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Proc. R. Soc. Lond. A 1995 **448**, 237-244
doi: 10.1098/rspa.1995.0014

Subharmonic dynamo action of fluid motions with two-dimensional periodicity

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Kinematic dynamos based on steady velocity fields with two-dimensional periodicity are analysed numerically. The velocity fields of the study by G. O. Roberts (1972) are used and the analysis is extended to the case when the spatial periodicity of the magnetic field differs from that of the velocity field not only in the homogeneous third direction. While the solutions of Roberts correspond to the most efficient dynamos in most cases, there are some cases in which spatially subharmonic dynamos are preferred.

1. Introduction

The spatially periodic magnetic fields obtained as solutions of the kinematic dynamo problem by G. O. Roberts (1969, 1970, 1972) and by Childress (1969) have become a classical part of dynamo theory because of their mathematical simplicity. The complications of dynamos in spheres or in spherical shells which are of direct geophysical relevance can be avoided to a considerable extent in the spatially periodic case because of the absence of boundaries. Moreover, a particular case studied by Roberts, which is a first order dynamo in his notation, has turned out to be a very efficient dynamo. A modified version of it has therefore been proposed as the basis for a laboratory dynamo experiment (Busse 1992) which is likely to be realized in the near future (Busse & Müller 1994). Nearly two-dimensional velocity fields of a similar kind are believed to be driven by thermal or chemical buoyancy in the liquid metallic cores of the Earth and other planets. The analytical model of the geodynamo proposed by Busse (1975) thus incorporates features similar to those exhibited in the first order dynamo of Roberts.

In all cases treated by Roberts, the computations of dynamos have been restricted to magnetic fields which exhibit the same periodicity as the velocity field, except for an additional periodic dependence on the coordinate of which the velocity field is independent. While it is plausible in many cases that this restricted class of solutions includes the preferred dynamo corresponding to a minimum magnetic Reynolds number, there is no proof for such a mathematical property. Indeed, counter examples can be constructed, as will be discussed in the following. There are thus several reasons to consider more general classes of solutions, as will be done in this paper.

Proc. R. Soc. Lond. A (1995) **448**, 237–244
Printed in Great Britain

TEX Paper

2. Mathematical formulation of the problem

Kinematic dynamos are defined as growing solutions of the induction equation

$$\left(\frac{\partial}{\partial t} - \lambda\nabla^2\right)\boldsymbol{B} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) \tag{2.1}$$

for the magnetic flux density $\boldsymbol{B}$ where λ is the magnetic diffusivity. The velocity field $\boldsymbol{u}$ is a given solenoidal vector field. According to Cowling's theorem growing two-dimensional solutions of equation (2.1) do not exist. But two-dimensional velocity fields can give rise to dynamo action. For details on the derivation of equation (2.1) and its properties we refer to books on dynamo theory; see, for instance Moffatt (1978), Roberts & Gubbins (1987) and others.

We use a Cartesian system of coordinates and assume the solenoidal velocity field in the form

$$\boldsymbol{u} = \nabla\psi(x,y)\times\boldsymbol{k} + \boldsymbol{k}w(x,y), \tag{2.2}$$

where $\boldsymbol{k}$ is the unit vector in the z-direction. Following Roberts we shall choose ψ and w in the form

$$\psi = \sum_{p,q=-1}^{1} a_{pq}\exp\{\mathrm{i}px+\mathrm{i}qy\},\quad w = \sum_{p,q=-1}^{1} b_{pq}\exp\{\mathrm{i}px+\mathrm{i}qy\}, \tag{2.3a}$$

where $a_{pq} = (-\delta_{p0}+\delta_{q0})/2$ in the cases I through III, while the coefficients b_{pq} are given by

$$\left.\begin{array}{lll} \text{case I} & : & b_{pq} = (\delta_{q0}-\delta_{p0})/2, \\ \text{case II} & : & b_{pq} = (\delta_{p0}+\delta_{q0})/2 - \delta_{p0}\delta_{q0}, \\ \text{case III} & : & b_{pq} = (1-\delta_{p0}-\delta_{q0}+\delta_{p0}\delta_{q0})/2. \end{array}\right\} \tag{2.3b}$$

