Spherical Geometry

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1 Spherical coordinates

In order to simplify notation, we define the operators

$$\mathrm{d}_r = \partial_r \tag{1}$$
$$\mathrm{d}_\theta = r^{-1}\partial_\theta \tag{2}$$
$$\mathrm{d}_\phi = \varpi^{-1}\partial_\phi \tag{3}$$

where $\varpi = r\sin\theta$ is the cylindrical radius, and write the gradient as

$$\operatorname{grad}\Psi = \begin{pmatrix} \mathrm{d}_r\Psi \\ \mathrm{d}_\theta\Psi \\ \mathrm{d}_\phi\Psi \end{pmatrix} \tag{4}$$

The divergence and curl operators are written as

$$\operatorname{div}\boldsymbol{A} = \mathcal{D}_r A_r + \mathrm{D}_\theta A_\theta + \mathrm{d}_\phi A_\phi \tag{5}$$

and

$$\operatorname{curl}\boldsymbol{A} = \begin{pmatrix} \mathrm{D}_\theta A_\phi - \mathrm{d}_\phi A_\theta \\ \mathrm{d}_\phi A_r - \mathrm{D}_r A_\phi \\ \mathrm{D}_r A_\theta - \mathrm{d}_\theta A_r \end{pmatrix}, \tag{6}$$

and the laplacian of a scalar, $\nabla^2 \equiv \operatorname{div}\operatorname{grad}$, is

$$\nabla^2\Psi = \mathrm{D}_r^2\Psi + \mathrm{D}_\theta^2\Psi + \mathrm{d}_\phi^2\Psi; \tag{7}$$

see Table 1 for the definition of additional operators, and Table 2 for the relations between them.

Table 1: Useful operators in spherical coordinates.

$\mathcal{D}_r = r^{-2}\partial_r(r^2\,\cdot)$	
$\mathrm{D}_r = r^{-1}\partial_r(r\,\cdot)$	$\mathrm{D}_\theta = r^{-1}\sin^{-1}\theta\,\partial_\theta(\sin\theta\,\cdot)$
$\mathrm{D}_r^{-1} = r\partial_r(r^{-1}\,\cdot)$	$\mathrm{D}_\theta^{-1} = r^{-1}\sin\theta\,\partial_\theta(\sin^{-1}\theta\,\cdot)$

Express all spatial derivatives in fully non-conservative form. Arrange mixed operators such that the innermost derivative operator is in the r-direction, the next one in the θ-direction, and the ϕ-operator is the outermost one.

Table 2: Some useful identities. The '≡' symbol indicates the definition of an operator, so $\mathcal{D}_r^2$ is not meant to be the same as the operator $\mathcal{D}_r$ applied twice.

$\mathrm{d}_r^2 = \mathrm{d}_r\mathrm{d}_r$	$\mathrm{d}_\theta^2 = \mathrm{d}_\theta\mathrm{d}_\theta$	$\mathrm{d}_\phi^2 = \mathrm{d}_\phi\mathrm{d}_\phi$
$\mathrm{D}_r^2 \equiv \mathrm{D}_r\mathrm{D}_r = \mathcal{D}_r\mathrm{d}_r$	$\mathrm{D}_\theta^2 \equiv \mathrm{D}_\theta\mathrm{d}_\theta$	
$\mathcal{D}_r^2 \equiv \mathrm{d}_r\mathcal{D}_r = \mathrm{D}_r^{-1}\mathrm{d}_r$	$\mathcal{D}_\theta^2 \equiv \mathrm{d}_\theta\mathrm{D}_\theta$	
	$\mathrm{d}_r\mathrm{D}_\theta = \mathrm{D}_\theta\mathrm{D}_r^{-1}$	$\mathrm{d}_\theta\mathrm{d}_\phi = \mathrm{d}_\phi\mathrm{D}_\theta^{-1}$

