Small-Scale-Field Dynamo

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Generation of magnetic field energy, without mean field generation, is studied. Isotropic mirror-symmetric turbulence of a conducting fluid amplifies the energy of small-scale magnetic perturbations if the magnetic Reynolds number is high, and the dimensionality of space d satisfies $2.103 < d < 8.765$. The result does not depend on the model of turbulence, incompressibility, and isotropy being the only requirements. [S0031-9007(96)01744-9]

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Solar and galactic magnetic fields are generated by turbulent motions of their electrically conducting quasifluid constituents. Generation of large-scale or mean magnetic fields is described by the kinematic dynamo theory of Steenbeck, Krause, and Radler [1]. Small-scale magnetic fields can come from the large-scale fields because of turbulent stretching and twisting and play an important role in saturation of the dynamo generated mean fields [2–4]. It was noted long ago [5–7] that small-scale fields (SSF) can also be generated in the absence of mean fields. For example, mirror-symmetric turbulence is incapable of the mean field dynamo, but can amplify SSF perturbations. A nice qualitative picture of this process was given by Moffatt [8]. When the magnetic Reynolds number is large, the magnetic field is frozen into the fluid. The field magnitude grows as the infinitesimal line element, that is, exponentially. However, the characteristic wave number of the magnetic perturbation also grows exponentially and finally reaches the dissipation scale. It is a delicate question which one of these effects wins on the average.

In many astrophysically interesting cases the viscosity is much larger than the resistivity, and the turbulent velocities cut off at a scale much larger than the scale at which resistivity destroys magnetic field. The SSF dynamo exists on scales between the viscous and resistive cutoffs. In high Reynolds number turbulence the SSF dynamo growth rate exceeds the growth rate of the mean field dynamo by a factor of $\mathrm{Re}^{1/2}$. Thus, if SSF dynamo does operate, it should be important for the large-scale dynamo saturation [3]. Also of interest are the consequences of the SSF dynamo for the applicability of the kinematic mean field dynamo as discussed by Kulsrud and Anderson [9]. In this Letter, we show that the SSF dynamo indeed operates for spatial dimension d in the interval $2.103 < d < 8.765$. The result is model independent; in particular, turbulence is not assumed to be δ-correlated in time.

To get this result, we use a zero-dimensional representation of the induction equation. Consider a small element of fluid with dimensions much smaller than the smallest scale of the turbulent flow, but much larger than the scale of the magnetic field. We may then approximate the velocity field of the fluid element by a Taylor expansion around an arbitrary center $\mathbf{r}_0$. Thus, $v_a = \dot{r}_{0a} + V_{ab}(r_b - r_{0b}) + \dots$. Here V is the rate of strain tensor moving with the fluid element, $V_{ab} = \partial_a v_b$. Note that the rate of strain tensor is dominated by the smallest scales close to the viscous cutoff [9]. We assume incompressibility, $\mathrm{tr}\, V = 0$. Then the magnetic field evolution is given by equations for the field B and the wave number k [10],

$$\dot{B} = VB\,, \tag{1}$$

$$\dot{k} = -V^t k\,. \tag{2}$$

Here V^t is the transpose of V; the molecular magnetic diffusivity will be taken into account later. Equations (1) and (2) are applicable for SSF dynamics only; at large length scales they are invalid. More precisely, (1) and (2) are always exact, being just equations for covectors and contravectors. However, if the length scale of the magnetic field becomes comparable to the length scale of the turbulent velocity field, it takes an infinite number of such equations to describe the time evolution of the magnetic field. This infinite system of equations is equivalent to the standard induction equation $\partial_t B = \nabla \times (v \times B)$. Formal solution to this linear equation can be written, but we do not know how to average the resulting magnetic energy. The linear (kinematic) dynamo is a nontrivial problem.

The system (1) and (2) is much simpler than the standard induction equation, all the relevant properties of turbulence being concentrated in the stretching tensor V. We wish to know whether the energy of the SSF grows or decays for a random stretching rate V with given statistics.

As an introduction to the problem, consider the Gaussian δ-correlated turbulence. Then turbulence is specified by the pair correlator of the stretching tensor. From isotropy and incompressibility,

$$\langle V_{ab} V_{cd} \rangle = 2\gamma \delta(t)[(d+1)\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}], \tag{3}$$

