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On revolutionizing quantum field theory with Tomita's modular theory

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In the book of Haag [*Local Quantum Physics* (Springer Verlag, Berlin, 1992)] about local quantum field theory the main results are obtained by the older methods of C^*- and W^*-algebra theory. A great advance, especially in the theory of W^*-algebras, is due to Tomita's discovery of the theory of modular Hilbert algebras [*Quasi-standard von Neumann algebras*, Preprint (1967)]. Because of the abstract nature of the underlying concepts, this theory became (except for some sporadic results) a technique for quantum field theory only in the beginning of the nineties. In this review the results obtained up to this point will be collected and some problems for the future will be discussed at the end. In the first section the technical tools will be presented. Then in the second section two concepts, the half-sided translations and the half-sided modular inclusions, will be explained. These concepts have revolutionized the handling of quantum field theory. Examples for which the modular groups are explicitly known are presented in the third section. One of the important results of the new theory is the proof of the PCT theorem in the theory of local observables. Questions connected with the proof are discussed in Sec. IV. Section V deals with the structure of local algebras and with questions connected with symmetry groups. In Sec. VI a theory of tensor product decompositions will be presented. In the last section problems that are closely connected with the modular theory and that should be treated in the future will be discussed. [S0022-2488(00)00906-3]

CONTENTS

PRELIMINARY REMARKS

The original version of this review has been twice as long and contained all necessary proofs. On request of the editors I had to shorten it. Therefore, most of the proofs have only been sketched or dropped completely. Readers interested in details can get the entire script from the server of **l**ocal **q**uantum **p**hysics:

http://www.lpq.uni-goettingen.de/papers/99/04/99042900.html

I. INTRODUCTION

In this section we start with some statements of general interest, and add the main concepts and notations to be used in this note.

I.1. Some general remarks

Shortly after the invention of quantum mechanics, several scientists tried to generalize this theory to systems of infinite many degrees of freedom. (See, e.g., P.A.M. Dirac,[1] Jordan and Wigner,[2] Heisenberg and Pauli.[3,4]) In many of these attempts the authors wanted to incorporate the principle of special relativity at the same time. The combination of these two aspects is called relativistic quantum field theory, for which the term QFT will be used as short form in this note.

Nonrelativistic quantum field theory and QFT are usually used in different branches of physics. The area of application for the first is quantum statistical mechanics, solid state physics, and liquids. The latter theory is mainly used for elementary particle physics. Quantum electrodynamics and the standard model are two theories where the concepts of QFT are used. These examples do not imply that the concepts of one form of the field theory cannot be useful for the other. The investigation of Bros and Buchholz[5] on the relativistic KMS condition is such a case.

QFT has several different facets:

1. Lagrangian quantum field theory together with perturbation theory.
2. L.S.Z. theory, which is useful for scattering problems.[6]
3. Wightman's quantum field theory[7] and its derivative, the Euclidean field theory.
4. The theory of local observables in the sense of Araki, Haag, and Kastler.[8]

The Lagrangean QFT is closest to physical intuition. But it has the disadvantage that the expressions which appear in this theory have only a formal meaning. Up to now there is no convincing scheme which puts the formal expressions onto a solid and consistent mathematical basis. The existing perturbation and renormalization theory does not, in most cases, indicate anything about the quality of the approximation. Therefore, only comparison with experiment can indicate the quality of the Lagrange function and the approximation. Not in all cases is one as lucky as in quantum electrodynamics, where the agreement between calculations and experiment is excellent. If, as it is the case in the standard model, the Lagrange function depends on too many parameters, then some sceptics are not satisfied, since some experimentalists say: ''With three parameters one can fit an elephant and with a fourth parameter one can make him wiggle his tail.'' Probably the right mathematics has still to be invented in order to make Lagrangean QFT acceptable for everyone.

Before and during World War II perturbation and renormalization theory consisted largely of formal manipulations. This led R. Jost to the sarcastic remark: ''In the thirties, under the demoralizing influence of quantum theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets.'' In the fifties there have been several attempts to put QFT on an axiomatic basis. This was possible since new mathematics had been developed, for instance the theory of distributions (see, e.g., Schwartz[9,10]) and the theory of C^*-algebras (see, e.g., Naimark[11]). The theory of distributions is needed for the LSZ[6] and the Wightman[7] approach, and the theory of C^*-algebras for the concept of local observables. While the LSZ and the Wightman formalisms are still close to the ideas of Lagrangein QFT, a new road was taken in the theory of local observables.

Since von Neumann[12,13] it is known that in quantum mechanics one can replace the unbounded physical observables by bounded functions of them. This has the advantage that, for many problems of a general nature, the annoying operator domain questions disappear. In 1947, Segal[14] proposed to use this method also for QFT. This idea has been taken up by R. Haag, and it developed between 1959 and 1964[15–17] into the theory of local observables.

The increase of knowledge in functional analysis led also to partial progress in Lagrangean QFT. With new techniques those theories which are superrenormalizable, could be rigorously handled. Glimm and Jaffe (see, e.g., Ref. 18) have been the main promotors of this subject. The number of scientists who have contributed to this field is enormous, and it is impossible to mention them all.

Reviewing the past, the situation is as follows: The analyticity properties of the Wightman functions allow one to choose the time coordinates to be purely imaginary. The functions obtained in this way are called Schwinger functions. These are (real) analytic for noncoinciding points and, in the case of Bose fields, symmetric in all variables. With help of the Hahn–Banach theorem one can extend these functions to the coincidence points as symmetric distributions. It was the idea of Symanzik[19] to identify these symmetric functions with the vacuum expectation value of a commutative and hence classical field. He also assumed that the representation of this field is on a Hilbert space with positive metric. In so doing the Schwinger functions can be considered as the moments of a positive measure on the space of tempered distributions $\mathcal{S}'$. Since many approximation theorems exist for positive measures, one can, in favorable situations, first approximate the dynamics on a lattice in a box and take the continuum limit and the limit for the box tending to the whole space.

Unfortunately, the positivity of the Hilbert space for the Wightman theory does not imply that the Schwinger functions define a positive linear functional (on the symmetrized test function algebra). The positivity of the Wightman functional implies only the restricted Osterwalder–Schrader positivity[20,21] (see, also, Glaser[22]). This is the positivity condition for nonoverlapping functions. If one uses the Osterwalder–Schrader condition also for overlapping functions, then one calls it extended positivity. If a theory fulfills extended Osterwalder–Schrader positivity and Euclidean covariance at the same time, then, by a result of Yngvason,[23] the Schwinger functions define a positive functional.

It is well known that broken time reversal (which is the case in nature) is not compatible with a positive measure for describing the Schwinger functions. A generalization would be to work with a signed (complex) measure. Borchers and Yngvason[24] have derived necessary and sufficient conditions implying that the Schwinger functions are moments of a complex measure. These conditions are closely related to the existence of the Wilson–Zimmermann[25,26] decomposition of products of field operators. The restricted Osterwalder–Schrader positivity still has to hold. In my opinion one has to learn to draw conclusions from this condition before one can handle convergence problems for signed measures. It is not known whether or not the Wilson–Zimmermann product expansion holds for every Lagrangean QFT. If this is not the case, one has to generalize the measure theory on Montel spaces (the test function space) as one has generalized the measure theory on $\mathbb{R}^n$ to distributions, except, one must find a completely different method to handle Lagrangean QFT.

In the theory of local observables the theories of von Neumann and C^*-algebras are the main tools for the investigation. In 1967 the theory of von Neumann algebras made a big step forward in Tomita's discovery of the theory of modular von Neumann algebras. In this paper I will focus my attention on results obtained by this new theory. In the theory of local observables, abbreviated QFTLO, many results have been obtained with the standard theory of von Neumann algebras. Most of them are described in the book of R. Haag.[8]

This article is structured into several sections. Each of them is centered around one concept or idea. The order of these sections does not follow some logical concept, but is done in such a manner that the number of references to succeeding sections is minimized. Each section is split into subsections. This is done in order to facilitate the search for special topics. The last section is reserved to open problems.

I.2. Assumptions of the theory of local observables

The investigations of this paper are based on the following assumptions:

In the theory of local observables one associates to every bounded open region $\mathcal{O}$ in Minkowski space $\mathbb{R}^d$ a C^*-algebra $\mathcal{A}(\mathcal{O})$. For any unbounded open set G the C^*-algebra $\mathcal{A}(G)$ is defined as the C^* inductive limit of the $\mathcal{A}(\mathcal{O})$ with $\mathcal{O}\subset G$. These algebras are subject to the following conditions:

(1) They fulfill isotony, i.e., if $\mathcal{O}_1\subset\mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1)\subset\mathcal{A}(\mathcal{O}_2)$.

(2) They fulfill locality, i.e., if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated regions then the corresponding algebras commute, i.e.,

$$A\in\mathcal{A}(0_1),\quad B\in\mathcal{A}(\mathcal{O}_2)\quad \text{implies}\quad [A,\ B]=0.$$

(3) They fulfill translational covariance, i.e., the translation group of $\mathbb{R}^d$ acts as automorphisms on $\mathcal{A}(\mathbb{R}^d)$. For every $a\in\mathbb{R}^d$ there exists an automorphism $\alpha_a\in\mathrm{Aut}\,\mathcal{A}(\mathbb{R}^d)$ with

$$\alpha_a\mathcal{A}(\mathcal{O})=\mathcal{A}(\mathcal{O}+a).$$

A representation π of $\mathcal{A}(\mathbb{R}^d)$ is called a particle representation if:
(i) π is a nondegenerate representation on a Hilbert space $\mathcal{H}$.
(ii) There exists a strongly continuous unitary representation of the translation group $a \mapsto U(a)$, such that:

(α) The spectrum of $U(a)$ is contained in the forward light-cone.

(β) The representation $U(a)$ implements the automorphism α_a, which means that for every $A \in \mathcal{A}(\mathbb{R}^d)$ one has

$$\mathrm{Ad}\, U(a)\pi(A) := U(a)\pi(A)U^*(a) = \pi(\alpha_a A).$$

(iii) A representation π is called a vacuum representation if:

(α) π is a particle representation.

(β) In $\mathcal{H}$ exists a vector Ω with $U(a)\Omega = \Omega \quad \forall a \in \mathbb{R}^d$.