In case IV the representation

$$\psi = \frac{1}{4}\sum_{p,q=-1}^{1}(\delta_{q0}-\delta_{p0})\exp\{2\mathrm{i}(px+qy)\},\quad w = \frac{\mathrm{i}}{4}\sum_{p,q=-1}^{1}(q+p)\delta_{pq}\exp\{\mathrm{i}px+\mathrm{i}qy\} \tag{2.3c}$$

is used where δ_{pq} denotes the Kronecker symbol, $\delta_{pq}=1$ for $p=q$ and $\delta_{pq}=0$ for $p\neq q$. Cases I through IV correspond to the velocity fields,

$$\left.\begin{aligned} \boldsymbol{u}^{\mathrm{I}} &= (\sin y, \sin x, \cos x - \cos y), \\ \boldsymbol{u}^{\mathrm{II}} &= (\sin y, \sin x, \cos x + \cos y), \\ \boldsymbol{u}^{\mathrm{III}} &= (\sin y, \sin x, 2\cos x\cos y), \\ \boldsymbol{u}^{\mathrm{IV}} &= (\sin 2y, \sin 2x, \sin(x+y)), \end{aligned}\right\} \tag{2.3d}$$

in Roberts's (1972) formulation.

To solve equation (2.1) for $\boldsymbol{B}$ the general Floquet ansatz,

$$\boldsymbol{B} = \sum_{m,n=-N}^{N}\boldsymbol{B}_{mn}\exp\{\mathrm{i}(m+f_x)x+\mathrm{i}(n+f_y)y+\mathrm{i}f_z z+\sigma t\}, \tag{2.4}$$

is introduced which gives rise to the following equations for the x- and y-components of the coefficient vector $\boldsymbol{B}_{mn}$,

$$(\sigma + Rm^{-1}[(m+f_x)^2 + (n+f_y)^2])\hat{\boldsymbol{B}}_{mn}$$
$$= \sum_{p,q=-1}^{1} [a_{pq}(q(m-q+f_x) - p(n-p+f_y))) - \mathrm{i}b_{pq}f_z]\hat{\boldsymbol{B}}_{m-q,n-p}$$
$$+ \sum_{p,q=-1}^{1} a_{pq}\boldsymbol{A}\cdot\hat{\boldsymbol{B}}_{m-p,n-q}, \qquad (2.5a)$$

where the vector $\hat{\boldsymbol{B}}_{mn}$ includes only the x- and y-components of $\boldsymbol{B}_{mn}$ and where the matrix $\boldsymbol{A}$ is defined by

$$\boldsymbol{A} = \begin{pmatrix} 0 & q^2 \\ p^2 & 0 \end{pmatrix}.$$

Since we have chosen the amplitude of motion and its wavenumber equal to unity it is appropriate to replace the diffusivity λ by the inverse of the magnetic Reynolds number Rm. The z-component of $\boldsymbol{B}_{mn}$ is determined through the condition $\nabla\cdot\boldsymbol{B} = 0$ once equation (2.5) for $\hat{\boldsymbol{B}}_{mn}$ has been solved. The latter equation represents an eigenvalue problem for the eigenvalue σ as a function of the Floquet vector $\boldsymbol{f}$.

While equation (2.5) has been derived for cases I through III, a somewhat more complex equation is obtained in case IV:

$$(\sigma + Rm^{-1}[(m+f_x)^2 + (n+f_y)^2])\hat{\boldsymbol{B}}_{mn}$$
$$= \sum_{p,q=-1}^{1} \{\tfrac{1}{2}(\delta_{q0} - \delta_{p0})[q(m-2q+f_x) - p(n-2p+f_y)]\hat{\boldsymbol{B}}_{m-2q,n-2p}$$
$$+\tfrac{1}{4}(q+p)\delta_{pq}f_z\hat{\boldsymbol{B}}_{m-q,n-p} + (\delta_{q0} - \delta_{p0}\boldsymbol{A}\cdot\hat{\boldsymbol{B}}_{m-2q,n-2p}\}. \qquad (2.5b)$$

