}$$

and taking derivative w.r.t. conformal time $\tilde{t} = \int dt/a$ one can rewrite the equation as

$$\partial_t F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} =$$
$$= \frac{1}{a}\partial_{\tilde{t}}\left(\epsilon_{ijk} a^2 B_k\right) + \frac{1}{a}\left(\partial_i\left(a^2 E_j\right) - \partial_j\left(a^2 E_i\right)\right) = 0$$
$$\epsilon_{ijk}\partial_{\tilde{t}}\tilde{B}_k + \left(\partial_i \tilde{E}_j - \partial_j \tilde{E}_i\right) = 0$$

The last equation is equivalent to Minkowski space homogeneous equation.



Exercise 5. Demonstrate that the homogeneous Maxwell equations in expanding Universe can be rewritten in the form

$$\frac{\partial \tilde{\boldsymbol{B}}}{\partial \tilde{t}} + \nabla \times \tilde{\boldsymbol{E}} = \boldsymbol{0}$$
$$\nabla \cdot \tilde{\boldsymbol{B}} = \boldsymbol{0}$$

identical to that of Minkowski space-time.

Comoving charge and current densities

The inhomogeneous Maxwell equations

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\,F^{\mu\nu}) = 4\pi J^\mu$$

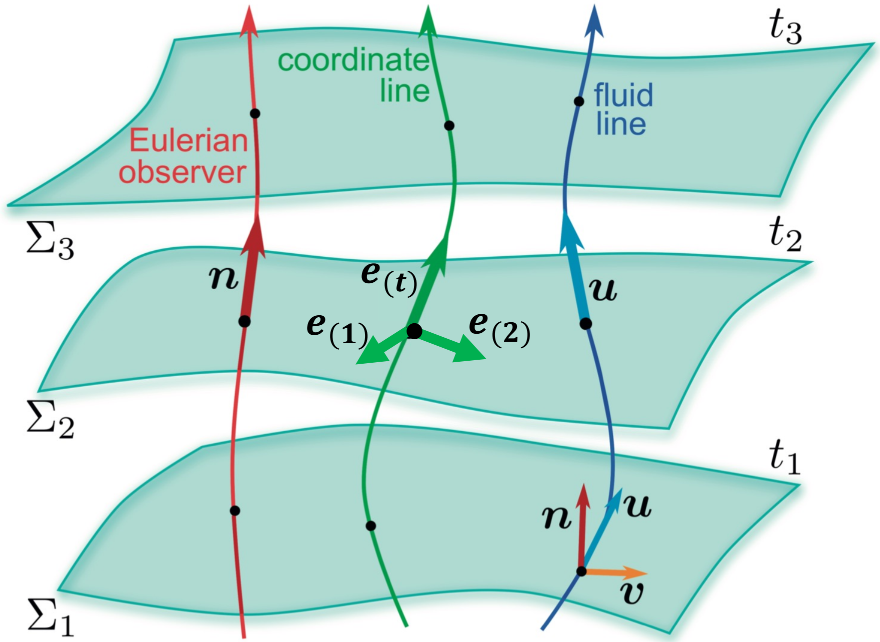
The $\mu = 0$ equation:

$$\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{00}g^{ij}F_{0j}) = \frac{1}{a^3}\partial_i\left(a^3\left(\frac{-1}{a^2}\right)(-aE^i_{ph})\right) = 4\pi\,J^0$$

$$\partial_i\left(\boldsymbol{a^2 E^i_{ph}}\right) = 4\pi\,\boldsymbol{a^3 J^0}$$

$$\boldsymbol{\nabla\cdot\widetilde{E} = 4\pi\widetilde{\rho}_e}$$

also takes the same form as in Minkowski space if **comoving charge density $\widetilde{\rho}_e = a^3 J^0$** is introduced.