In the following we will always deal with vacuum representations and we set

$$\mathcal{M}(\mathcal{O}) = \pi(\mathcal{A}(\mathcal{O}))''.$$

(γ) We require weak additivity, i.e., for every $\mathcal{O}$ there holds

$$\{\bigcup_{a \in \mathbb{R}d} \mathcal{M}(\mathcal{O}+a)\}'' = \mathcal{M}(\mathbb{R}^d).$$

(4) Very often also the covariance under the whole Poincaré group will be assumed. This means there shall exist a continuous unitary representation $U(\Lambda)$ of the Lorentz group obeying the correct relations with the translations and

$$(\alpha) \quad U(\Lambda)\Omega = \Omega,$$

$$(\beta) \quad U(\Lambda)\mathcal{M}(\mathcal{O})U(\Lambda)^* = \mathcal{M}(\Lambda\mathcal{O}).$$

For the physical interpretation of these assumptions see the book of Haag[8] or the lecture notes of Borchers.[27]

I.3. Tomita–Takesaki theory

As already mentiond this representation is mainly based on the Tomita–Takesaki theory. At the Baton Rouge conference in 1967 Tomita[28] distributed a preprint containing his theory on the standard form of von Neumann algebras. At the same time Haag, Hugenholtz, and Winnink[29] published their paper on the description of thermodynamic equilibrium states using the KMS condition. Probably Hugenholtz and Winnink have been the first realizing the similarity between certain aspects of their approach and Tomita's theory and hence the importance of this new mathematical theory for theoretical physics. (See, e.g., the thesis of Winnink.[30]) But general awareness of Tomita's theory, only by Takesaki's[31] treatment, published in the Lecture Notes in Mathematics. Since then this theory is usually called the Tomita–Takesaki theory.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Neumann algebra acting on this space with commutant $\mathcal{M}'$. A vector Ω is cyclic and separating for $\mathcal{M}$ if $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$. If these conditions are fulfilled then a modular operator Δ and a modular conjugation J is associated to the pair $(\mathcal{M}, \Omega)$ such that:
(i) Δ is self-adjoint, positive, and invertible

$$\Delta\Omega = \Omega, \quad J\Omega = \Omega.$$

(ii) The unitary group Δ^{it} defines a group of automorphisms of $\mathcal{M}$

$$\mathrm{Ad}\,\Delta^{it}\mathcal{M} = \mathcal{M} \quad \forall t \in \mathbb{R}.$$

This automorphism group will often be denoted as

$$\mathrm{Ad}\,\Delta^{it}A =: \sigma^t(A). \tag{I.3.1}$$

(iii) For every $A\in\mathcal{M}$ the vector $A\Omega$ belongs to the domain of $\Delta^{1/2}$.
(iv) The operator J is a conjugation, i.e., J is antilinear and $J=J^*=J^{-1}$, where J commutes with Δ^{it}. This implies the relation

$$\mathrm{Ad}\, J\Delta=\Delta^{-1}. \tag{I.3.2}$$

(v) J maps $\mathcal{M}$ onto its commutant

$$\mathrm{Ad}\, J\mathcal{M}=\mathcal{M}'.$$

(vi) The operators $S:=J\Delta^{1/2}$ and $S^*=J\Delta^{-1/2}$ have the property

$$SA\Omega=A^*\Omega \quad \forall A\in\mathcal{M},$$

$$S^*A'\Omega=A'^*\Omega \quad \forall A'\in\mathcal{M}'.$$

This implies that $A\Omega$, $A\in\mathcal{M}$ is in the domain of $\Delta^{1/2}$ and $B\Omega$, $B\in\mathcal{M}'$ is in the domain of $\Delta^{-1/2}$. S will be called the Tomita conjugation of $(\mathcal{M},\Omega)$.
(vii) From (iii) one concludes that for $A\in\mathcal{M}$ the vector valued function

$$t\mapsto\Delta^{it}A\Omega$$

has an analytic continuation into the strip $S(-\frac{1}{2},0):=\{z\in\mathbb{C};-\frac{1}{2}<\Im m z<0\}$. Property (vi) implies

$$\Delta^{i(t-i/2)}A\Omega=\Delta^{it}JA^*\Omega, \quad A\in\mathcal{M}. \tag{I.3.3}$$

For elements $B\in\mathcal{M}'$ Eq. (I.3.1) implies that $\Delta^{it}B\Omega$ has an analytic continuation into the strip $S(0,\frac{1}{2})$ and one gets by (vi)

$$\Delta^{i(t+i/2)}B\Omega=\Delta^{it}JB^*\Omega, \quad B\in\mathcal{M}'. \tag{I.3.3$'$}$$

(viii) Using Eq. (I.3.3) and the fact that J is a conjugation one obtains that for A, $B\in\mathcal{M}$ the function $(\Omega,B\sigma^t(A)\Omega)$ can be analytically continued into the strip $S(-1,0)$. One finds at the lower boundary the relation

$$(\Omega,B\sigma^{(t-i)}(A)\Omega)=(\Omega,\sigma^t(A)B\Omega), \quad A,B\in\mathcal{M}. \tag{I.3.4a}$$

or equivalently

$$(\Omega,B\Delta^{i(t-i)}A\Omega)=(\Omega,A\Delta^{-it}B\Omega), \quad A,B\in\mathcal{M}. \tag{I.3.4b}$$

The last two relations are called the KMS condition. They characterize the modular group uniquely. If a unitary group fulfills the KMS condition for $\mathcal{M}$ then it is the modular group of $\mathcal{M}$. (See Ref. 33 Thm. 9.2.16.)

For the proofs see Takesaki[31] or textbooks as Bratteli and Robinson[32] or Kadison and Ringrose[33] or Stratila.[34]

A central role in this theory is played by faithful normal states of von Neumann algebras. As a consequence of the Reeh–Schlieder theorem[35] we know that the vacuum state has this property for every local algebra in quantum field theory.

Faithful normal states do not exist for every von Neumann algebra. The generalization of this concept are the weights. With so called normal, faithful, semifinite weights the Tomita–Takesaki theory can be developed also (see, e.g., Haagerup[36]). The concept of weights will not be explained for the moment, but only when it has to be used. Also the mathematical results obtained by the Tomita–Takesaki theory will be mentioned when needed.

I.4. Remarks on the edge of the wedge problem

In this section we want to collect some results from the theory of analytic functions of several complex variables. All the results are given without proofs.

The theory of several complex variables is an important tool in quantum field theory and we assume familiarity with these methods. The situation appearing here (and often in other physical cases) is the edge of the wedge problem. One deals with two analytic functions $f^+(z)$ and $f^-(z)$, $z\in\mathbb{C}^n$ defined in a tube T^+ and $T^-=-T^+$, respectively. The tube T^+ is based on a convex cone $C\subset\mathbb{R}^n$ with apex at the origin and defined by

$$T(C)=T^+=\{z\in\mathbb{C}^n;z=x+iy,y\in C,x\in\mathbb{R}^n\}.$$

One assumes that $f^+(z)$ and $f^-(z)$ both have boundary values $f^+(x)$, $f^-(x)$, respectively (in the sense of distributions) and that these boundary values coincide on some open set $G\subset\mathbb{R}^n$. In this situation one knows from the edge of the wedge theorem[37] that both functions are analytic continuations of each other and are analytic also in a complex neighborhood of G.

I.4.1. Theorem: (Edge of the Wedge)
Denote by B the ball

$$B=\left\{z;\|z\|:=\left(\sum |z_i|^2\right)^{1/2}<1\right\}$$

and define $B_C^+=B\cap T(C)$ and $B_C^-=B\cap T(-C)$. Assume $f^+(z)$ and $f^-(z)$ are functions holomorphic in B_C^+ and *B_C^-, respectively, with f^+ and f^- having continuous boundary values at real points $\|x\|<1$ and assume that these boundary values coincide. Then there exists a complex neighborhood $\mathcal{N}$ of $\mathbb{R}^n\cap B$ and a function f holomorphic in $B_C^+\cup B_C^-\cup\mathcal{N}$ such that*

$$f=f^+ \ \text{on}\ B_C^+ \quad \text{and} \quad f=f^- \ \text{on}\ B_C^- .$$

In several applications one has functions depending on several real variables. One knows that one can analytically continue in one variable if the others are fixed. One would like to know conditions which imply that one can analytically continue in all variables simultaniously. An important result on this question is the Malgrange–Zerner theorem. (For details see Epstein.[38]) Since we need the result only for two variables, we will formulate it only for this situation. The generalization to more than two variables is straightforward.

I.4.2. Theorem: (Malgrange–Zerner)
Let $f(x_1,x_2)$ be a continuous function of two variables defined on $(-1,1)\times(-1,1)$. Assume for fixed x_2 the function $f(x_1,x_2)$ has an analytic continuation $f(z_1,x_2)$ holomorphic in $z_1\in D^+=\{z;|z|<1,\Im m z>0\}$, and for fixed x_1 an analytic continuation $f(x_1,z_2)$ holomorphic in $z_2\in D^+$. Assume $f(z_1,x_2)$ and $f(x_1,z_2)$ are bounded and continuous, i.e., $f(z_1,x_2)$ is a continuous function in x_2 with values in the bounded analytic functions on D^+, and the same for $f(x_1,z_2)$. Then exists a function $f(z_1,z_2)$ holomorphic in some neighborhood $\mathcal{N}\cap D^+\times D^+$, where $\mathcal{N}$ is some neighborhood of $D^+\times(-1,1)\cup(-1,1)\times D^+$. This function has boundary values on $(-1,1)\times(-1,1)$ which coincide with $f(x_1,x_2)$.

The importance of holomorphic functions of several complex variables is the following fact: Not every domain G is a natural domain in $\mathbb{C}^n$. In such a situation every function holomorphic in G can be analytically continued into a larger domain. The domain into which every function, holomorphic in G, can be analytically continued is called the envelope of holomorphy $H(G)$ of G. We will need the tube theorem, the double cone theorem, and the Jost–Lehmann–Dyson theorem. The tube theorem can be found in every text book on several complex variables.

I.4.3. Theorem: (Tube Theorem)
Let G be a connected domain $G\subset\mathbb{R}^n$ and let $T(G)=\{z\in\mathbb{C}^n;\Im m z\in G\}$. Then

$$H(T(G))=T(\mathrm{Co}\ G),$$

where Co G *denotes the convex hull of* G.

Another result of importance in QFT is the double cone theorem discovered independently by Vladimirov[39] and Borchers.[40]

I.4.4. Theorem: (Double Cone Theorem)
Let G *be a subdomain of* $\mathbb{R}^d$, *and let* $\mathcal{N}(G)$ *be some complex neighborhood of* G. *Let* $\Gamma = T(C)\cup T(-C)\cup\mathcal{N}(G)$ *and* $H(\Gamma)$ *be its envelope of holomorphy. Assume* $c,d\in G$ *such that* $d-c\in C$ *and* $c+\lambda(d-c)\in G$ *for* $0\leqslant\lambda\leqslant 1$. *Then*

$$D_{c,d}\subset H(\Gamma)\cap\mathbb{R}^n,$$

where $D_{c,d}$ *denotes the double cone* $(c+C)\cap(d-C)$.

We also need a result of Bros, Epstein, Glaser, and Stora,[41] which deals with the edge of the wedge theorem in two variables.

I.4.5. Theorem: (Bros, Epstein, Glaser, Stora)
Let T^+ *and* T^- *be tubes based on the first and third quadrant, respectively. Assume the coincidence domain is the first quadrant. If a real line* $ax_1+bx_2=c, a,b,c\in\mathbb{R}$ *intersects interior of the first quadrant, then all complex, nonreal points*

$$az_1+bz_2=c,\ \ z_1,z_2\quad \text{not both in}\ \mathbb{R}$$

belong to the envelope of holomorphy of the edge of the wedge problem.

Many results in QFT are based on the Jost–Lehmann–Dyson representation. This characterizes the envelope of holomorphy in case the cone C is the forward light cone and the coincidence domain has some special properties. Jost and Lehmann have solved a special case.[42] The general solution is due to Dyson.[43] In this proof one uses tempered distributions. But that the answer is general has first been shown by Bros, Messiah, and Stora.[44] For more details on the Jost–Lehmann–Dyson representation see Ref. 27, Sec. III.4.

I.4.6. Theorem: (Jost, Lehmann, Dyson)
Define $h(u,m)$ *to be the hyperboloid*

$$h(u,m)=\{z\in\mathbb{C}^d;(z-u)^2=m^2,u\in\mathbb{R}^d,m\in\mathbb{R}\}.$$

Let $G\subset\mathbb{R}^d$ *be a domain bounded by two spacelike hypersurfaces. The complement of the envelope of holomorphy of the edge of the wedge problem for*

$$G\cup T(V^+)\cup T(-V^+)$$

consists of the closure of the union of all real and complex points of the hyperboloids $h(u,m)$ *which do not intersect* G.