$$\operatorname{curl}^2\boldsymbol{A} = \begin{pmatrix} \mathrm{D}_\theta\mathrm{D}_r A_\theta + \mathrm{d}_\phi\mathrm{D}_r A_\phi - \mathrm{D}_\theta^2 A_r - \mathrm{d}_\phi^2 A_r \\ \mathrm{d}_\phi\mathrm{D}_\theta A_\phi + \mathrm{d}_\theta\mathrm{D}_r A_r - \mathrm{d}_\phi^2 A_\theta - \mathrm{D}_r^2 A_\theta \\ \mathrm{d}_\phi\mathrm{D}_r A_r + \mathrm{d}_\theta\mathrm{d}_\phi A_\theta - \mathrm{D}_r^2 A_\phi - \mathcal{D}_\theta^2 A_\phi \end{pmatrix} \tag{8}$$

is the double-curl.

$$\operatorname{grad}\operatorname{div}\boldsymbol{A} = \begin{pmatrix} \mathrm{d}_r\mathcal{D}_r A_r + \mathrm{d}_r\mathrm{D}_\theta A_\theta + \mathrm{d}_r\mathrm{d}_\phi A_\phi \\ \mathrm{d}_\theta\mathcal{D}_r A_r + \mathrm{d}_\theta\mathrm{D}_\theta A_\theta + \mathrm{d}_\theta\mathrm{d}_\phi A_\phi \\ \mathrm{d}_\phi\mathcal{D}_r A_r + \mathrm{d}_\phi\mathrm{D}_\theta A_\theta + \mathrm{d}_\phi\mathrm{d}_\phi A_\phi \end{pmatrix} \tag{9}$$

or, written such that r-operators come first and ϕ operators last,

$$\operatorname{grad}\operatorname{div}\boldsymbol{A} = \begin{pmatrix} \mathcal{D}_r^2 A_r + \mathrm{D}_\theta\mathrm{D}_r^{-1} A_\theta + \mathrm{d}_\phi\mathrm{D}_r^{-1} A_\phi \\ \mathrm{d}_\theta\mathcal{D}_r A_r + \mathcal{D}_\theta^2 A_\theta + \mathrm{d}_\phi\mathrm{D}_\theta^{-1} A_\phi \\ \mathrm{d}_\phi\mathcal{D}_r A_r + \mathrm{d}_\phi\mathrm{D}_\theta A_\theta + \mathrm{d}_\phi^2 A_\phi \end{pmatrix} \tag{10}$$

The laplacian of a vector is evaluated as $\boldsymbol{\nabla}^2 = \operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl}$ and can be written as

$$\boldsymbol{\nabla}^2\boldsymbol{A} = \begin{pmatrix} (\nabla^2 - 2r^{-2})A_r - 2r^{-1}(\mathrm{D}_\theta A_\theta + \mathrm{d}_\phi A_\phi) \\ (\nabla^2 - \varpi^{-2})A_\theta + 2r^{-1}(\mathrm{d}_\theta A_r - \cot\theta\,\mathrm{d}_\phi A_\phi) \\ (\nabla^2 - \varpi^{-2})A_\phi + 2r^{-1}(\mathrm{d}_\phi A_r + \cot\theta\,\mathrm{d}_\phi A_\theta) \end{pmatrix} \tag{11}$$

Table 3: Spatial operators relevant in spherical coordinates. $\varpi = r\sin\theta$ is the cylindrical radius.

$\mathrm{d}_r = \partial_r$	$\mathrm{d}_\theta = r^{-1}\partial_\theta$	$\mathrm{d}_\phi = \varpi^{-1}\partial_\phi$
$\mathrm{D}_r = r^{-1} + \mathrm{d}_r$	$\mathrm{D}_\theta = r^{-1}\cot\theta + \mathrm{d}_\theta$	
$\mathrm{D}_r^{-1} = -r^{-1} + \mathrm{d}_r$	$\mathrm{D}_\theta^{-1} = -r^{-1}\cot\theta + \mathrm{d}_\theta$	
$\mathcal{D}_r = 2r^{-1} + \mathrm{d}_r$		
$\mathrm{d}_r^2 = \partial_r^2$	$\mathrm{d}_\theta^2 = r^{-2}\partial_\theta^2$	$\mathrm{d}_\phi^2 = \varpi^{-2}\partial_\phi^2$
$\mathrm{D}_r^2 = 2r^{-1}\mathrm{d}_r + \mathrm{d}_r^2$	$\mathrm{D}_\theta^2 = r^{-1}\cot\theta\,\mathrm{d}_\theta + \mathrm{d}_\theta^2$	
$\mathcal{D}_r^2 = -2r^{-2} + \mathrm{D}_r^2$	$\mathcal{D}_\theta^2 = -\varpi^{-2} + \mathrm{D}_\theta^2$	