The $\mu = i$ equations

$$\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}g^{00}g^{ij}F_{j0}) + \frac{1}{\sqrt{-g}}\partial_j(\sqrt{-g}g^{ik}g^{jl}F_{kl}) =$$

$$= \frac{1}{a^3}\partial_t\left(a^3\left(\frac{-1}{a^2}\right)aE^i_{ph}\right) + \frac{1}{a^3}\partial_j\left(a^3\left(\frac{1}{a^4}\right)\epsilon_{ijk}a^2B^k_{ph}\right) =$$

$$= -\frac{1}{a^4}\partial_{\tilde{t}}(a^2E^i_{ph}) + \frac{1}{a^4}\epsilon_{ijk}\partial_j(a^2B^k_{ph}) = 4\pi J^i$$

Introducing **comoving current density $\tilde{J}^i = a^4 J^i$** one finds

$$\boldsymbol{-\partial_{\tilde{t}}\widetilde{E} + \nabla\times\widetilde{B} = 4\pi\widetilde{J}}$$

Comoving magnetic field in primordial plasma

The energy density of radiation is

$$\rho = \frac{\pi^2}{30} g_* T^4 \sim 1 \left[\frac{g_*}{3}\right] \left[\frac{T}{1\ \mathrm{eV}}\right]^4 \ \mathrm{eV}^4$$

Magnetic field with comparable energy density

$$\rho_B = \frac{B_{eq}^2}{8\pi} \sim \rho$$

has the strength

$$B_{eq} \sim 15 \left[\frac{g_*}{3}\right]^{1/2} \left[\frac{T}{1\ \mathrm{eV}}\right]^2 \mathrm{G}$$

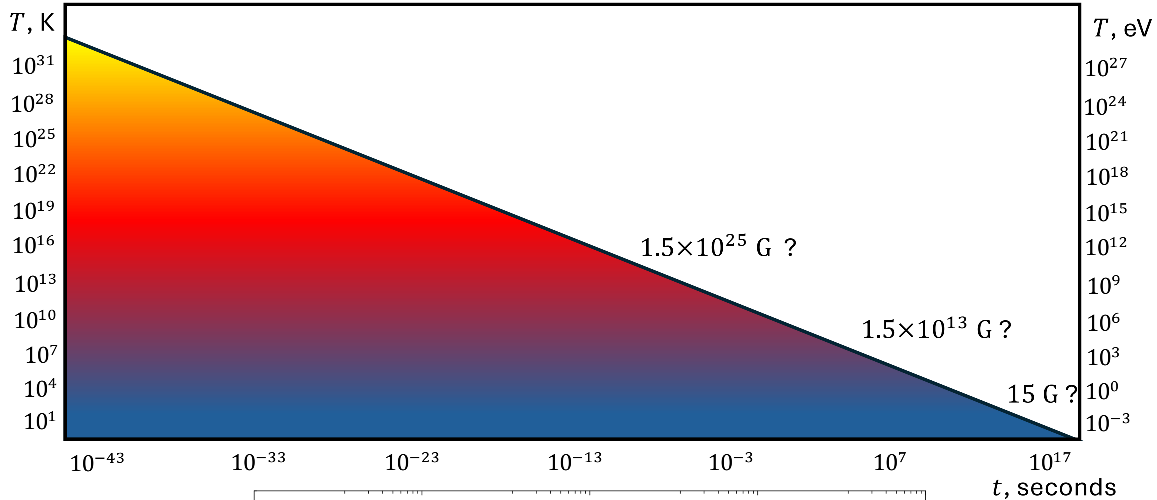
Temperature evolution (from entropy conservation $g_* T^3 a^3 = const$)