I.5. Some notations

(i) If $\mathcal{O}$ is some open domain in the Minkowski space then $\mathcal{O}'$ denotes the interior of the spacelike complement of $\mathcal{O}$.
(ii) A domain of special importance is the *wedge*. Such a domain can be characterized in two ways:
(α) First characterization: Let t, s be two perpendicular vectors in $\mathbb{R}^d$, i.e., $(t,s)=0$, such that $t^2=1$ and t belongs to the forward light-cone and $s^2=-1$ is spacelike. In this situation one defines

$$W(t,s):=\{a\in\mathbb{R}^d;|(a,t)|<-(a,s)\}. \tag{I.5.1}$$

If, for instance, t is the time direction and s is the 1-direction then this becomes $W_R=\{a;|a_0|<a_1\}$.
(β) Second characterization: Every two-plane containing a timelike direction must cut the bound-

ary of the forward light cone in two light rays. Let these light rays be described by the two lightlike vectors l_1, l_2 belonging to the forward light-cone. These vectors are different. Now define:

$$W(l_1,l_2) := \{\lambda_1 l_1 - \lambda_2 l_2 + \tilde{a}; \lambda_1, \lambda_i > 0, (\tilde{a}, l_i) = 0, i = 1,2\}. \tag{I.5.2}$$

It is easy to see that the two definitions result in the same set of wedges. The two definitions coincide if $\{t, s\}$ and $\{l_1, l_2\}$ span the same two-plane and if $s = \lambda_1 l_1 - \lambda_2 l_2$ with positive coefficients.

The opposite wedge of a wedge W is the negative of W and it is usually denoted by W'. It is obtained by replacing s by $-s$ in the first description and by interchanging the two lightlike vectors in the second description.

(iii) Given a wedge W there is exactly a one-parametric subgroup of the Lorentz boosts which maps this wedge onto itself. In the above example of the zero- and one-direction, the Lorentz transformations are the boosts in the (0,1)-plane. We will write these transformations (in case the wedge is the right wedge W_R in the (0,1)-plane) as

$$\Lambda(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{I.5.3}$$

(iv) Let $\mathcal{A}$ be a C^*-algebra and π_1, π_2 be two equally faithful representations. These representations are called *quasi-equivalent* if the isomorphism between $\pi_1(\mathcal{A})$ and $\pi_2(\mathcal{A})$ extends to an isomorphism of the associated von Neumann algebras

$$\pi_1(\mathcal{A})'' \cong \pi_2(\mathcal{A})''.$$

Two representations π_1 and π_2 of a theory of local observables are called locally normal if $\pi_1(\mathcal{A}(\mathcal{O}))$ and $\pi_2(\mathcal{A}(\mathcal{O}))$ are quasi-equivalent for every bounded open region $\mathcal{O}$.

I.6. Things not treated

It is clear that I am not able to handle all subjects of QFTLO which are not in the book of Haag. There is the reason of space, and more important, there are others who are more expert on that particular field than myself.

(i) Low dimensional QFT's:

If the dimension of the Minkowski space is two, then the set of points spacelike to the origin is no longer connected. This has for the definition of statistics the consequence that not only the permutation—but also the braid group is of importance.

It is well known that in the classical theory, the solution of the free wave equation is the sum of two functions depending only on one light-cone coordinate. A similar phenomenon appears in two-dimensional conformal QFT's. This means there exist quantum fields depending only on one of the light-cone coordinates. These are often called right- or left-movers. One can map the real line onto the circle and often one finds that such theory has an additional symmetry, namely, the rigid rotation of the circle. Such theories are usually called chiral field theories.

The braid group and the additional symmetry of chiral field theories opens a ''wonderland'' of new possibilities. Whether or not it is possible to get some important inspiration for the four-dimensional QFT from these theories will only be answered in the future.

(ii) General relativistic quantum fields:

It is a dream that one day it will be possible to combine quantum field theory with general relativity. As a first step it is probably reasonable to treat the QFT of test particles. These are theories where the quantum fields are influenced by the gravitational field (which is treated classically), but where the energy of the quantum field does not appear as a source of the gravitational field.

The main problem of this theory is the replacement of the spectrum condition. At the moment it is not clear whether or not there exist states describing a finite number of particles. At least in theories with a horizon the Hawking–Unruh effect[45,46] seems to indicate that no such states exist in this situation. Therefore, the main stream of investigation focuses on the aspect that the speed of particles should not be higher than that of light (defined by the gravitational field). These investigations use extensively the theory of wavefront sets.

(iii) Renormalization group:

For a long time the renormalization group method has been used mainly in connection with perturbation theory. This theory is designed in order to understand the physics at very low or very high energies. Not long ago Buchholz and Verch[47] were able to transcribe the renormalization group technique to QFTLO. In this scheme there are no serious obstructions, that means their method uses a sound mathematical basis. In examples they could show that the limiting theories can be different from the original theory. In some cases there is even more than one limiting theory. In my opinion this is an important new aspect of QFT which deserves one's attention. Buchholz will give a representation of this theory in the same volume.

An appendix to the references will be added containing a list of papers on the subjects not treated. This incomplete list might be a help for a start for those interested in some more details on one or more of these fields. I am obliged to K. H. Rehren and R. Verch for preparing these lists.

II. ON VON NEUMANN SUBALGEBRAS

From the axioms of QFTLO there has been extracted a large number of beautiful results. All of them are in accordance with our physical intuition. Examples are the collision theory and the theory of superselection sectors described in the book of Haag,[8] or the properties of the spectrum of the translations presented in the lecture notes by Borchers.[27]

However, up to now it is not clear how to distinguish the theories with different dynamics from each other. Since for two different theories the local nets as a whole are not isomorphic to each other, one should look (as a start) at the embedding of the algebra of one region $\mathcal{O}_1$ into the algebra of a bigger region $\mathcal{O}_2$. What is known about this question will be collected in this section.

II.1. Order by inclusion and order of modular operators

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ acting on the Hilbert space $\mathcal{H}$. Assume that both algebras have a common cyclic and separating vector Ω. Then one has $\mathcal{N}\Omega \subset \mathcal{M}\Omega$ and hence the Tomita conjugation $S_{\mathcal{M}}$ of $\mathcal{M}$ is an extension of the Tomita conjugation $S_{\mathcal{N}}$ of $\mathcal{N}$.

Dropping the index $\mathcal{M}$ of the Tomita conjugation, the operator S has the following properties (see I.3):

(*i*) S is a densely defined closed antilinear operator with domain of definition $\mathcal{D}(S)$ and $\mathcal{M}\Omega$ is a core for S.

(*ii*) $S^2 = \mathbb{1}$ on $\mathcal{D}(S)$.

(*iii*) $\Omega \in \mathcal{D}(S)$ and $S\Omega = \Omega$.

Since S is closed it has a polar decomposition $S = J\Delta^{1/2}$. The modular operator Δ is invertible and J is a conjugation. Equation (I.3.2) reads

$$J\Delta J = \Delta^{-1}, \quad J = J^* = J^{-1}.$$

These properties follow from the condition $S^2 = \mathbb{1}$. (See, e.g, Bratteli and Robinson, Ref. 32, Prop. 2.5.11.)

Usually a Tomita conjugation will be a densely defined unbounded operator. The best way of describing an unbounded operator X is by its graph. This is the set $\{[\psi, X\psi] \in \mathcal{H} \oplus \mathcal{H}; \psi \in \mathcal{D}(X)\}$. If the operator is closed then the graph of X is a closed linear manifold of $\mathcal{H} \oplus \mathcal{H}$.

Therefore, it can be characterized by the projection $P(X)$ onto the graph. The projection $P(X)$ can be written as a two by two matrix $p_{i,k}, i,k=1,2$ of operators on $\mathcal{H}$ fulfilling

$$p^*_{i,k}=p_{k,i}, \quad \sum_j p_{i,j}p_{j,k}=p_{i,k}. \tag{II.1.1}$$

If the operator X is antilinear then $p_{1,2}$ and $p_{2,1}$ are antilinear also. The domain of X is given by $\mathcal{D}(X)=p_{1,1}\psi+p_{1,2}\varphi,\ \psi,\ \varphi\in\mathcal{H}$ and its range $p_{2,1}\psi+p_{2,2}\varphi$. Therefore, one gets $p_{2,1}=Xp_{1,1}$ and $p_{2,2}=Xp_{1,2}$. From these relations and from Eq. (II.1.1) one can easily express $p_{i,k}$ in terms of X. Of interest is $p_{1,1}$ which has the form

$$p_{1,1}=(1+X^*X)^{-1}. \tag{II.1.2}$$

If X_1 is an extension of X then the graph of X is a subset of the graph of X_1. This implies in particular $P(X_1)\geqslant P(X)$. If E_1 is the projection onto the first Hilbert space then we get $E_1P(X_1)E_1\geqslant E_1P(X)E_1$, and with Eq. (II.1.2)

$$(1+X_1^*X_1)^{-1}\geqslant(1+X^*X)^{-1}.$$

The matrix representing the projection onto the graph has been introduced by Stone.[48] It is often called the Stone- or characteristic matrix of the operator. More details can be found in Nussbaum.[49]

If the operator X is antilinear, then one has to replace the second Hilbert space by the conjugate complex Hilbert space. In this case the operators $p_{1,2}$ and $p_{2,1}$ are antilinear. With this change one can deal with the graph in the same manner as if the operator would be linear. If one feels uneasy with this procedure one can fix a conjugation K on $\mathcal{H}$ and multiply the antilinear operator X by K. Since KX is a linear operator the usual arguments can be applied. In the case $\mathcal{N}\subset\mathcal{M}$ one obtains

$$(\mathbb{1}+\Delta_{\mathcal{N}})^{-1}\leqslant(\mathbb{1}+\Delta_{\mathcal{M}})^{-1},$$

or

$$\Delta_{\mathcal{N}}\geqslant\Delta_{\mathcal{M}}. \tag{II.1.3}$$

This implies in particular that the domain of $\Delta_{\mathcal{N}}^{1/2}$ is contained in the domain of $\Delta_{\mathcal{M}}^{1/2}$. Since the domain of $\Delta_{\mathcal{N}}^{1/2}$ is the range of $\Delta_{\mathcal{N}}^{-1/2}$, the expression

$$\Delta_{\mathcal{N}}^{-1/2}\Delta_{\mathcal{M}}\Delta_{\mathcal{N}}^{-1/2}$$

is a densely defined bounded and hence a closable operator, and one gets

$$\text{closure } \Delta_{\mathcal{N}}^{-1/2}\Delta_{\mathcal{M}}\Delta_{\mathcal{N}}^{-1/2}\leqslant\mathbb{1}. \tag{II.1.4}$$

As an application of this discussion we obtain:

II.1.1. Theorem:
Let $\mathcal{M}_i$ be an increasing family of von Neumann algebras, i.e., $\mathcal{M}_i\subset\mathcal{M}_{i+1}$. Let

$$\mathcal{M}=\{\cup_i\mathcal{M}_i\}''.$$

Assume Ω is cyclic and separating for $\mathcal{M}_i$ and for $\mathcal{M}$. Denote by (Δ_i, J_i) and (Δ, J) the modular operators and modular conjugations of $\mathcal{M}_i$ and $\mathcal{M}$*, respectively. Then Δ_i converges to Δ in the resolvent sense and J_i converges strongly to J.*