In principle the summation limit N in expression (2.4) is infinite. But in order to solve equations (2.5) numerically we have to assume a finite truncation parameter N. Coefficients $\hat{\boldsymbol{B}}_{mn}$ with $|m|$ or $|n|$ exceeding N are neglected. By replacing N by $N-2$ the accuracy of the numerical solution can be checked. As the magnetic Reynolds number Rm is increased the truncation parameter N must be increased as well. For the results described in the following section it was sufficient to use $N = 7$ in case I, $N = 11$ in cases II and III. But in case IV values of N as high as 13 were required to reduce the difference between results for N and $N-2$ to less than 2%.

3. Discussion of the numerical results

It is not feasible to plot the magnetic Reynolds number Rm as a function of all three parameters f_x, f_y, f_z. Since the structure of basic equation (2.1) is that of a Mathieu equation in the x- and in the y-coordinate, it seems likely that in addition to the case $f_x = f_y = 0$ and its neighbourhood the cases with $f_x = \frac{1}{2}$ or $f_y = \frac{1}{2}$ will lead to the lowest values of Rm for which growing fields $\boldsymbol{B}$ occur. Since this expectation was confirmed in preliminary computations we shall focus on the 'subharmonic' cases in the following while f_z will be regarded as a continuous parameter.

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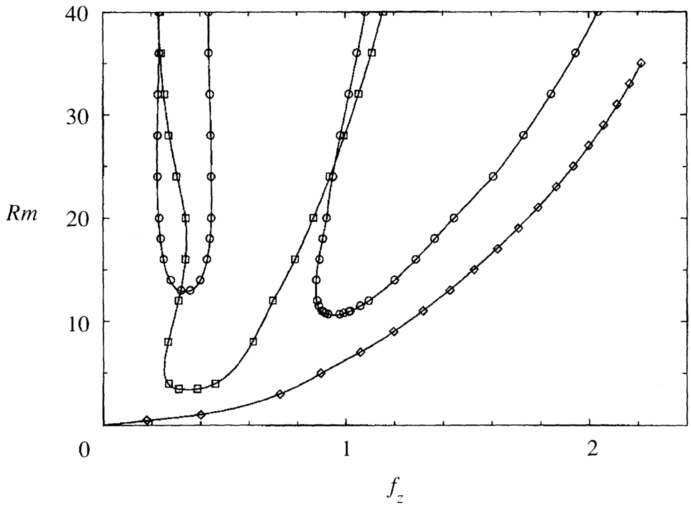

Figure 1. The magnetic Reynolds number Rm for onset of dynamo action in case I as a function of the wavenumber f_z for the example $f_x = f_y = 0$ (◇), $f_x = 0.5$, $f_y = 0$ (□), and $f_x = f_y = 0.5$ (○). In the latter two examples oscillatory dynamos occur.

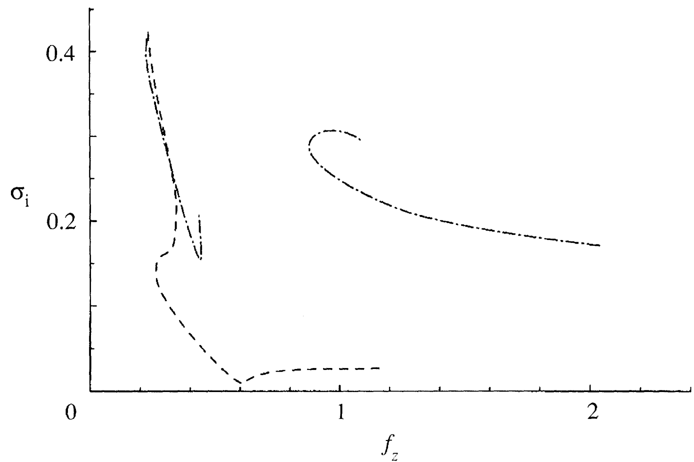

Figure 2. The imaginary parts σ_i of the growth rates σ for the examples of figure 1 in which oscillatory onset of dynamo action occurs: $f_x = 0.5$, $f_y = 0$ (– –), $f_x = f_y = 0.5$ (–·).