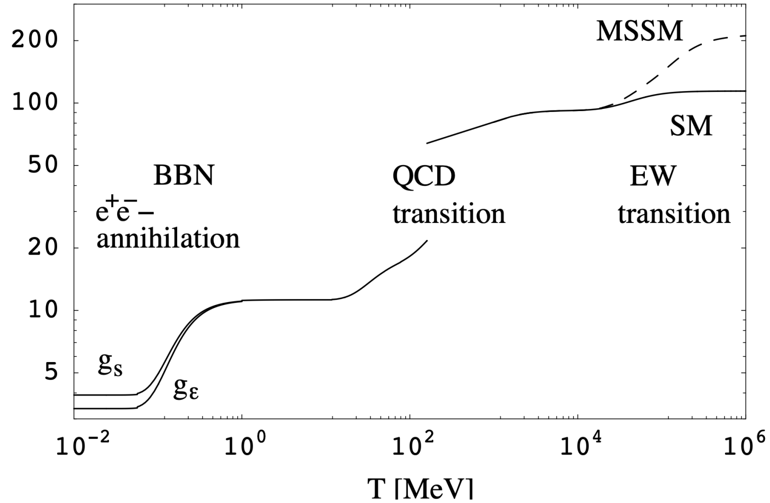
$$T = \frac{1}{a}\left[\frac{g_*}{4}\right]^{-\frac{1}{3}} T_0$$

where $T_0 = 2.7\ \mathrm{K} = 2.3\times10^{-4}$ eV is the CMB temperature. This gives

$$\tilde{B}_{eq} = a^2 B_{eq} = \sqrt{\left(\frac{32\,\pi^3}{30}\right)}\left[\frac{g_*}{4}\right]^{-\frac{1}{6}} T_0^2 \simeq 4\left[\frac{g_{*S}}{4}\right]^{-\frac{1}{6}} \mu\mathrm{G}$$

The number of effective entropy degrees of freedom g_{*S} evolves by a factor of ~ 30 over the history of the Universe, so that $g_{*S}^{1/6}$ changes only be a factor 1.7. The comoving "equipartition" field strength remans almost constant all over the Universe history.





Magnetic field power spectrum, strength, correlation length

The early Universe is homogeneous and isotropic and magnetic field produced by motions of particles in such Universe is also expected to be statistically homogeneous and isotropic: it should have stochastic structure with direction randomly changing from place to place.

Fourier transform of (comoving) magnetic field $\tilde{B}(t,\vec{x})$:

$$\widetilde{\boldsymbol{B}}(t,\vec{k}) = \int d^3x\, \tilde{B}(t,\vec{x})\, e^{i\vec{k}\vec{x}}$$

Statistical homogeneity and isotropy:

$$\langle \widetilde{\boldsymbol{B}}_i(t,\vec{k})\widetilde{\boldsymbol{B}}_j(t,\vec{k}')\rangle$$
$$= (2\pi)^3\delta(\vec{k}-\vec{k}')\left[\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)P_B(k) - i\epsilon_{ijm}\frac{k_m}{k}P_{aB}(k)\right]$$