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where γ is a constant with dimension of frequency. The physical meaning of γ is the characteristic stretching rate, γ is determined by the smallest eddies. Let $P(k, B)$ be the probability distribution for the pairs (k, B). Since V of (1) and (2) is δ correlated in time, one can derive a Fokker-Planck equation for P using standard procedures (see, eg., [11]),

$$\partial_t P = (1/2)[\partial_{k_a}\partial_{k_b} D_{ab}^{kk} + 2\partial_{k_a}\partial_{B_b} D_{ab}^{kB} + \partial_{B_a}\partial_{B_b} D_{ab}^{BB}]P, \tag{4}$$

where the mean velocities vanish and the diffusivities are

$$D_{ab}^{kk} = \gamma[(d+1)k^2\delta_{ab} - 2k_a k_b], \tag{5}$$

$$D_{ab}^{BB} = \gamma[(d+1)B^2\delta_{ab} - 2B_a B_b], \tag{6}$$

$$D_{ab}^{kB} = -\gamma[(d+1)k_b B_a - k_a B_b - k_c B_c \delta_{ab}]. \tag{7}$$

For our purposes, it suffices to now define the magnetic energy spectrum $W(k)$ as a B^2 moment of the probability distribution P. From (4)–(7), taking into account solenoidality $k_c B_c = 0$, and assuming isotropy, one gets the evolution equation for the magnetic energy spectrum,

$$\dot{W} = (d-1)k^2 W'' + (d^2-5)kW' + 2(d^2-d-2)W, \tag{8}$$

where the prime stays for the modulus k derivative, and time was normalized to exclude a factor proportional to γ. For $d = 3$ this equation coincides with those obtained in [7,9].

Boundary conditions for (8) is the immediate question. First of all, molecular magnetic diffusivity will eat up all the modes with very large k. Thus, the boundary condition at $k = \infty$ is simply $W = 0$. It is assumed that the limit $k_{\max} \to \infty$ is taken after all other limits. The order of performing limits can be important. For example, magnetic energy $E = \int dk\, W$ in two dimensions without resistivity grows exponentially,

$$\lim_{t\to\infty} \lim_{k_{\max}\to\infty} E = \infty .$$

At the same time, magnetic energy finally decays in two dimensions if the resistivity is positive, thus

$$\lim_{k_{\max}\to\infty} \lim_{t\to\infty} E = 0 .$$

This follows from the Zeldovich antidynamo theorem, and our calculations will reproduce this result.

Second, the very model (1) and (2) is not applicable at small k, where the turbulence is not reducible to the corresponding rate of stretching. In fact, at large scales all we have is a usual turbulent magnetic diffusivity. The magnetic field is thrown out of the large-scale region into turbulent scales. The corresponding boundary condition should be $W' = 0$ at some $k = k_{\min}$. From what follows and because $k_{\min}$ is small, this boundary condition is equivalent to $W = 0$ at $k = 0$.

The eigenmode of (8) is simply k^y, and the growth rate is

$$\gamma(y) = (d-1)y(y-1) + (d^2-5)y + 2(d^2-d-2). \tag{9}$$

To satisfy the boundary conditions, one has to have a pair of solutions with the exponent $y = \mathrm{Re}\, y \pm i0$ giving the same growth rate γ (then the linear combination of these modes will vanish at $\ln k = \pm\infty$). This means that y is determined by the requirement

$$\gamma'(y) = 0 . \tag{10}$$

Calculating y and then calculating the corresponding growth rate, we get the SSF dynamo ($\gamma > 0$) for dimensionalities d satisfying

$$d(d-1)(9-d) > 16 , \tag{11}$$

or $2.103 < d < 8.765$ as advertised.

Certainly we are mostly interested in the $d = 3$ case, which happens to be a dynamo case according to (11). However, working in d dimensions is useful for what is coming. Suppose, that the δ-correlation supposition is dropped. Then the coefficients in the diffusion equation (8), and, in fact, the very form of the equation, will change. Perhaps we will still have SSF dynamo in some dimensionalities interval, but the lower critical dimension might turn out to be greater than three, and there will be no SSF dynamo in three dimensions. As Vainshtein [12] puts it, the SSF dynamo is a quantitative rather than a qualitative problem. It was our original idea to show that the critical dimensions do depend on the correlation properties of turbulence, and the δ-correlated result of [7,9] is not reliable. To our surprise, we found that the critical dimensions are structurally stable and probably even universal as far as the turbulence is incompressible and isotropic. We give a short account of our investigations in what follows.

The idea is to solve (1) and (2) for a long rather than a short time interval. The solution is

$$B = UB_0 , \tag{12}$$

$$k = (U^t)^{-1}k_0 . \tag{13}$$

Here U is the evolution matrix given by $\dot{U} = VU$, $U(t=0) = I$; in terms of Euler $\mathbf{r}$ and Lagrange $\mathbf{a}$ coordinates $U = \partial\mathbf{r}/\partial\mathbf{a}$. If time of integration is long enough, the matrix U can be written as $R_1 D R_2$, where R are random rotation matrices, and D is diagonal. Physically speaking, after several correlation times, the orientation of the fluid element is randomized. Now suppose that at time $t = 0$ magnetic energy spectrum was isotropic, $W_0 = W_0(k)$, where k now means the magnitude. Using randomness of the rotation matrices R, one can calculate the new spectrum $W(k)$ after the transformation (12) and (13) as

$$W(k) = \langle |D\hat{k}||D^{-1}\hat{b}|^{-(d+1)} W_0(|D\hat{k}|k)\rangle , \tag{14}$$

where $\langle\cdots\rangle$ means averaging over the angles of the perpendicular d-dimensional unit vectors $\hat{k}$ and $\hat{b}$. The integral transformation (14) generalizes the Fokker-Planck equation (8). Equation (14) is the main result; it can be derived as follows.