2 ... without radial derivatives

Useful for solar surface observations

$$\mathrm{curl}\,\boldsymbol{A} = \begin{pmatrix} \mathrm{D}_\theta A_\phi - \mathrm{d}_\phi A_\theta \\ +\mathrm{d}_\phi A_r \\ -\mathrm{d}_\theta A_r \end{pmatrix}, \tag{12}$$

Compute $A_r \equiv T\hat{\boldsymbol{r}}$ from $J_r = L^2 A_r$.

$$\mathrm{curl}^2\boldsymbol{A} = \begin{pmatrix} -\mathrm{D}_\theta^2 A_r - \mathrm{d}_\phi^2 A_r \\ +\mathrm{d}_\phi \mathrm{D}_\theta A_\phi - \mathrm{d}_\phi^2 A_\theta \\ +\mathrm{d}_\theta \mathrm{d}_\phi A_\theta - \mathcal{D}_\theta^2 A_\phi \end{pmatrix} \tag{13}$$

3 Modified cartesian operators

Consider first

$$\mathrm{grad}\,\mathrm{div}\,\boldsymbol{A} = \begin{pmatrix} \mathrm{d}_r\mathcal{D}_r A_r + \mathrm{d}_r \mathrm{D}_\theta A_\theta + \mathrm{d}_r \mathrm{d}_\phi A_\phi \\ \mathrm{d}_\theta\mathcal{D}_r A_r + \mathrm{d}_\theta \mathrm{D}_\theta A_\theta + \mathrm{d}_\theta \mathrm{d}_\phi A_\phi \\ \mathrm{d}_\phi \mathrm{D}_r A_r + \mathrm{d}_\phi \mathrm{D}_\theta A_\theta + \mathrm{d}_\phi \mathrm{d}_\phi A_\phi \end{pmatrix} \tag{14}$$

Write explicitly

$$\mathrm{d}_r\mathcal{D}_r = -2r^{-2} + 2r^{-1}\partial_r + \partial_{rr} \tag{15}$$

$$\mathrm{d}_r\mathrm{D}_\theta = -r^{-2}\cot\theta + r^{-1}\cot\theta\,\partial_r - r^{-2}\partial_\theta + r^{-1}\partial_{r\theta} \tag{16}$$

$$\mathrm{d}_r\mathrm{d}_\phi = -r^{-1}\varpi^{-1}\partial_\phi + \varpi^{-1}\partial_{r\phi} \tag{17}$$

$$\mathrm{d}_\theta\mathcal{D}_r = 2r^{-2}\partial_\theta + r^{-1}\partial_{r\theta} \tag{18}$$

$$\mathrm{d}_\theta\mathrm{D}_\theta = -r^{-2}(1+\cot^2\theta) + r^{-2}\cot\theta\,\partial_\theta + r^{-2}\partial_{\theta\theta} \tag{19}$$

$$\mathrm{d}_\theta\mathrm{d}_\phi = -r^{-1}\varpi^{-1}\cot\theta\partial_\phi + r^{-1}\varpi^{-1}\partial_{\theta\phi} \tag{20}$$

$$\mathrm{d}_\phi\mathrm{D}_r = r^{-1}\varpi^{-1}\partial_\phi + \varpi^{-1}\partial_{r\phi} \tag{21}$$

$$\mathrm{d}_\phi\mathrm{D}_\theta = r^{-1}\varpi^{-1}\cot\theta\partial_\phi + r^{-1}\varpi^{-1}\partial_{\theta\phi} \tag{22}$$