In figure 1 the critical Reynolds number for case I is shown for various combinations of f_x and f_y. As expected the case $f_x = f_y = 0$ analysed by Roberts (1972) corresponds to the lowest value of Rm. An oscillatory onset of dynamo action occurs for the other cases. The imaginary part σ_i of the growth rate at the point $\sigma_r = 0$ is shown in figure 2. Since the dynamos correspond to a Hopf bifurcation, the complex conjugate of any eigenvalue σ also is an eigenvalue. Even in the supercritical regime the real part σ_r of the growth rate assumes its maximum value in the case $f_x = f_y = 0$ as is indicated in figure 3 where the growth rates have been plotted for a given value of Rm. For reasons of symmetry the cases $f_y = \frac{1}{2}$, $f_x = 0$ and $f_x = \frac{1}{2}$, $f_y = 0$ give the same results and the former case has not been plotted in any of the figures.

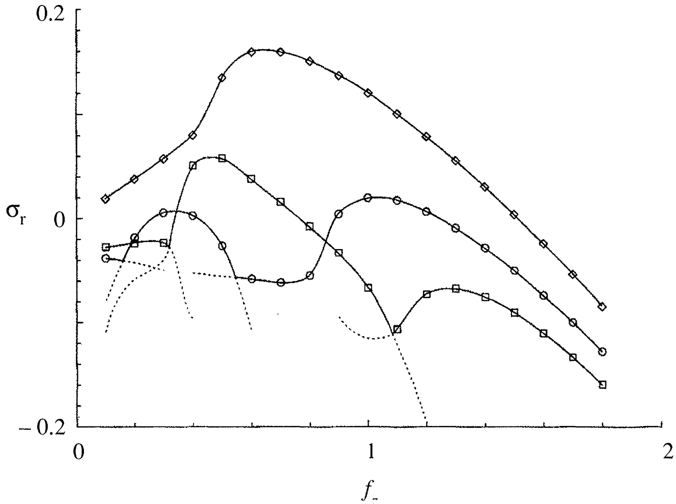

Figure 3. The growth rate σ_r for the examples of figure 1 at $Rm = 15$. The strongest growing magnetic fields may correspond to different eigensolutions as a function of f_z as indicated by the intersection in the figure.

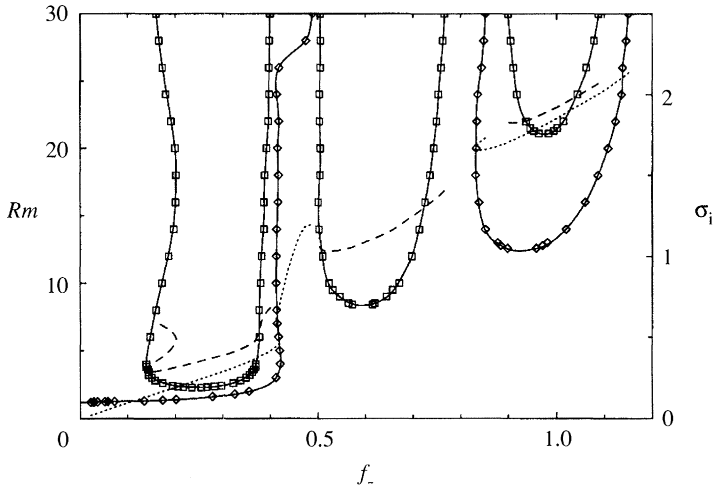

Figure 4. The magnetic Reynolds number Rm and frequency σ_i (left ordinate) for the onset of dynamo action in case II for the examples $f_x = f_y = 0$ (◇, dotted line for σ_i) and $f_x = 0.5, f_y = 0$ (□, dashed line for σ_i). No dynamo was found for $f_x = f_y = 0.5$ for $Rm \leqslant 20$.