If P_0 was the probability distribution of pairs $(\mathbf{k}, \mathbf{B})$ before the transformation (12) and (13), then the new pdf P after the transformation is

$$P(\mathbf{k}, \mathbf{B}) = P_0(U^t\mathbf{k}, U^{-1}\mathbf{B}). \tag{15}$$

Because of isotropy, the magnetic energy spectrum $W(k)$ can be calculated as

$$W(k) = \int B^{d+1} dB \langle P(k\hat{k}, B\hat{b})\rangle, \tag{16}$$

where $\langle\cdots\rangle$ is the angle average. Now plug (15) into (16)

$$W(k) = \int B^{d+1} dB \langle P_0(kU^t\hat{k}, BU^{-1}\hat{b})\rangle. \tag{17}$$

As a consequence of isotropy and solenoidality, the probability distribution has a form

$$P_0(\mathbf{k}, \mathbf{B}) = Q(k, B)\delta(\hat{k}\cdot\hat{b}), \tag{18}$$

and (17) can be written as

$$W(k) = \int B^{d+1} dB \langle Q(|U^t\hat{k}|k, |U^{-1}\hat{b}|B) \times \delta(\hat{k}\cdot\hat{b}/(|U^t\hat{k}||U^{-1}\hat{b}|))\rangle. \tag{19}$$

Now change the integration variable in (19), $B \to B/|U^{-1}\hat{b}|$, and get

$$W(k) = \int B^{d+1} dB \langle |U^t\hat{k}||U^{-1}\hat{b}|^{-(d+1)} \times Q(|U^t\hat{k}|k, B)\delta(\hat{k}\cdot\hat{b})\rangle. \tag{20}$$

Recalling (18) and the definition of W_0, we write (20) in a form (14). Only the diagonal part of U matters because of the angle averaging.

A nice property of (14) is that the eigenmode is still k^y. The amplification factor after a single transformation (14) is given by

$$\Gamma(y) = \langle |D\hat{k}|^{(y+1)} |D^{-1}\hat{b}|^{-(d+1)}\rangle. \tag{21}$$

Write the diagonal part as $D = \exp\operatorname{diag}(\lambda_1/2, \ldots, \lambda_d/2)$, the incompressibility implies $\lambda_1 + \cdots + \lambda_d = 0$. Then,

$$|D\hat{k}|^2 = \sum_j e^{\lambda_j} k_j^2, \tag{22}$$

$$|D^{-1}\hat{b}|^2 = \sum_j e^{-\lambda_j} b_j^2. \tag{23}$$

As before, to satisfy the boundary conditions one chooses the exponent y in such a way as to minimize Γ,

$$\Gamma'(y) = 0. \tag{24}$$

Thus, the necessary and sufficient condition for the SSF dynamo is that Γ be greater than 1 for arbitrary y and λ_j. Since Γ is equal to 1 at $\lambda_1 = \cdots = \lambda_d = 0$, this point should be a minimum for the dynamo case. Calculating (21) in the vicinity of zero, one gets

$$\Gamma(y) = 1 + [4d(d-1)(d+2)]^{-1}\Big(\sum_j \lambda_j^2\Big) \times [(d-1)y(y-1) + (d^2-5)y + 2(d^2-d-2)]. \tag{25}$$

Comparing to (9), we see that Γ is greater than 1 in the vicinity of the point $\lambda = 0$ if (11) is satisfied. Thus the structural stability of the result (11) is proved. We suspect that (11) ensures that Γ is greater than 1 everywhere, not only in the vicinity of the zero point. We do not have a proof. However, numerical calculations of the integral (21) in three dimensions confirm that $\lambda = 0$ is a global minimum and Γ is greater than 1 everywhere else.

To conclude, we have shown that SSF dynamo operates in three dimensional isotropic turbulence. The result seems to be model independent and is definitely structurally stable. The SSF dynamo is of great importance to the understanding of the growth of large-scale fields. Galaxies and upper layers of stars are characterized by $R_m \gg \mathrm{Re}$ which is the necessary condition for a SSF dynamo to be present. The large growth rate of the SSF dynamo—it is $\mathrm{Re}^{1/2}$ bigger than the mean field growth rate—ensures that self-consistency effects first enter at the small scales, when the large-scale fields are negligible [9]. Clearly large-scale fields are observed in galaxies and stars, and the dynamo process must involve considerable evolution where the small-scale fields are dynamically important but the large scales are not. The SSF dynamo saturation mechanisms and the effects of SSF dynamo on the conventional mean field dynamo should be subjects of further studies [9,13].

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