$$\mathrm{d}_\phi\mathrm{d}_\phi = \varpi^{-2}\partial_{\phi\phi} \tag{23}$$

curl2

$$\mathrm{D}_r^2 = 2r^{-1}\partial_r + \partial_{rr} \tag{24}$$

$$\mathrm{grad}\,\mathrm{div}\,\boldsymbol{A} = (\mathrm{grad}\,\mathrm{div}\,\boldsymbol{A})_{\mathrm{cart}} + (\mathrm{grad}\,\mathrm{div}\,\boldsymbol{A})_{\mathrm{extr}} \tag{25}$$

$$(\mathrm{grad}\,\mathrm{div}\,\boldsymbol{A})_{\mathrm{cart}} = \tag{26}$$

$$\begin{pmatrix} A_{r,rr} + r^{-1}A_{\theta,\theta r} + \varpi^{-1}A_{\phi,\phi r} \\ r^{-1}A_{r,r\theta} + r^{-2}A_{\theta,\theta\theta} + r^{-1}\varpi^{-1}A_{\phi,\phi\theta} \\ \varpi^{-1}A_{r,r\phi} + r^{-1}\varpi^{-1}A_{\theta,\theta\phi} + \varpi^{-2}A_{\phi,\phi\phi} \end{pmatrix} \tag{27}$$

$$(\mathrm{grad}\,\mathrm{div}\,\boldsymbol{A})_{\mathrm{extr}} = (\,-2r^{-2}A_r + 2r^{-1}A_{r,r}\,) \tag{28}$$

4 Legendre polynomials

$$A_\theta = \mathrm{d}_\phi S, \quad A_\phi = -\mathrm{d}_\theta S. \tag{29}$$

Backward transformation

$$S_m(r,\theta) = \sum_{\ell=1}^{L} \sqrt{N_{\ell m}} S_{\ell m}(r) P_\ell^m(\cos\theta), \tag{30}$$

where

$$N_{\ell m} = \tfrac{1}{2}(2\ell+1)\frac{(\ell-m)!}{(\ell+m)!} \tag{31}$$

are the normalization coefficients of the Legendre polynomials. Forward transformation

$$S_{\ell m}(r) = \sqrt{N_{\ell m}} \int_0^\pi P_\ell^m(\cos\theta) S(r,\theta)\sin\theta\,\mathrm{d}\theta \tag{32}$$

Useful relation

$$\sin\theta\frac{\mathrm{d}P_\ell^m}{\mathrm{d}\theta}=-\frac{(\ell+1)(\ell+m)}{2\ell+1}P_{\ell-1}^m+\frac{\ell(\ell-m+1)}{2\ell+1}P_{\ell+1}^m \tag{33}$$

$$\cos\theta P_\ell^m=\frac{\ell+m}{2\ell+1}P_{\ell-1}^m+\frac{\ell-m+1}{2\ell+1}P_{\ell+1}^m \tag{34}$$

5 Spherical Bessel functions

$$j_0(z)=\frac{\sin z}{z}\equiv\sqrt{\pi/2z}J_{1/2}(z) \tag{35}$$

$$j_1(z)=\frac{\sin z}{z^2}-\frac{\cos z}{z} \tag{36}$$

$$j_2(z)=\left(\frac{3}{z^3}-\frac{1}{z}\right)\sin z-\frac{3}{z}\cos z \tag{37}$$

Zeros: $j_l(z_l)=0$, $z_0=\pi$, $z_1=4.493409$, $z_2=5.763459$. Note that

$$\mathrm{d}j_1/\mathrm{d}\ln z|_{z_1}=-1. \tag{38}$$

6 Relative magnetic helicity

$$H_{\rm rel}=\int(\boldsymbol{A}+\boldsymbol{A}_{\rm P})\cdot(\boldsymbol{B}-\boldsymbol{B}_{\rm P})\,\mathrm{d}V, \tag{39}$$

where $\boldsymbol{B}_{\rm P}=\boldsymbol{\nabla}\times\boldsymbol{A}_{\rm P}$ is a potential (current-free) reference field in the interior with $\boldsymbol{B}_{{\rm P}r}=\boldsymbol{B}_r$ on $r=R$. We express $\boldsymbol{A}_{\rm P}$ in terms of the poloidal part of the superpotential,

$$\boldsymbol{A}_{\rm P}=\boldsymbol{\nabla}\times(\hat{\boldsymbol{r}}\Phi). \tag{40}$$