A similar result holds for decreasing sequences. This result has first been obtained by D'Antoni, Doplicher, Fredenhagen, and Longo.[50]

II.2. The first fundamental relation

There are other aspects of the relation (II.1.4) which give some more information. Since the result is needed several times, I quote it as in the report at the IAMP conference in Paris.[51]

Theorem A:
Let $\mathcal{M},\mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω. Denote the modular operators and conjugations by $\Delta_{\mathcal{M}}, J_{\mathcal{M}}$ and $\Delta_{\mathcal{N}}, J_{\mathcal{N}}$, respectively. Let $V\in\mathcal{B}(\mathcal{H})$ be a unitary operator with
(*i*) $V\Omega=\Omega$, *and*
(*ii*) $\mathrm{Ad}\, V\mathcal{N}\subset\mathcal{M}$,
then the function $V(t):=\Delta_{\mathcal{M}}^{-it}V\Delta_{\mathcal{N}}^{it}$ *has the properties*:
(*a*) $V(t)$ *is* $*$*-strong continuous in* $t\in\mathbb{R}$.
(*b*) $V(t)$ *possesses an analytic extension into the strip* $S(0,\frac{1}{2})=\{t\in\mathbb{C};0<\Im m\, t<\frac{1}{2}\}$ *as holomorphic function with values in the normed space* $\mathcal{B}(\mathcal{H})$.
(*c*) *In this strip we have the estimate*

$$\|V(\tau)\|\leqslant 1. \tag{II.2.1}$$

(*d*) $V(\tau)$ *has boundary values at* $\Im m\,\tau=0$ *and at* $\Im m\,\tau=\frac{1}{2}$ *in the* $*$*-strong topology*.
(*e*) *On the upper boundary the value is given by*

$$V(t+i\tfrac{1}{2})=J_{\mathcal{M}}V(t)J_{\mathcal{N}}, \tag{II.2.2}$$

hence by (a) also this function is $*$*-strong continuous in t.*

II.2.1. Remarks:
(*i*) With $\mathcal{N}'\supset V^*\mathcal{M}'V$ one obtains

$$V^*(t)=V(-t)^*.$$

Notice that the function $V(\overline{z})^*$ is again an analytic function holomorphic in $S(-\frac{1}{2},0)=\{t\in\mathbb{C};-\frac{1}{2}<\Im m\,t<0\}$. Therefore, the last relation reads in the complex

$$V^*(z)=V(-\overline{z})^*. \tag{II.2.3}$$

Idea of the proof: The continuity properties are shown by standard methods. The interesting parts are the analyticity properties. Let us identify for a moment $V\mathcal{N}V^*$ with $\mathcal{P}\subset\mathcal{M}$. Since $A\to A^{\alpha}, 0\leqslant\alpha\leqslant 1$ is an operator monotone function on positive operators (see, e.g., Pedersen, Ref. 52, Prop. 1.5.8.) we obtain from Eq. (II.1.3)

$$\Delta_{\mathcal{P}}^{\alpha}\geqslant\Delta_{\mathcal{M}}^{\alpha},\quad 0\leqslant\alpha\leqslant 1$$

and hence

$$\text{closure }\{\Delta_{\mathcal{P}}^{-\alpha}\Delta_{\mathcal{M}}^{2\alpha}\Delta_{\mathcal{P}}^{-\alpha}\}\leqslant\mathbb{1},\quad 0\leqslant\alpha\leqslant\tfrac{1}{2}.$$

This implies

$$\|\text{closure }\Delta_{\mathcal{M}}^{\alpha}\Delta_{\mathcal{P}}^{-\alpha}\|\leqslant 1,\quad 0\leqslant\alpha\leqslant\tfrac{1}{2}.$$

From this one easily derives the statements of the theorem except Eq. (II.2.2). This relation is obtained by applying $V(t)$ to vectors of the form $A'\Omega$ with $A'\in\mathcal{N}'$ and then using the properties of the operators $S_{\mathcal{N}}^*$ and $S_{\mathcal{M}}$. □

A proof of the last theorem can be found in Ref. 51. A short version of the proof is due to Florig.[53]

II.3. Characteristic functions and von Neumann subalgebras

In the special case $V = \mathbb{1}$ one uses the following notations:

II.3.1. Definition:
Assume $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and Ω is cyclic and separating for both algebras. We set

$$D_{\mathcal{M},\mathcal{N}}(t) = \Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it}. \tag{II.3.1}$$

The function $D_{\mathcal{M},\mathcal{N}}(t)$ satisfies the following relations:

II.3.2. Lemma:
For the function $D(t) := D_{\mathcal{M},\mathcal{N}}(t)$ *defined in Eq.* (II.3.1) *the following holds*:
(1) $D(t)$ *is unitary and strongly continuous in t. Moreover* $D(0) = \mathbb{1}$.
(2) $D(t)\Omega = \Omega$, *for all* $t \in \mathbb{R}$.
(3) $D(t)$ *has a bounded analytic continuation into the strip* $S(0,\frac{1}{2})$ *and has strongly continuous boundary values at* $\Im m t = 0$ *and* $\Im m t = \frac{1}{2}$.
(4) $D(t + \mathrm{i}/2)$ *is unitary and strongly continuous in t.*
(5) $D(t)$ *fulfills the following cocycle relation*:

$$D(s+t) = \sigma_{\mathcal{M}}^{-t}(D(s))D(t). \tag{II.3.2}$$

(6) *For complex values of the arguments one finds*

$$D(t+\mathrm{i}/2)^* J_{\mathcal{M}} D(t) = D(t)^* J_{\mathcal{M}} D(t+\mathrm{i}/2)$$

is independent of t.
(7) $\mathrm{Ad}\{D(t)D(\mathrm{i}/2)^*\}\mathcal{M} \subset \mathcal{M}$ *holds for all* $t \in \mathbb{R}$.

All these properties follow from the definition of $D_{\mathcal{M},\mathcal{N}}(t)$.
Notice that the properties of $D(t)$ described in Lemma II.3.2 do not contain any reference to the algebra $\mathcal{N}$. Therefore, we introduce the following notation:

II.3.3. Definition:
Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with a cyclic and separating vector Ω.
1. By $\mathcal{S}ub(\mathcal{M})$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ which have Ω as cyclic vector.
2. An operator-valued function $D(t)$ which fulfills the properties (1)–(7) of Lemma II.3.2 will be called a characteristic function of $\mathcal{M}$.
3. The set of characteristic functions belonging to $\mathcal{M}$ will be denoted by $\mathcal{C}har(\mathcal{M})$.

II.3.4. Theorem:
Let $\mathcal{M}$ *be a von Neumann algebra with a cyclic and separating vector* Ω. *Then to every characteristic function* $D(t)$ *of* $\mathcal{M}$ *exists a von Neumann subalgebra* $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$ *such that* $D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}$. *The correspondence*

$$\mathcal{S}ub(\mathcal{M}) \Leftrightarrow \mathcal{C}har(\mathcal{M})$$

is one to one.

Idea of the proof: First one defines

$$U(t) = \Delta_{\mathcal{M}}^{it} D(t) \quad \text{and} \quad K = J_{\mathcal{M}} D(\mathrm{i}/2), \tag{II.3.3}$$

and shows $K = K^* = K^{-1}$ and $U(t)K = KU(t)$. For obtaining this one uses the cocycle relation (II.3.2) and Lemma II.3.6 (6). Next one wants to construct the von Neumann algebra $\mathcal{N}$ or better the algebra $\mathcal{N}'$ which we define

$$\mathcal{N}' = \bigvee_{t\in\mathbb{R}} \mathrm{Ad}\, U(t)\mathcal{M}'. \tag{II.3.4}$$

This algebra is invariant under $\mathrm{Ad}\, U(t)$. ($\sigma^t := \mathrm{Ad}\, U(t)$.) First one shows that $KJ_{\mathcal{M}}\mathcal{M}J_{\mathcal{M}}K$ commutes with $\sigma^t(\mathcal{M}')$ and hence with $\mathcal{N}'$. Since $KJ_{\mathcal{M}}$ is unitary and maps Ω onto itself it follows that Ω is cyclic for $\mathcal{N}$. The demonstration of this needs property (7) of Lemma II.3.2 and the cocycle relation. Consequences are the relations

$$[\sigma^{t_1}(A_1'), K\sigma^{t_2}(A_2')K] = 0, \quad A_1', A_2' \in \mathcal{M}'; t_1, t_2 \in \mathbb{R}. \tag{II.3.5}$$

If one writes $U(t) = \Delta^{it}$, one shows for $A' \in \mathcal{M}'$

$$\Delta^{-1/2}\sigma^t(A')\Omega = K\sigma^t(A'^*)\Omega. \tag{II.3.6}$$

The proof of this relation needs the analyticity property of $D(t)$. In order to extend this relation to all of $\mathcal{N}'$ one defines by integration the elements $\sigma^f(A')$ for $A' \in \mathcal{M}'$ and $f \in \mathcal{L}^1(\mathbb{R})$ entire analytic. For such elements $\mathrm{Ad}\, \Delta^{it}\sigma^f(A')$ is an entire analytic operator valued function. With such elements one can extend Eq. (II.3.6) to products and by density arguments to arbitrary elements in $\mathcal{N}'$. With this result it is easy to show that $U(-t)$ fulfills the KMS-condition for $\mathcal{N}'$ which implies that it is the modular group for that algebra. It remains to show the uniqueness of the mapping. If $D_1(t)$ and $D_2(t)$ are different then follows from the construction used above that the algebras are different. Conversely assume $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{S}ub(\mathcal{M})$ and $D_1(t)$ and $D_2(t)$ coincide. Then Δ_1^{it} and Δ_2^{it} coincide and also J_1 and J_2 coincide by Eq. (II.3.3). This implies that $\mathcal{N}_1 \cap \mathcal{N}_2$ is invariant under $\Delta_1^{it} = \Delta_2^{it}$. Since $J_1\mathcal{M}'J_1$ is contained in the intersection it follows that Ω is cyclic for $\mathcal{N}_1 \cap \mathcal{N}_2$. Hence $\mathcal{N}_1$ and also $\mathcal{N}_2$ coincide with $\mathcal{N}_1 \cap \mathcal{N}_2$. (See Ref. 33, Thm. 9.2.36.) Hence the map $\mathcal{S}ub(\mathcal{M}) \Leftrightarrow \mathcal{C}har(\mathcal{M})$ is one to one.

The content of this subsection is taken from Ref. 54.

II.4. The second fundamental relation

There is a second fundamental relation which has to be used several times also. A special case appeared first in Ref. 55. The present formulation is taken from Ref. 51 and this proof is due to Florig.[53] It uses only functions of one variable and not of two variables as in the original demonstration.