The results for case II of (2.3) are shown in figure 4. The subharmonic dynamo almost matches the critical value of Rm for $f_x = f_y = 0$. No growing magnetic fields could be found in the doubly subharmonic case $f_x = f_y = \frac{1}{2}$ in the regime $Rm < 20$, $|f_z| < 1$. Both types of dynamos are of oscillatory nature and the parallel dependence of σ_i on f_z for $f_z > 0.1$ indicates a close correspondence in the dynamo mechanisms. The approximate linear dependence of σ_i on f_z suggests that the dynamo waves propagating in the z-direction exhibit a nearly constant phase velocity.

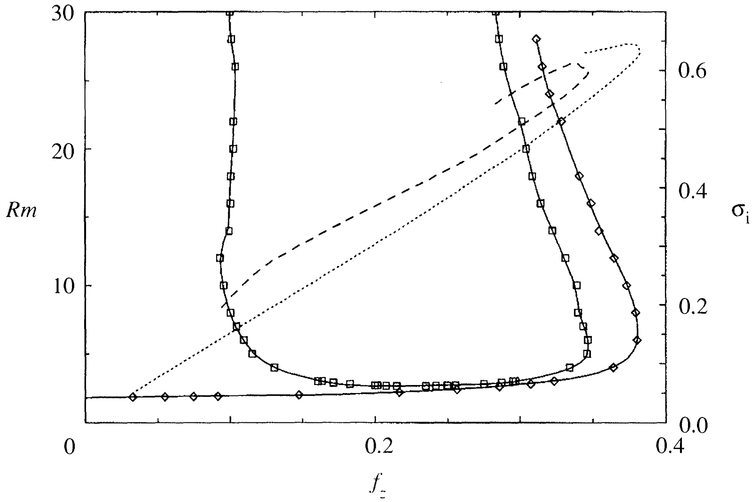

Figure 5. Same as figure 4 in case III. Dynamo action was also found in the case $f_x = f_y = 0.5$ in the ranges $0.15 \leqslant f_z \leqslant 0.25$ and $0.5 \leqslant f_z \leqslant 0.75$ for $Rm = 20$. But the corresponding curves are not included in the figure, because the values of Rm are higher than in the other examples.

In figure 5 the results for case III are shown. Here the subharmonic dynamo does indeed correspond to the critical magnetic Reynolds number within the intermediate regime $0.5 \leqslant f_z \leqslant 0.8$ of the z-wavenumber. As shown in the figure there are two other subharmonic branches, but these are preceded by the onset of magnetic fields with the same periodicity as the velocity field. In addition there are doubly subharmonic dynamos, but they occur only in the regions $0.15 \leqslant f_z \leqslant 0.25$ and $0.5 \leqslant f_z \leqslant 0.75$. They correspond to relatively high values of Rm of the order 20 or larger and are not shown in figure 5 for this reason. Again, all of the dynamos are oscillatory. The imaginary part of the growth rate plotted in the same figure exhibits a similar approximate linear trend as in the case II. Case IV is unusual in that the basic wavenumbers of the two components of the velocity fields (2.2) differ by a factor two. Dynamos that are subharmonic with respect to $w(x, y)$ exist for sufficiently high values of f_z and the dynamo with $f_x = f_y = \frac{1}{2}$ actually precedes the dynamo with the same periodicity as the velocity field in the regime $0.7 \leqslant f_z \leqslant 1.4$ as shown in figure 6.

When the dynamos of the cases I through IV are compared for a given value of the wavenumber f_z which may correspond to the finite size of the domain of electrically conducting fluid, then case I appears to offer the lowest magnetic Reynolds number for dynamo action. Except for case IV the differences in the critical values Rm are not substantial for f_z of the order 0.5 or larger. But in the limit $f_z \to 0$ the dynamo with $f_x = f_y = 0$ of case I is unique in that the magnetic Reynolds number tends to zero. This property is caused by the fact that case I represents a first order dynamo in the language of Roberts (1972). This property is lost as soon as finite values of f_x or f_y are used and a finite value Rm of the order of f_x or f_y must be expected in the limit $f_z \to 0$. The first order dynamo of Roberts (1971) has been studied in considerable detail by Childress (1979) and Soward (1989) with a special emphasis on the limit of large Rm. Unfortunately such analytical studies can not easily be extended to subharmonic cases.