The corresponding current is

$$\boldsymbol{J}_{\rm P}=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\hat{\boldsymbol{r}}\Phi)=-\boldsymbol{\nabla}\times(\hat{\boldsymbol{r}}\nabla^2\Phi). \tag{41}$$

The potential field is current-free, i.e. $\boldsymbol{J}_{\rm P}=0$, so we require

$$\nabla^2\Phi=0. \tag{42}$$

We solve this equation in spectral space and define

$$\Phi_\ell(r)=\sqrt{N_{\ell m}}\int_0^\pi P_\ell(\cos\theta)\Phi(r,\theta)\sin\theta\,\mathrm{d}\theta. \tag{43}$$

where $m=0$. The backward transform is

$$\Phi(r,\theta)=\sum_{\ell=1}^{L}\sqrt{N_{\ell m}}\;\Phi_\ell(r)P_\ell(\cos\theta), \tag{44}$$

and

$$N_{\ell m}=\tfrac{1}{2}(2\ell+1)\frac{(\ell-m)!}{(\ell+m)!} \tag{45}$$

are the normalization coefficients of the Legendre polynomials.

It is convenient to define $\tilde{\Phi}_\ell=r\Phi_\ell$, so the potential equation in spectral space reads

$$\tilde{\Phi}_\ell''-\frac{\ell(\ell+1)}{r^2}\tilde{\Phi}_\ell=0, \tag{46}$$

where and primes denote radial derivatives. Since the radial components of $\boldsymbol{B}$ and $\boldsymbol{B}_{\rm P}$ coincide on $r=R$, we have

$$B_r=-\boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{r}\Phi=-\nabla_\perp^2\tilde{\Phi}. \tag{47}$$

on $r=R$. In spectral space this corresponds to the boundary condition

$$\tilde{\Phi}_\ell(R)=\frac{R^2}{\ell(\ell+1)}b_\ell \tag{48}$$

where

$$b_\ell=\sqrt{N_{\ell m}}\int_0^\pi P_\ell(\cos\theta)B_r(R,\theta)\sin\theta\,\mathrm{d}\theta. \tag{49}$$

7 Useful relations in axisymmetry

A useful relation in connection with axisymmetry is

$$\mathcal{D}_\theta^2\mathrm{d}_\theta=\mathrm{d}_\theta\mathrm{D}_\theta\mathrm{d}_\theta=\mathrm{d}_\theta\mathrm{D}_\theta^2. \tag{50}$$

It allows to show that the vacuum condition in axisymmetry, $D^2A_\phi=0$, is equivalent to the general vacuum condition, $\nabla^2\Phi=0$, where Φ is the poloidal superpotential of the vacuum field, so $\boldsymbol{A}=\boldsymbol{\nabla}\times(\hat{\boldsymbol{r}}\Phi)$. Here we have introduced the operator

$$D^2=-\varpi^{-2}+\nabla^2=\mathrm{D}_r^2+\mathcal{D}_\theta^2, \tag{51}$$

which is frequently used in axisymmetry. In axisymmetry, only the ϕ component of the $\boldsymbol{A}$ of a potential field is finite, and $A_\phi=r\mathrm{d}_\theta\Phi$. Thus,

$$D^2A_\phi=D^2r\mathrm{d}_\theta\Phi=r\mathrm{d}_\theta\mathrm{D}_r^2\Phi+r\mathrm{d}_\theta\mathrm{D}_\theta^2\Phi=r\mathrm{d}_\theta\nabla^2\Phi=0 \tag{52}$$

Another useful relation in axisymmetry is

$$\frac{\mathrm{d}P_\ell(\cos\theta)}{\mathrm{d}\theta}=P_\ell^1(\cos\theta) \tag{53}$$