Theorem B:
Let $\mathcal{M}$, $\mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω. Let $W(s) \in \mathcal{B}(\mathcal{H})$ be an operator family fulfilling the following requirements with respect to the triple $(\mathcal{M},\mathcal{N},\Omega)$.
(i) For $s \in \mathbb{R}$ the operators $W(s)$ are unitary and strongly continuous and fulfill the equation $W(s)\Omega = \Omega$.
(ii) The function $W(s)$ possesses an analytic continuation into the strip $S(0,\frac{1}{2})$ with strongly continuous boundary values.
(iii) The operators $W(i/2+t)$ are again unitary.
(iv) The function $W(\sigma)$ is bounded, hence $\|W(\sigma)\| \leq 1$.
(v) For $t \in \mathbb{R}$ one has $W(t)\mathcal{N}W(t)^* \subset \mathcal{M}$ and $W(i/2+t)\mathcal{N}'W(i/2+t)^* \subset \mathcal{M}'$.
In this situation the modular operator and the transformations $W(s)$ fulfill the following transformation rules:

$$\Delta_{\mathcal{M}}^{it} W(s)\Delta_{\mathcal{N}}^{-it} = W(s-t),$$

$$J_{\mathcal{M}}W(s)J_{\mathcal{N}} = W(i/2+s).$$

II.4.1. Remark:
In some applications one has to face the situation that $W(t+\mathrm{i}/2)$ has eventually a discontinuity at

one point, but all other properties remain valid. Such singularity is harmless. The reason is as follows: The proof of Theorem B is based on the continuation across a line, applied to matrix elements of the operator valued function

$$(t,s)\mapsto\Delta_{\mathcal{M}}^{it}W(s+t)\Delta_{\mathcal{N}}^{-it}. \tag{II.4.1}$$

These matrix elements have bounded analytic continuations, which are continuous at the boundary of their domain with the possible exception of one point with $\Im m\, t = \mathrm{i}/2$. By the dominated convergence theorem and the boundedness of Eq. (II.4.1), this piece-wise continuity is sufficient to ensure coincidence of boundary values in the sense of distributions. The edge-of-the-wedge theorem, Thm. I.4.1, then implies analyticity in the coincidence region, so continuity in the exceptional point holds *a fortiori*.

Proof: Choose $A\in\mathcal{N}$ and $B\in\mathcal{M}'$ and define for fixed s the two functions of the variable t:

$$F^+(t)=(\Omega,B\Delta_{\mathcal{M}}^{it}W(s+t)\Delta_{\mathcal{N}}^{-it}A\Omega),$$

$$F^-(t)=(\Omega,A\Delta_{\mathcal{N}}^{it}W^*(s+t)\Delta_{\mathcal{M}}^{-it}B\Omega).$$

Since $B\in\mathcal{M}'$ and $A\in\mathcal{N}$ and since $W(t)$ has a bounded analytic extension into the strip $S(0,\frac{1}{2})$, also the two functions have bounded extensions, $F^+(t)$ into the strip $S(0,\frac{1}{2})$ and $F^-(t)$ into the strip $S(-\frac{1}{2},0)$. Using modular theory one can compute the functions $F^+(t+\mathrm{i}/2)$ and $F^-(t-\mathrm{i}/2)$. By the assumption about the mapping property of $W(s+t)$ and of $W(s+t+\mathrm{i}/2)$ we obtain:

$$F^+(t)=F^-(t),\quad\text{and}\quad F^+(t+\mathrm{i}/2)=F^-(t-\mathrm{i}/2).$$

By these coincidences we obtain a periodic entire analytic function. Since this function is bounded by $\max\{\|B^*\Omega\|\,\|A\Omega\|,\|A^*\Omega\|\,\|B\Omega\|\}$ it is constant. This implies

$$(\Omega,B\Delta_{\mathcal{M}}^{\mathrm{it}}W(s+t)\Delta_{\mathcal{N}}^{-\mathrm{it}}A\Omega)=(\Omega,BW(s)A\Omega).$$

Since Ω is cyclic for $\mathcal{N}$ and for $\mathcal{M}'$ follows the first statement of the theorem. The second statement is the same as Eq. (II.2.2). □

II.5. Half-sided translations

From the general theory of von Neumann subalgebras described in Subsec. II.3 we turn to special cases. We start with half-sided translations.

II.5.1. Definition:
Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector $\Omega\in\mathcal{H}$.
1. $\mathcal{H}str(\mathcal{M})^+$ denotes the set of one-parametric continuous unitary groups $U(t)$, $t\in\mathbb{R}$ with the properties:

α. $U(t)$ has a positive generator, i.e., we can write

$$U(t)=\exp\{iHt\},\quad\text{with}\quad H\geqslant 0.$$

β. $U(t)\Omega=\Omega\,\forall t\in\mathbb{R}$.
γ. $\mathrm{Ad}\,U(t)\mathcal{M}\subset\mathcal{M}$ for all $t\geqslant 0$.

2. $\mathcal{H}str(\mathcal{M})^-$ denotes the set of one-parametric continuous unitary groups $U(t)$, $t\in\mathbb{R}$ with γ replaced by

γ'. $\mathrm{Ad}\,U(t)\mathcal{M}\subset\mathcal{M}$ for all $t\leqslant 0$.

We call the groups belonging to $\mathcal{H}str(\mathcal{M})^\pm$ $\pm$half-sided translations associated with $\mathcal{M}$.

In the definition of the +half-sided translations it is not possible to replace $\mathbb{R}^+$ by $\mathbb{R}$ because

$$\mathrm{Ad}\,U(t)\mathcal{M}\subset\mathcal{M}\quad\forall t$$

implies together with the positivity of the spectrum and the invariance of the vacuum $U(t)=\mathbb{1}$ for all $t\in\mathbb{R}$.

An example where half-sided translations appear, is the algebra of the wedge $\mathcal{M}(W)$. If $W=W(l_1,l_2)$, then the translations along the direction l_1 fulfill the assumptions of +half-sided translations and those along the l_2 direction the assumptions of −half-sided translations.

II.5.2. Theorem:
Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω and let $U(t)\in\mathcal{H}str(\mathcal{M})^+$. Then holds:

$$\Delta^{it}U(s)\Delta^{-it}=U(e^{-\pi t}s),$$

$$JU(s)J=U(-s).$$

This theorem appeared first in Ref. 55. The following proof is based on Theorem B.

Proof: If $U(a)$ fulfills the assumptions of the theorem then it has an analytic continuation into the upper half plane. By assumption $U(a)$ maps $\mathcal{M}$ into itself for positive arguments and hence $U(a)$ maps $\mathcal{M}'$ into itself for negative arguments. Therefore, we can apply Theorem B to the family $W(s)=U(e^{2\pi s})$ and obtain together with the analyticity of $U(a)$

$$\mathrm{Ad}\,\Delta^{it}U(e^{2\pi s})=U(e^{2\pi(s-t)}),$$

$$\mathrm{Ad}\,\Delta^{it}U(a)=U(e^{-2\pi t}a),$$

$$\mathrm{Ad}\,JU(a)=U(-a).$$

This shows the theorem.

II.5.3. Remarks:
(i) If $U(t)\in\mathcal{H}str(\mathcal{M})^-$ then one obtains the relations

$$\Delta^{it}U(s)\Delta^{-it}=U(e^{2\pi t}s),$$

$$JU(s)J=U(-s).$$

(ii) For a wedge $W(l_1,l_2)$ the two lightlike directions span the characteristic two-plane of the wedge. If $\mathbf{x}$ is in this plane then one finds the transformation formula

$$\Delta^{it}U(\mathbf{x})\Delta^{-it}=U(\Lambda(t)\mathbf{x}),$$

where $\Lambda(t)$ are the Lorentz boosts of the wedge described in Eq. (I.5.3).
(iii) Let $U(t)\in\mathcal{H}str(\mathcal{M})^+$ and define $\mathcal{N}=\Delta^{i1}\mathcal{M}\Delta^{-i1}$ then one finds by the last theorem

$$\Delta^{it}\mathcal{N}\Delta^{-it}\subset\mathcal{N}\quad\text{for}\quad t\leqslant 0.$$

II.6. Half-sided modular inclusions

The last point of Remark II.5.3 led Wiesbrock[56,57] to introduce the concept of half-sided modular inclusions.

II.6.1. Definition:
Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector $\Omega\in\mathcal{H}$. The modular operator and conjugation of this pair will be denoted by Δ and J.

1. By $\mathcal{H}smi(\mathcal{M})^-$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ with the properties:
   α. Ω is cyclic for $\mathcal{N}$. It is also separating for $\mathcal{N}$ since $\mathcal{N}\subset\mathcal{M}$.
   β. $\Delta^{it}\mathcal{N}\Delta^{-it}:=\mathrm{Ad}\,\Delta^{it}\mathcal{N}\subset\mathcal{N}$ for $t\leqslant 0$.
2. By $\mathcal{H}smi(\mathcal{M})^+$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ with the properties:
   α. Ω is cyclic for $\mathcal{N}$. It is also separating for $\mathcal{N}$ since $\mathcal{N}\subset\mathcal{M}$.

β. $\Delta^{\mathrm{it}}\mathcal{N}\Delta^{-\mathrm{it}} := \mathrm{Ad}\,\Delta^{\mathrm{it}}\mathcal{N} \subset \mathcal{N}$ for $t \geqslant 0$.
The elements of $\mathcal{H}smi(\mathcal{M})^{\mp}$ will be called the von Neumann subalgebras fulfilling the condition of $\mp$half-sided modular inclusion.

It should be remarked, that one cannot replace $\mathbb{R}^{-}$ by $\mathbb{R}$ because

$$\mathrm{Ad}\,\Delta^{\mathrm{it}}\mathcal{N} \subset \mathcal{N} \quad \forall t$$

implies $\mathcal{N} = \mathcal{M}$. The principle of half-sided modular inclusion is closely related to the half-sided translations by the following result:

II.6.2. Theorem:
Let $\mathcal{N} \in \mathcal{H}smi(\mathcal{M})^{-}$. Then there exists a group $U(t) \in \mathcal{H}str(\mathcal{M})^{+}$ such that the equation

$$\mathcal{N} = \mathrm{Ad}\,U(1)\mathcal{M}$$

holds.

Theorem II.6.2 is in some sense the converse of Thm. II.5.2. In some cases where one can compute the modular group one can find subalgebras fulfilling the conditions of half-sided modular inclusion. In these cases the corresponding half-sided translations are known only if they are geometric groups. But this is not always the case.

Idea of the proof: If the theorem is true and $\mathcal{N} = U(1)\mathcal{M}U(-1)$ then one has $\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it} = \Delta_{\mathcal{M}}^{-it}U(1)\Delta_{\mathcal{M}}^{it}U(-1) = U(e^{2\pi t}-1)$. Therefore, one has to show that the product $\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it} := D(t)$ commutes for different values of the arguments. For this one uses Thm. B again. By Thm. A $D(t)$ has an analytic continuation into the strip $S(0,\frac{1}{2})$. On both boundaries the expression is unitary. By assumption of the modular inclusion one obtains:

$$D(t)\mathcal{N}D(t)^* \subset \mathcal{N}, \quad \text{for} \quad t \geqslant 0,$$

$$D(t)\mathcal{N}'D(t)^* \subset \mathcal{N}', \quad \text{for} \quad t \leqslant 0,$$

$$D(i/2+t)\mathcal{N}'D(i/2+t)^* \subset \mathcal{N}', \quad \text{for} \quad t \in \mathbb{R}.$$

The last statements follow from $D(i/2+t) = J_{\mathcal{M}}D(t)J_{\mathcal{N}}$. $J_{\mathcal{N}}$ maps $\mathcal{N}'$ onto $\mathcal{N}, D(t)$ maps this into $\mathcal{M}$ and finally $J_{\mathcal{M}}$ maps this into $\mathcal{M}' \subset \mathcal{N}'$. Consequently one can apply Thm. B to the expression

$$W(s) = D\left(\frac{1}{2\pi}\log(e^{2\pi s}+1)\right).$$

If we set $U(e^{2\pi t}-1) = D(t)$ then we obtain

$$U(e^{2\pi t}-1)U(e^{2\pi x}-1) = U(e^{2\pi x}+e^{2\pi t}-2).$$

This shows that $U(a)$ is additive for positive arguments and by analytic continuation it follows that it is an additive unitary group with positive generator. It remains to show that $\mathcal{N}$ is of the form $U(1)\mathcal{M}U(-1)$. To this end we introduce:

II.6.3. Definition:
Let $\mathcal{N}$ be a $-$modular inclusion then we set

$$\mathcal{N}(e^{-2\pi t}) = \Delta_{\mathcal{M}}^{it}\mathcal{N}\Delta_{\mathcal{M}}^{-it},$$

$$\mathcal{N}(-e^{-2\pi t}) = \{\Delta_{\mathcal{M}}^{it}J_{\mathcal{M}}\mathcal{N}J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-it}\}'.$$

$$\mathcal{N}(0) = \{\bigcup_t \mathcal{N}(e^{-2\pi t})\}''.$$

With this one finds

II.6.4. Lemma:
The von Neumann algebras $\mathcal{N}(t)$, defined above, fulfill the following relations:

$$t_1 < t_2 \quad \text{implies} \quad \mathcal{N}(t_1) \supset \mathcal{N}(t_2),$$

$$\mathcal{N}(0) = \mathcal{M}.$$

The first line follows from the definition of half-sided modular inclusions. The algebra $\mathcal{N}(0)$ is a subalgebra of $\mathcal{M}$ which is invariant under the modular group of $\mathcal{M}$, has Ω as cyclic vector and hence coincides with $\mathcal{M}$. (See. Ref. 33, Thm. 9.2.36.)