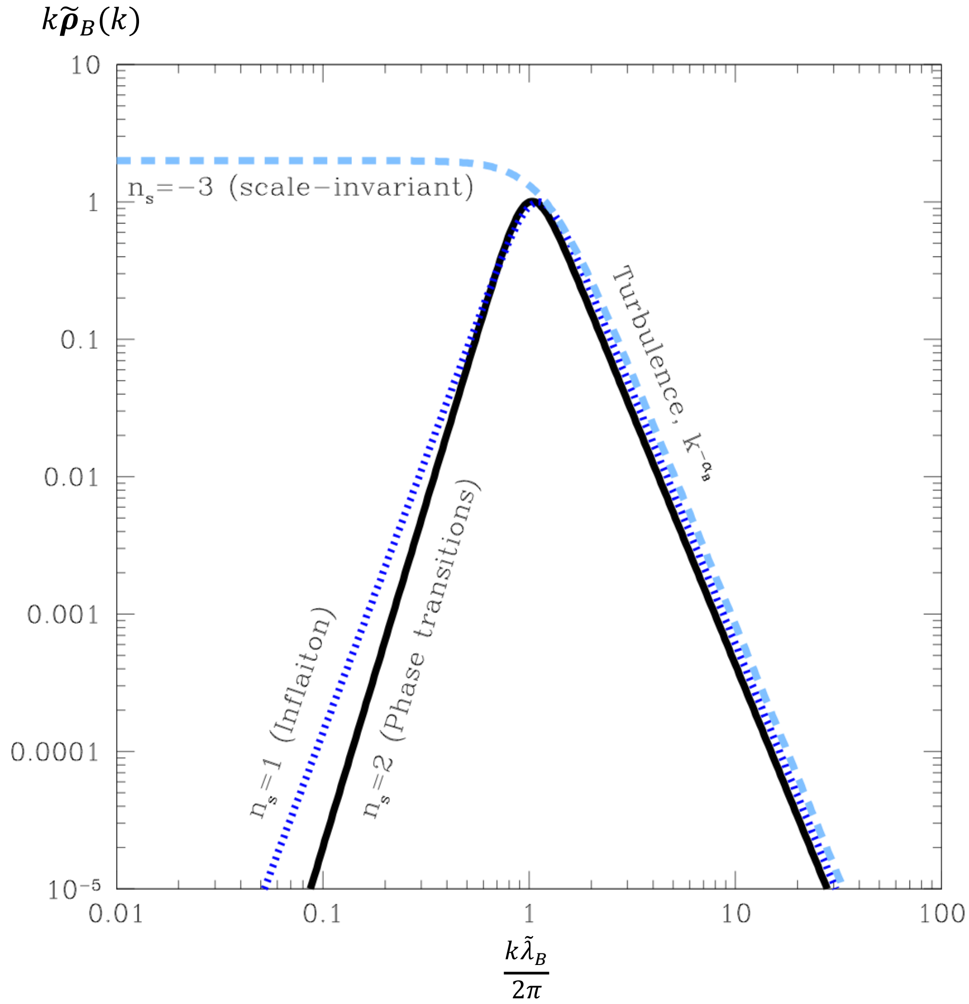
The angle brackets are for “enseble average” over many realisations of the stochastic magnetic field. P_B and P_{aB} are symmetric and anti-symmetric parts of the magnetic field Fourier spectrum.

Magnetic field energy density

$$\tilde{\rho}_B = \frac{\tilde{B}^2}{8\pi} = \frac{1}{(2\pi)^3}\int d^3k\, P_B(k) = \frac{1}{2\pi^2}\int \frac{dk}{k}\, k^3 P_B(k) = \int \tilde{\boldsymbol{\rho}}_B(k)dk$$

Magnetic field strength and direction are changing in space. It is convenient to introduce the volume-average strength and correlation length of the field derived from its power spectrum:

$$\langle\tilde{B}\rangle = \sqrt{8\pi\tilde{\rho}_B}\,, \qquad \tilde{\lambda}_B = \frac{1}{\rho_B}\int\left(\frac{2\pi}{k}\right)\tilde{\boldsymbol{\rho}}_B(k)dk$$



Magnetic field power spectrum shape(s)

$$\tilde{\rho}_B = \frac{\tilde{B}^2}{8\pi} = \frac{1}{(2\pi)^3}\int d^3k\, P_B(k) = \frac{1}{2\pi^2}\int \frac{dk}{k}\; k^3 P_B(k) = \int \tilde{\boldsymbol{\rho}}_B(k) dk$$

Scale-invariant power spectrum: energy density per decade of k is constant:

$$P_B \propto k^{n_s}, \qquad n_s = -3$$
$$k\tilde{\boldsymbol{\rho}}_B(k) \propto k^0$$

Batchelor power spectrum (or “causal” power spectrum, Caprini&Durrer astro-ph/0603476, see Caprini’s lectures next week): $n_s = 2$.

Power spectrum of Kolmogorov type turbulence: $n_s = -11/3$.