7.1 Examples

$$\mathrm{D}_r^2 j_1(kr) = \left(2r^{-2} - k^2\right) j_1(kr) \tag{54}$$

$$\mathcal{D}_\theta^2 \sin\theta = -2r^{-2}\sin\theta \tag{55}$$

Thus

$$b(r,\theta) = j_1(kr)\sin\theta \tag{56}$$

is an eigenfunction of D^2 with

$$D^2 b + k^2 b = 0. \tag{57}$$

A particular solution of the potential equation

$$D^2 a = 0 \tag{58}$$

is

$$a = \begin{cases} \varpi & \text{for } r \le 1, \\ \varpi/r^3 & \text{for } r > 1. \end{cases} \tag{59}$$

where $\varpi = r\sin\theta$.

7.2 Constant α sphere

The axisymmetric steady state dynamo equations for $\alpha =$ const and $\eta_\mathrm{T} = R = 1$ are

$$\alpha b + D^2 a = 0, \quad -\alpha D^2 a + D^2 b = 0. \tag{60}$$

A solution that satisfies the boundary condition

$$\mathrm{d}a/\mathrm{d}r + 2a = 0 \quad \text{on } r = 1 \tag{61}$$

is

$$a = k^{-1}[j_1(kr) + \tfrac{1}{3}r]\sin\theta, \quad b = j_1(kr)\sin\theta \tag{62}$$

(with $k = 4.493409$) in $r \le 1$ and

$$a = \tfrac{1}{3}k^{-1}r^{-2}\sin\theta, \quad b = 0 \tag{63}$$

in $r > 1$.

7.3 Rewrite

$$\boldsymbol{A} = a\hat{\boldsymbol{\phi}} + \boldsymbol{\nabla}\times c\hat{\boldsymbol{\phi}}, \tag{64}$$

$$\boldsymbol{B} = b\hat{\boldsymbol{\phi}} + \boldsymbol{\nabla}\times a\hat{\boldsymbol{\phi}}, \tag{65}$$

where $D^2 c = -b$ and $c = 0$ on $r = 1$ is *assumed*. Therefore,

$$c = k^{-2} j_1(kr)\sin\theta \tag{66}$$

7.4 The potential part of $\boldsymbol{B}$

Both in the interior and the exterior the field has a potential part, $\boldsymbol{\nabla}\Psi$. This can be seed by splitting

$$a = a_0 + a_1 \tag{67}$$

with

$$a_0 = k^{-1} j_1(kr)\sin\theta, \quad a_1 = \tfrac{1}{3}k^{-1} r\sin\theta, \tag{68}$$

where a_1 satisfies

$$\boldsymbol{B}_1 = \boldsymbol{\nabla}\times a_1\hat{\boldsymbol{\phi}} = \boldsymbol{\nabla}\Psi. \tag{69}$$

In this case, $\boldsymbol{B}_1 = \frac{2}{3}k^{-1}\boldsymbol{z}$ (in $r < 1$), so

$$\Psi = \tfrac{2}{3}k^{-1} r\cos\theta \quad r < 1, \tag{70}$$

and

$$\Psi = \tfrac{1}{3}k^{-1} r^{-2}\cos\theta \quad r > 1. \tag{71}$$

7.5 Vacuum reference field

$$\boldsymbol{A}_\mathrm{P} = a_\mathrm{P}\hat{\boldsymbol{\phi}} + \boldsymbol{\nabla}\times c_\mathrm{P}\hat{\boldsymbol{\phi}}, \tag{72}$$

$$\boldsymbol{B}_\mathrm{P} = b_\mathrm{P}\hat{\boldsymbol{\phi}} + \boldsymbol{\nabla}\times a_\mathrm{P}\hat{\boldsymbol{\phi}}, \tag{73}$$

$$a_\mathrm{P} = \tfrac{1}{3} r k^{-1}\sin\theta, \quad b_\mathrm{P} = 0, \quad c_\mathrm{P} = 0. \tag{74}$$

7.6 Integral over P

$$\int_0^\pi P_\ell(\cos\theta)\sin\theta\,\mathrm{d}\theta = \frac{2}{2\ell+1} \tag{75}$$