Proof of the theorem, continuation: From the observation that $U(a)$ is a continuous group it follows that the family $\mathcal{N}(t)$ is also continuous at zero. Hence we obtain

$$\mathcal{M} = U(-1)\mathcal{N}U(1).$$

This shows the theorem.

We end this subsection with some uniqueness result which is taken from Ref. 58.

II.6.5. Theorem:
Let $\mathcal{M}_a$ and $\mathcal{N}_a$, $a \in \mathbb{R}$ be two families of von Neumann algebras on the Hilbert spaces $\mathcal{H}_m, \mathcal{H}_n$ with the cyclic and separating vector Ω_m, Ω_n, respectively. Assume there are continuous unitary one-parametric groups $U_{\mathcal{M}}(a), U_{\mathcal{N}}(a)$ both fulfill spectrum condition and leave Ω_m, Ω_n unchanged. Assume

$$\mathcal{M}_a = U_{\mathcal{M}}(a)\mathcal{M}_0 U_{\mathcal{M}}(-a), \quad \mathcal{N}_a = U_{\mathcal{N}}(a)\mathcal{N}_0 U_{\mathcal{N}}(-a).$$

Let moreover

$$\mathcal{M}_a \subset \mathcal{M}_b, \quad \mathcal{N}_a \subset \mathcal{N}_b \quad \text{for} \quad a > b.$$

If there exists a unitary map W with $W\mathcal{H}_n = \mathcal{H}_m$ and $W\Omega_n = \Omega_m$ and in addition

$$\mathcal{M}_0 = W\mathcal{N}_0 W^* \quad \text{and} \quad \mathcal{M}_1 = W\mathcal{N}_1 W^*,$$

then follows

$$\mathcal{M}_a = W\mathcal{N}_a W^* \quad \forall \;\; a \in \mathbb{R},$$

$$U_{\mathcal{M}}(a) = W U_{\mathcal{N}}(a) W^*.$$

The same is true if we require that $\mathcal{M}_0$ and $\mathcal{M}_1$ as well as $\mathcal{N}_0$ and $\mathcal{N}_1$ both fulfill modular inclusion for negative arguments of the modular groups.

The proof follows from Thm. II.5.2 and the uniqueness of the modular groups.

II.7. Remarks, additions, and problems

(I) For the definition of half-sided translations one has used that the group $U(s)$ maps the cyclic and separating vector onto itself and that $U(s)$ has a positive generator. From this one concluded Thm. II.5.2. The arguments can be reversed and one finds

II.7.1. Theorem:
Let $U(s)$ be a continuous unitary group fulfilling $U(s)\mathcal{M}U(-s) \subset \mathcal{M}$ for $s \geqslant 0$. Then any two of the three conditions imply the third

a. $U(s) = e^{iHs}$ *with* $H \geqslant 0$.
b. $U(s)\Omega = \Omega$ *for all* $s \in \mathbb{R}$.
c. $\mathrm{Ad}\,\Delta^{it}(U(s)) = U(e^{-2\pi t}s)$,
$JU(s)J = U(-s)$.

The implication $b+c \rightarrow a$ has been shown by Wiesbrock[59] and $a+c \rightarrow b$ can be found in Ref. 60.

II.7.2. Remark: The conditions a, b, and c of Thm. II.7.1 do not imply the relation $\mathrm{Ad}\, U(s)\mathcal{M} \subset \mathcal{M}$ for $s \geqslant 0$. This is due to the fact that the modular group Δ^{it} does not determine the algebra $\mathcal{M}$. But if we know $\mathrm{Ad}\, U(s_0)\mathcal{M} \subset \mathcal{M}$ for one $s_0 \neq 0$ then one finds $s_0 > 0$ and $\mathrm{Ad}\, U(s)\mathcal{M} \subset \mathcal{M}$ for all $s > 0$. The first line of c implies the inclusion for a half-line, and the conditions a and b imply, together with the proof of Thm. II.5.2, that this is the positive half-line.

(II) Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Assume there exists a unitary group $U^+(x^+) \in \mathcal{H}str(\mathcal{M})^+$ and a unitary group $U^-(x^-) \in \mathcal{H}str(\mathcal{M})^-$. If these groups commute, then one can construct a two-dimensional theory, which eventually does not fulfill the weak additivity property.

We set:

(α) $U(\mathbf{x}) = U^+(x^+)U^-(x^-)$ where $\mathbf{x} \in \mathbb{R}^2$ and x^+, x^- are the light-cone coordinates. This $U(\mathbf{x})$ fulfills the spectrum condition since U^+ and U^- are half-sided translations.

(β) $\mathcal{M}(W_R) = \mathcal{M}$ and $\mathcal{M}(W_L) = \mathcal{M}'$. The algebras of the shifted wedges are defined by the translations: $\mathcal{M}(W_R + \mathbf{x}) = \mathrm{Ad}\, U(\mathbf{x})\mathcal{M}(W_R)$ and $\mathcal{M}(W_L + \mathbf{x}) = \mathrm{Ad}\, U(\mathbf{x})\mathcal{M}(W_L)$.

(γ) Notice that in the two-dimensional Minkowski space a double cone is the intersection of a shifted right-wedge with a shifted left-wedge. For $\mathbf{a} - \mathbf{b} \in W_R$ we put $D_{b,a} = (W_R + \mathbf{b}) \cap (W_L + \mathbf{a})$ and

$$\mathcal{M}(D_{b,a}) = \mathcal{M}(W_R + \mathbf{b}) \cap \mathcal{M}(W_L + \mathbf{a}).$$

It is easy to check that this defines a Poincaré covariant net on the two-dimensional Minkowski space. We only do not know whether or not Ω is cyclic for $\mathcal{M}(D_{b,a})$.

II.7.3. Problem: Can one characterize those algebras $\mathcal{M}$ which fulfill the assumption described under (II) and for which Ω is also cyclic for $\mathcal{M}(D_{b,a})$?

(III) The space $\mathcal{C}har(\mathcal{M})$ can easily be furnished with a topology.

II.7.4. Definition:

Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. We introduce on $\mathcal{C}har(\mathcal{M})$ the topology τ of simultaneous*-strong convergence of $D_\alpha(t)$ and $D_\alpha(t + \mathrm{i}/2)$, and this uniformly on every compact K of the real line. The neighborhoods of an element $D(t)$ are given by

$$\begin{aligned} U(\psi_1, \ldots, \psi_n, K, D(t)) = \{ D'(t) \in \mathcal{C}har(\mathcal{M}); & \|(D(t) - D'(t))\psi_i\| + \|(D(t)^* - D'(t)^*)\psi\| \\ & + \|(D(t+\mathrm{i}/2) - D'(t+\mathrm{i}/2))\psi\| + \|(D(t+\mathrm{i}/2)^* - D'(t+\mathrm{i}/2)^*)\psi\| \leqslant 1, \\ & i = 1, \ldots, n; t \in K \}. \end{aligned}$$

With this topology one obtains:

II.7.5. Theorem:

The space $\mathcal{C}har(\mathcal{M})$ is τ complete.

For details see Ref. 54.

(IV) Using the modular automorphisms of $\mathcal{M}$ one sees that $\mathcal{S}ub(\mathcal{M})$ contains a continuous family of different elements if it contains a nontrivial element. With help of the Longo endomorphism one can construct a decreasing family (by inclusion) of elements. (For $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$ the Longo endomorphism applied to $\mathcal{N}$ is $\mathrm{Ad}\, (J_{\mathcal{N}} J_{\mathcal{M}})\mathcal{N}$.)

If $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$, then there is a natural injection of $\mathcal{S}ub(\mathcal{N})$ into $\mathcal{S}ub(\mathcal{M})$. Hence if $\mathcal{S}ub(\mathcal{M})$ is nontrivial it must have a rich structure.

II.7.6. Problems: (α) Since finite algebras have a trace it follows that the set $\mathcal{S}ub(\mathcal{M})$ consists of only one point, namely, $\mathcal{M}$ itself. That this is not the case for local algebras has first been shown by Kadison[61] and by Guenin and Misra.[62] If the von Neumann algebra is infinite, does then $\mathcal{S}ub(\mathcal{M})$ contain nontrivial points?

(β) The definition of $\mathcal{S}ub(\mathcal{M})$ (Def. II.3.3) depends on the cyclic and separating vector Ω. If Ω

and Ψ are two different cyclic and separating vectors of $\mathcal{M}$, does this imply that $\mathcal{C}har_{\Omega}(\mathcal{M})$ and $\mathcal{C}har_{\Psi}(\mathcal{M})$ are homeomorphic?

(V) *II.7.7. Problem:* If the algebra $\mathcal{N}\in\mathcal{S}ub(\mathcal{M})$ is connected with a half-sided translation (or a half-sided inclusion) then the characteristic function $D(t)$ is Abelian. Assume $D(t)$ is Abelian, then do there exist two commuting half-sided translations $U_1\in\mathcal{H}str(\mathcal{M})^+$, $U_1\in\mathcal{H}str(\mathcal{M})^-$, such that $\mathcal{N}=\mathrm{Ad}\,(U_1(1)U_2(-1))\mathcal{M}$ holds? (One of the factors could be trivial.)

III. ON LOCAL MODULAR ACTION, EXAMPLES

Since the modular group of the pair $(\mathcal{M}(O),\ \Omega)$ is defined but not very concrete, one would like to have examples where this group can be computed explicitly. These are those where the modular group of the algebra, associated with some domain in the Minkowski space, defines a geometric transformation. We start with the result of Bisognano and Wichmann[63,64] at which we look in some detail. Afterwards the other examples known up to now will be discussed. Since it promotes the feeling for the modular groups, if they act local, it is interesting to look for other possibilities. As the result of Trebels[65] shows, there are no other cases in the vacuum sector.