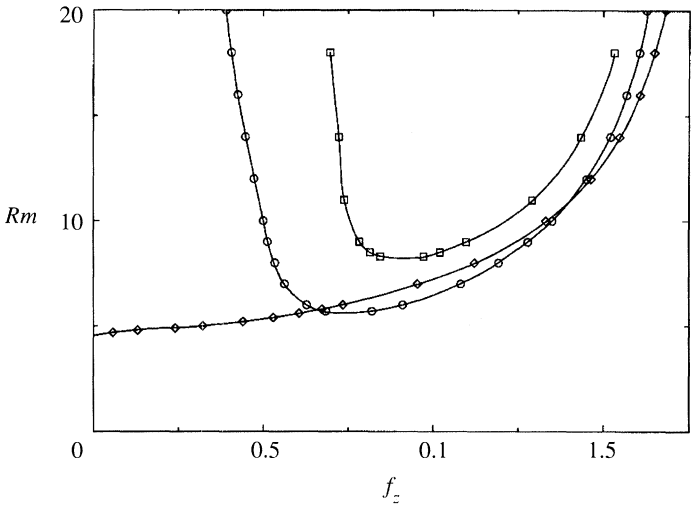

Figure 6. Same as figure 1 in case IV. The dynamos are characterized by a monotone onset in this case.

4. Concluding remarks

The subharmonic dynamos discussed in the preceding section have not received much attention since they do not exhibit a mean component of the magnetic field in the sense of an average over the x, y-plane. In the case of a spherical dynamo subharmonic dynamos driven by a velocity field which is periodic in the azimuthal direction would not show an axisymmetric component of the magnetic field. In general, however, there is an entire spectrum of motions available such that a mean or an axisymmetric component of the magnetic field can be generated through secondary interactions. For this reason the subharmonic route of dynamo action deserves some attention.

A characteristic feature of all subharmonic dynamos displayed in figures 1–6 is the property that they require a wavenumber in excess of a positive minimum value f_z. The cause of this feature is not obvious and analytical treatments of the problem are difficult.

Another question that can not be answered in a general sense concerns the necessary properties of two-dimensional, spatially periodic steady velocity fields of the form (2.2) that are required for dynamo action. The four cases considered in this paper have been selected originally by Roberts on the basis of heuristic arguments. Other cases obtained by phase shifts of 90° of w relative to ψ typically lead to disappearance of dynamo action. This property is not surprising, since the velocity field (2.2) can be written in the form,

$$\boldsymbol{v} = \nabla\psi \times (\boldsymbol{k} + (\boldsymbol{i} + \boldsymbol{j})/\sqrt{2}), \tag{4.1}$$

if x and y are replaced by $x + \frac{1}{2}\pi$ and $y - \frac{1}{2}\pi$ in the expansion (2.3) for w in the case I and by $x + \frac{1}{2}\pi$ and $y + \frac{1}{2}\pi$ in the case II. The velocity field (4.1) is a toroidal velocity field and thus incapable of dynamo action. A similar shift of the vertical component w in case III also leads to an apparent disappearence of dynamo action at least for $Rm < 20$. But the streamlines of the velocity field do not lie in parallel planes in this case. Actually the undulating surfaces spanned by the streamlines resemble the surface tangential to the velocity field of case II. It is not obvious why dynamo action is obtained in one case and not in the other. It is hoped that antidynamo theorems for certain types of periodic velocity fields of the form (2.2) will eventually lead to a better understanding for the restrictions on dynamo action.

The research reported in this paper has been supported by the Deutsche Forschungsgemeinschaft.

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Received 8 March 1994; accepted 12 May 1994