8 Magnetic helicity in spheres embedded in a vacuum

The helicity integral can be split into toroidal and poloidal components, i.e.

$$\boldsymbol{A}\cdot\boldsymbol{B} = ab + \boldsymbol{A}_\mathrm{p}\cdot\boldsymbol{B}_\mathrm{p} \tag{76}$$

The poloidal part of the helicity integral can be written in the form

$$\boldsymbol{A}_\mathrm{p}\cdot\boldsymbol{B}_\mathrm{p} = ab + \boldsymbol{\nabla}\cdot(a\hat{\boldsymbol{\phi}}\times\boldsymbol{A}_\mathrm{p}). \tag{77}$$

This shows first of all that the magnetic helicity for the infinite volume (surface integral vanishes) can be written as

$$H=\int_{V+V_{\rm ext}}(ab+\boldsymbol{A}_{\rm p}\cdot\boldsymbol{B}_{\rm p})\;{\rm d}V=2\int_V ab\;{\rm d}V. \qquad (78)$$

This is gauge-invariant, because there is no surface term. Since $b=0$ outside the sphere, this integral is only over the sphere. Except for the case of the constant α sphere, H will always be zero because of cancelation between the two hemispheres. Thus, the magnetic helicity integral for the infinite upper half space is, in the Coulomb gauge,

$$H_{\rm N}^{\rm (Cou)}=2\int_{\rm N} ab\;{\rm d}V+\int_{\theta=\pi/2} a\boldsymbol{A}_{\rm p}\cdot{\rm d}\boldsymbol{S}. \qquad (79)$$

The second integral vanishes for fields with dipolar symmetry ($A_r=0$ on the midplane).

8.1 Relative helicity

We make use of the fact that for the vacuum field only $a_{\rm P}$ is finite. The contribution from the toroidal components to the relative helicity integral is just $\int(a+a_{\rm P})b\,{\rm d}V$, because the vacuum field does not have a toroidal field. For the poloidal contribution we note that the vacuum field has a purely toroidal vector potential, so we have

$$\boldsymbol{A}_{\rm p}\cdot(\boldsymbol{B}-\boldsymbol{B}_{\rm P})_{\rm p}=b(a-a_{\rm P})+\boldsymbol{\nabla}\cdot[(a-a_{\rm P})\hat{\boldsymbol{\phi}}\times\boldsymbol{A}_{\rm p}]. \qquad (80)$$

However, on the boundary we always have

$$a=a_{\rm P}\quad\mbox{on }\partial V, \qquad (81)$$

so the relative magnetic helicity is always (both for toroidal and poloidal fields)

$$H^{\rm (rel)}=2\int_V ab\;{\rm d}V. \qquad (82)$$

In particular, we have

$$H_{\rm N}^{\rm (rel)}=2\int_{\rm N} ab\;{\rm d}V. \qquad (83)$$

8.2 Mean-field $\alpha\Omega$ dynamo

We consider here the case without meridional circulation, so we have

$$\left(\partial_t-\eta_{\rm T}D^2\right)a=\alpha b \qquad (84)$$

$$\left(\partial_t-\eta_{\rm T}D^2\right)b=\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\nabla}\times\alpha\boldsymbol{B}_{\rm p}+\varpi\boldsymbol{B}_{\rm p}\cdot\boldsymbol{\nabla}\Omega. \qquad (85)$$

We are interested in the evolution of ab, so from the first equation we have

$$b\partial_t a=\alpha b^2-\eta_{\rm T}jb, \qquad (86)$$

where $j=-D^2a$. From the second equation we have

$$a\partial_t b=\alpha\boldsymbol{B}_{\rm p}^2-\eta_{\rm T}\boldsymbol{J}_{\rm p}\cdot\boldsymbol{B}_{\rm p}-\tfrac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{F}_{\rm H}. \qquad (87)$$