III.1. The result of Bisognano and Wichmann for the wedge domain

The first explicit determination of a modular group is due to Bisognano and Wichmann. They assumed that the local algebras are generated by Wightman fields, and that the Lorentz transformations act on the indices of the fields by finite dimensional representations of the Lorentz group, i.e.,

$$U(\Lambda)A_i(x)U^*(\Lambda)=\sum_j D_i^j(\Lambda)A_j(\Lambda x),$$

where $D_i^j(\Lambda)$ is the direct sum of finite dimensional representations. In this situation the theory is also PCT invariant (Jost[66]). Here the case of only one scalar field will be treated. For the general case see Ref. 64. All our calculations use the $\mathbb{R}^4$.

III.1.1. Remark:

(1) Let $\Lambda(t)$ as in Eq. (I.5.3) and let the forward tube T^+ be defined by

$$T^+=\{z;\Im m z\in V^+\}.$$

Then we have:

For $x\in W_R$ one has $\Lambda(t)x\in T^+$ in the range $-\frac{1}{2}<\Im m t<0$, and if $x\in W_L$, one has $\Lambda(t)x\in T^-$ for $0<\Im m t<\frac{1}{2}$.

For $\Im m t=0$, or $\pm\frac{1}{2}$, the vector $\Lambda(t)x$ belongs again to $\mathbb{R}^4$.

(2) Let $A(x)$ be the field operator, then $U(\mathrm{i}y)A(x)\Omega=A(x+\mathrm{i}y)\Omega$ is defined for $y\in V^+$.

(3) Let $x=(x_0,x_1,x_2,x_3)\in W_R$ then

$$\Lambda(-\mathrm{i}/2)(x_0,x_1,x_2,x_3)=(-x_0,-x_1,x_2,x_3),$$

and hence

$$U(\Lambda(-\mathrm{i}/2))A(x)\Omega=A(-x_0,-x_1,x_2,x_3)\Omega.$$

(4) On the other hand the PCT operator Θ gives

$$\Theta A(x)\Omega=A(-x)\Omega.$$

This suggests for the modular conjugation the representation

$$J=\Theta U(\pi,e_1)=U(\pi,e_1)\Theta,$$

where $U(\pi,e_1)$ represents the rotation around the x_1 axis and π is the angle of rotation.
(5) If $U(\Lambda(-\mathrm{i}/2))$ is the square root of the modular operator of the wedge algebra then this leads for any testfunction to the relation

$$JU(\Lambda(+i\pi))A(f)\Omega=A(\overline{f})\Omega.$$

To show that all the remarks are true we need some notations from the theory of the tensor algebra.

III.1.2. Notations:
(1) $\underline{\mathcal{S}}$ denotes the tensor algebra generated by $\mathcal{S}(\mathbb{R}^4)$.
(2) For $\underline{f}\in\underline{\mathcal{S}}$, $A(\underline{f})$ denotes the smeared field operator. As domain of definition for the field operators we choose

$$\mathcal{D}=\{A(\underline{f})\Omega;\underline{f}\in\underline{\mathcal{S}}\}.$$

(3) If G is a domain, then we denote by $\mathcal{P}(G)$ the algebra generated by elements $A(f)$, where f has its support in G.
(4) We call a point y right of x, if $y\in x+W_R$. If G_1,G_2 are two domains, then we say G_1 is right of G_2 if this is true for all pairs of points in G_1 and G_2.

Now we are in the position to formulate the result of Bisognano and Wichmann.

III.1.3. Theorem:
Let $A(x)$ be a scalar quantum field. Set $\Delta=U(\Lambda(-\mathrm{i}/2))$ and $J=\Theta U(\pi,e_1)$, as introduced in III.1.1.(4). *Then holds:*
(a) $J\mathcal{P}(W_R)J=\mathcal{P}(W_L)$,
(b) $\Delta^{\mathrm{it}}\mathcal{P}(W_{R,L})\Delta^{-\mathrm{it}}=\mathcal{P}(W_{R,L}),\quad t\in\mathbb{R}$,
(c) $J\Delta^{1/2}X\Omega=X^*\Omega\quad\forall X\in\mathcal{P}(W_R)$,
$J\Delta^{-1/2}Y\Omega=Y^*\Omega\quad\forall Y\in\mathcal{P}(W_L)$,
(d) $\mathcal{P}(W_R)\Omega$ *is a core for* $\Delta^{1/2}$.

Statement (a) is Jost's PCT theorem. Statement (b) is nothing else but the Lorentz covariance of the theory. We have added (d) because this is an important aspect of the Tomita–Takesaki theory.

Idea of the proof: If G_{i+1} is right of G_i and $x_i\in G_i$ then $A(\Lambda(t)x_1)A(\Lambda(t)x_2)\cdots A(\Lambda(t)x_n)\Omega$ has an analytic continuation in t into the strip $S(-1/2,0)$. At the lower boundary it has the value $A(\Lambda(t)x_1^j)\cdots A(\Lambda(t)x_n^j)\Omega$ with $x^j=(-x_0,-x_1,x_2,x_3)$.

Since the domains G_i are chosen in such a way that the field operators $A(x_i)$ and also $A(x_i^j)$ commute for different lower index one finds for supp $f_i\subset G_i$ and with the above definition of J the relation

$$U(\Lambda(\mathrm{i}/2))A(f_1)\cdots A(f_n)\Omega=J\{A(f_1)\cdots A(f_n)\}^*\Omega. \tag{III.1.1}$$

Now let us denote by $\mathcal{Q}$ the set of operators $A(\underline{f})$ where the f's have the following properties:
(a) To $\underline{f}$ exists a sequence of domains G_i, $i=1,...,n$ such that $G_i\in W_R$ and G_{i+1} is right of G_i.
(b) f is a product function with support of $\underline{f}\subset G_1\times\cdots\times G_n$. Then Eq. (III.1.1) holds also for this set. It remains to show that one can extend (III.1.1) to all of $\mathcal{P}(W_R)$. To this end one first has to show that $\mathcal{Q}\Omega$ is a core for $U(\Lambda(-\mathrm{i}/2))$. Knowing this one can extend Eq. (III.1.1) to all of $\mathcal{P}(W_R)$. For the first problem one observes that $\mathcal{Q}\Omega$ is invariant under $U(\Lambda(t))$ so that one can use Nelsons theorem. For the second question one uses the commutativity of $JA(f)$ with $\mathcal{P}(W_R)$ which follows from the fact that x^j belongs to the left wedge. □

III.1.4. Definition:
A representation of a QFTLO fulfills the Bisognano–Wichmann property if the modular group of every wedge acts local, like the associated group of Lorentz boosts, on the underlying space.

III.2. Other examples

(i) In a field theory of massless, noninteracting particles every influence travels along the boundary of the light-cone. Therefore, there holds not only spacelike, but also timelike commutativity. This implies that the vector Ω is cyclic and separating also for the algebra of the forward light-cone V^+. In 1978, Buchholz[67] determined the modular group for this situation. It coincides with the dilatations.

III.2.1. Theorem:
In a field theory of noninteracting massless particles the modular group of the algebra of the forward light-cone V^+ acts as follows:

$$\Delta_{V^+}^{it} = V(e^{-2\pi t}),$$

where

$$V(\lambda)A(x)V^+(\lambda) = A(\lambda x), \quad \lambda > 0$$

holds. This means $V(\lambda)$ implements the dilatations.

Since the calculation is similar to that of the Bisognano–Wichmann case, it will not be repeated here.

(ii) If the theory is conformally covariant then the algebra of the double cone can be transformed onto the algebra of the wedge or the forward light-cone. Since the modular groups are known for the last two algebras, the modular group for the algebra of the double cone can be obtained by transformation. The result is:

III.2.2. Theorem:
Assume we are dealing with a conformal covariant theory. Let D be the double cone

$$D = \{x : |x_0| + \|\vec{x}\| < 1\}$$

and denote by

$$x^{\pm} = x_0 \pm \|\vec{x}\|.$$

Then the modular group of the pair $(\mathcal{M}(D), \Omega)$ induces on D a geometric transformation given by the formula:

$$x^{\pm}(\lambda) = \frac{-(1-x^{\pm}) + e^{-2\pi\lambda}(1+x^{\pm})}{(1-x^{\pm}) + e^{-2\pi\lambda}(1+x^{\pm})}.$$

The modular group of the double cone has first been computed by Hislop and Longo.[68]

(iii) The examples treated before and those of the next subsection are based on the vacuum representation. There are also situations where one can compute the modular groups for thermal representations. These investigations are due to Borchers and Yngvason.[69] In these cases the domain is the forward light-cone or the wedge in two-dimensional models that factorize in light-cone coordinates. In order that one obtains local action for the modular groups one has to deal with Wightman fields of scale dimension 1. The results are as follows:

III.2.3. Theorem:
Assume we are dealing with a Weyl system over the two-dimensional Minkowski space that factorize in light-cone coordinates. Let ω be the quasi free KMS state and π the corresponding representation of the Weyl algebra for a field of scale dimension 1. Then the modular groups of the forward light-cone and the wedge act local on the corresponding algebras. The transformations are:

For the forward light-cone:

$$\mathbf{x} \mapsto \varphi_{V^+}(t,\mathbf{x}), \quad \mathbf{x} \in V^+.$$

For the wedge:

$$\mathbf{x} \mapsto \varphi_W(t,\mathbf{x}), \quad \mathbf{x} \in W.$$

Here t is the element of the modular group and the functions φ *are given by*

$$\varphi_V + (u,\mathbf{x}) = (\varphi + (u,x^L), \varphi_+(u,x^R)),$$

$$\varphi_W(u,\mathbf{x}) = (\varphi_-(u,x^L), \varphi_+(u,x^R)),$$

with

$$\varphi_-(u,x) = -\varphi_+(-u,-x),$$

$$\varphi_+(u,x) = \frac{\beta}{2\pi} \log\{1 + \mathrm{e}^{-2\pi u}(e^{2\pi x/\beta} - 1)\}.$$

III.3. The counterexamples of Yngvason

The examples of Yngvason[70] are treated separately, because they show examples of theories with special properties. From the result on half-sided translations (Sec. II.5) we know, that the modular group of the wedge acts on the translations as the Lorentz boosts of the wedge. This might give the impression that the modular group of this algebra acts local. That this is not true in general is shown by the first example. If one defines the algebras of the double cones by intersection then the modular group acts local in the characteristic two plane of the wedge, but not necessarily local in the perpendicular direction, as shown by the last example.

Suppose Φ is a Hermitian Wightman field which transforms covariantly under space–time translations, but not necessarily under Lorentz transformations, and depends only on one light-cone coordinate, say x^+. Locality implies that the commutator $[\Phi(x^+), \Phi(y^+)]$ has support only for $x^+ = y^+$. Moreover, from the spectrum condition it follows that the generator for translations in the x^+ direction is positive semidefinite. This implies that the Fourier transform of the two-point function has the form

$$\tilde{\mathcal{W}}_2(p) = \theta(p) p Q(p^2) + c\,\delta(p).$$

In this formula Ω is the vacuum vector, $Q(p^2)$ is a positive, even polynomial in $p \in \mathbb{R}$ and $\theta(s) = 1$ for $s \geqslant 0$ and zero else, and $c = (\Omega, \Phi(x^+)\Omega)^2 \geqslant 0$ is a constant. Subtracting $c^{1/2}$ from Φ if necessary, we may drop the $\delta(p)$ term. For simplicity of notation from now on we write x,y instead of x^+, y^+.