The factor 1/2 enters in the expression for the helicity flux, because the relative magnetic helicity is given by $2ab$, so

$$\boldsymbol{F}_{\rm H}=-2(\alpha\boldsymbol{B}_{\rm p}-\eta_{\rm T}\boldsymbol{J}_{\rm p})\times\hat{\boldsymbol{\phi}}a-2a\varpi\Omega\,\boldsymbol{B}_{\rm p}, \qquad (88)$$

where we have made use of the identities

$$a\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\nabla}\times\alpha\boldsymbol{B}_{\rm p}=\alpha\boldsymbol{B}_{\rm p}^2+\alpha\boldsymbol{B}_{\rm p}\times\hat{\boldsymbol{\phi}}a, \qquad (89)$$

$$aD^2b=-\boldsymbol{J}_{\rm p}\cdot\boldsymbol{B}_{\rm p}-\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\rm p}\times\hat{\boldsymbol{\phi}}a, \qquad (90)$$

$$\varpi a\,\boldsymbol{B}_{\rm p}\cdot\boldsymbol{\nabla}\Omega=\boldsymbol{\nabla}\cdot(a\varpi\Omega\,\boldsymbol{B}_{\rm p}). \qquad (91)$$

The last one comes from the helicity flux due to the toroidal velocity, $\boldsymbol{U}=\varpi\Omega\hat{\boldsymbol{\phi}}$,

$$(\boldsymbol{U}\times\boldsymbol{B}_{\rm p})\times a\hat{\boldsymbol{\phi}}=a\varpi\Omega\,\boldsymbol{B}_{\rm p} \qquad (92)$$

We also note that in axisymmetry the poloidal magnetic field can be written in the form

$$\boldsymbol{B}_{\rm p}\equiv\boldsymbol{\nabla}\times a\hat{\boldsymbol{\phi}}=-\varpi^{-1}\hat{\boldsymbol{\phi}}\times\boldsymbol{\nabla}(\varpi a) \qquad (93)$$

Thus, the final magnetic helicity equation is

$$\frac{{\rm d}}{{\rm d}t}H^{\rm (rel)}=2\int_V\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}}-\oint_{\partial V}\boldsymbol{F}_{\rm H}\cdot{\rm d}\boldsymbol{S} \qquad (94)$$

where

$$\overline{\mathcal{E}}\cdot\overline{\boldsymbol{B}}=\alpha\boldsymbol{B}^2-\eta_{\rm T}\boldsymbol{J}\cdot\boldsymbol{B} \qquad (95)$$

where we have omitted the overbars again, and

$$\boldsymbol{F}_{\rm H}=2\boldsymbol{E}_{\rm p}\times\boldsymbol{A}_{\rm t}, \qquad (96)$$

where $\boldsymbol{A}_{\rm t}=\hat{\boldsymbol{\phi}}a$ and

$$\boldsymbol{E}_{\rm p}=-\boldsymbol{U}\times\boldsymbol{B}_{\rm p}-(\alpha\boldsymbol{B}_{\rm p}-\eta_{\rm T}\boldsymbol{J}_{\rm p}). \qquad (97)$$

This equation is true even if there was meridional circulation, because the poloidal flow does not enter, so that

$$\boldsymbol{E}_{\rm p}=-\boldsymbol{U}_{\rm t}\times\boldsymbol{B}_{\rm p}-(\alpha\boldsymbol{B}_{\rm p}-\eta_{\rm T}\boldsymbol{J}_{\rm p}) \qquad (98)$$

is the relevant expression even in the general case (although then the symbol $\boldsymbol{E}_{\rm p}$ is misleading, because one would think there would be the additional term $\boldsymbol{U}_{\rm p}\times\boldsymbol{B}_{\rm t}$ which is not the case). On the other hand, $\boldsymbol{U}_{\rm p}=0$ on the boundary, so there would not have been any confusion anyway. The expression (88) is therefore to be preferred.

8.3 Advective gauge

Assume $\boldsymbol{U}=(0,0,\varpi\Omega)$, so

$$\frac{\partial\boldsymbol{A}}{\partial t}=\boldsymbol{U}\times\boldsymbol{B}=\Omega\boldsymbol{\nabla}(\varpi A_\phi)=-\varpi A_\phi\boldsymbol{\nabla}\Omega-\boldsymbol{\nabla}\phi \qquad (99)$$

with $\phi=U_\phi A_\phi$.