For our future investigations we can restrict our attention to the one-particle Hilbert space $\mathcal{H}_{Q,1}$. We know that the modular group of the half line acts as a delatation by the factor $e^{-2\pi t}$. This amounts in momentum space to a dilatation by the factor $\lambda = e^{2\pi t}$. This information is sufficient to compute the action of Δ_+^{it} and of Δ_-^{it} on the one-particle Hilbert space. Since the modular groups do not change the support properties of the half-line, i.e., the analyticity property in momentum space, one obtains:

> *Since the algebra and its commutant have the same modular group we see that wedge duality is fulfilled iff* $Q(p)$ *has only real zeros.*

The duality condition for bounded intervals is a little more difficult. Yngvason has shown:

The duality condition is violated if $Q(p)$ *is not a constant.*

Finally we consider fields in n-dimensional Minkowski space. Guided by the low-dimensional examples considered above we shall compute the modular groups of the wedge algebras for generalized free fields on $\mathbb{R}^n$. We treat the special case where the two-point function has in Fourier space the form

$$\mathcal{W}_2(p) = M(p)\,d\mu(p),$$

where $d\mu$ is a positive Lorentz invariant measure with support in the forward light cone and M is a polynomial that is positive on the support of $d\mu$. The polynomial M allows a factorization,

$$M(p) = F(p)F(-p),$$

where $F(p)$ is a function (in general not a polynomial) with certain analyticity properties.

To describe the properties of F we use the light-cone coordinates $x^{\pm} = x^0 \pm x^1$ for $x = (x^0,\ldots,x^n) \in \mathbb{R}^n$ and denote $(x^2,\ldots,x^n)$ by $\hat{x}$. The Minkowski scalar product is

$$\langle x,y\rangle = \tfrac{1}{2}(x^+y^- + x^-y^+) - \hat{x}\cdot\hat{y}.$$

There is no lack of polynomials M allowing such a factorization; one example is

$$M(p) = (p^1)^2 + \cdots + (p^n)^2 + m^2$$

with

$$F(p) = \sqrt{\hat{p}\cdot\hat{p} + m^2} + \mathrm{i}p^1 = \sqrt{\hat{p}\cdot\hat{p} + m^2} + (\mathrm{i}/2)(p^+ - p^-).$$

By analogy with the first example we define for $\lambda > 0$ the unitary operators $V_R(\lambda)$ on the Fock space $\mathcal{H}$ over the one-particle space $\mathcal{H}_1 = L^2(\mathbb{R}^n, M(p)\,d\mu(p))$ by

$$\tilde{f}_\lambda(p) = \frac{\sqrt{\hat{p}\cdot\hat{p} + m^2} - (\mathrm{i}/2)(\lambda p^+ - \lambda^{-1}p^-)}{\sqrt{\hat{p}\cdot\hat{p} + m^2} - (\mathrm{i}/2)(p^+ - p^-)}\tilde{f}(\lambda p^+, \lambda^{-1}p^-, \hat{p}).$$

This example demonstrates also that the modular group of $\mathcal{M}(W_R)$ may act nonlocal in the $\hat{x}$ directions. In fact, let f be a test function with compact support in W_R. Under this transformation $\tilde{f}$ is no longer the Fourier transform of a function of compact support in the $\hat{x}$ directions, because it is not analytic in $\hat{p}$. From this lack of analyticity it is not difficult to deduce that $W(f_\lambda)$ does not belong to any wedge algebra generated by the field unless the wedge is a translate of W_R or W_L, but we refrain from presenting a formal proof. The operator $W(f_\lambda)$ is still localized in the x^0,x^1 directions in the sense that it is contained in

$$\mathcal{M}(W_R + a) \cap \mathcal{M}(W_R + b)'$$

for some $a,b \in W_R$.

III.4. The result of Trebels on local modular action

In the last subsections we saw that under special assumptions the modular groups of algebras, belonging to definite domains, can be computed. In many of these examples the modular transformations led to geometric transformations of the underlying sets. Therefore, it is natural to ask whether or not there might exist other cases where the modular group of the algebra of a set acts as geometric transformations on the underlying set. It is impossible to answer this question for arbitrary sets. Therefore we restrict the sets to the family of double cones and their limits, i.e., to wedges, forward and backward lightcones. The following results are taken from the thesis of Trebels.[65]

III.4.1. Definition:
A unitary transformation V which maps $\mathcal{M}(G)$ (G open) onto itself and which maps Ω onto itself is called geometric, causal and order preserving if there exists a one to one map $g: G \to G$ with the properties:
(i) $x \in G$ implies $x_g \in G, x_{g^{-1}} \in G$.

(*ii*) $x,y\in G$ and $x-y$ are spacelike, then x_g-y_g and $x_{g^{-1}}-y_{g^{-1}}$ are spacelike.
(*iii*) $x-y\in V^+$ implies x_g-y_g and $x_{g^{-1}}-y_{g^{-1}}$ belong to V^+.
(*iv*) For every $G^1\subset G$ one has

$$\mathrm{Ad}\, V\mathcal{M}(G^1)=\mathcal{M}(G^1_g), \quad \text{with} \quad G^1_g=\{x_g\,;x\in G\}.$$

Notice that $g\rightarrow x_g$ maps double cones onto double cones. Since double cones form a base of the topology of $\mathbb{R}^d$ we see that $x\rightarrow x_g$ is continuous. Our first observation is the following.

With this notation the following result holds:

III.4.2. Theorem:

Assume we are dealing with a quantum field theory in the vacuum sector, and that the dimension of the Minkowski space is larger than two. Let D be a double cone and let Δ^{it} be the modular group of $\mathcal{M}(D)$. Assume this group acts geometric and causal on D. Then Δ^{it} coincides with the group of Hislop–Longo transformations (up to a positive scale transformation of the group parameter).

If G is the generator of the Hislop–Longo transformation then we have shown that $g(t)$ is of the form $g(t)=\exp\{mGt\}$ where m is a positive constant. One would like to prove that $m=1$. To this end one has to use the KMS condition. (See Sec. I.3.) With the methods available up to now we are not able to give a general proof for the statement $m=1$. However, if we would deal with a finite number of Wightman fields then the modular transformation would be $\Delta^{it}\Phi_k(x)\Delta^{-it}=D^j_k(t)\Phi_j(g(t)x)$. Here $D^j_k(t)$ is a finite dimensional representation of the dilatations. In this situation one can at least show that m is bounded by one. We do not want to give the calculations.

III.4.3 Remark: The case $m=0$ can be excluded. This case would mean that the algebra of every subdomain $D_1\subset D$ is invariant under the modular group of D. But this implies by the cyclicity of Ω that $\mathcal{M}(D_1)$ and $\mathcal{M}(D)$ coincide. (See Ref. 33, Thm. 9.2.36.) Such situation is only possible if the theory is Abelian.

Idea of the proof: The dimension of the Minkowski space shall always be larger than 2. Let us start with a geometric causal and order preserving map g of the domain G. Since g is continuous it maps closed light cones $\overline{V^+}$ onto closed light cones. From this one concludes that g maps light like lines onto lightlike lines.

It is our intention to look at the possible geometric, causal and order preserving maps of the double cone. But, by an order preserving conformal transformation γ we can send the double cone onto the forward light cone. Then $\gamma g\gamma^{-1}$ is a geometric, causal and order preserving map of V^+. These are much easier to handle.

If we take a light ray l inside V^+ then the closure of the set $\cup\{V^-_x\cap V^+,x\in l\}$ is the intersection of a closed half-space with V^+. g maps such sets onto sets of the same kind. This is also true for the boundary of such half-spaces, since g is continuous. Hence g maps affine tangent hyperplanes intersected with V^+ onto sets of the same kind. Now one can draw the following conclusions:

(i) g maps parallel light rays onto parallel light rays.
(ii) Since spacelike straight lines are intersections of tangent hyperplanes g maps spacelike straight lines onto spacelike straight lines.
(iii) Since every two-plane containing a timelike direction is generated by a family of parallel light rays passing through one light ray one gets from (i) that every two-plane containing a timelike direction is mapped onto a two-plane of the same kind.
(iv) Since every timelike line is the intersection of two such two-planes we see that g maps also timelike straight lines onto timelike straight lines.

From this one deduces that g is a linear transformation. More precisely one finds

III.4.4. Proposition:

Every geometric, causal and order preserving map of the forward lightcone is an element of the Lorentz group extended by the dilatation.

The modular group is a one-parametric group. This implies that every element is the square of another element. Hence if the group acts geometric and causal on the underlying domain, then it

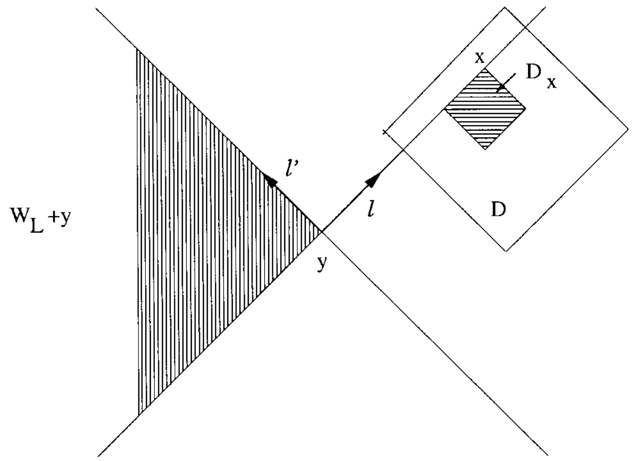


FIG. 1. Position of the double cones D, D_x and the wedge $W(l',l)+y$.

acts automatically order preserving. If the modular group induces a geometric and causal action on the underlying domain then we know from the last proposition that it is a one-parametric subgroup of the $(d(d-1)/2+1)$-dimensional Lie group generated by the Lorentz group and the dilatations. In order to restrict the possibilities we have to use the following properties:
1. The group $g(t)$ is induced by the modular group of $\mathcal{M}(D)$, where D is a double cone. This implies that for $A\in\mathcal{M}(D)$ the expression

$$\Delta^{\mathrm{i}t}\mathcal{A}\Omega$$

has an analytic continuation into the strip $S(-\frac{1}{2},0)$.
2. We are dealing with a quantum field theory in the vacuum sector. This implies in particular that the translations fulfill spectrum condition.

We want to compare the geometric modular action with the action of the translations. As technical tool we need the following result which can easily be proved with help of the double cone theorem, Thm. I.4.4. Here we will not present the proof.

III.4.5. Theorem:
Assume we are dealing with a quantum field theory in the vacuum sector, and that the dimension of the Minkowski space is larger than two. Let D_1, D_2 be two double cones with center x_1, x_2, respectively. If x_1-x_2 is lightlike and if $\mathcal{M}(D_1)$ and $\mathcal{M}(D_2)$ commute then the whole quantum field theory is Abelian.

We want to look at the modular group of the double cone D. Let $x\in D$ and if $\Delta^{\mathrm{i}t}$ acts geometric and causal on D then $g(t)x$ can be differentiated with respect to t since $g(t)$ is a subgroup of a Lie group. We want to investigate the direction of $g'(t)x$ and want to show $g'(0)x\in\overline{V^-}$. To this end we look at the situation described in Fig. 1.

With $B\in\mathcal{M}(W_L+y)$ and $A\in\mathcal{M}(D_x)$ we set

$$f^+(\lambda,\tau)=(\Omega,BU(\lambda t)\Delta^{\mathrm{i}\tau}A\Omega),$$

$$f^-(\lambda,\tau)=(\Omega,A\Delta^{-\mathrm{i}\tau}U(-\lambda t)B\Omega),$$

where $U(x)$ is the representation of the translations. Using techniques of analytic functions of two complex variables one can deduce that $g'(0)x$ must lie in $\overline{V^+}$.

Since the considerations about the structure of the coincidence are stable under conformal transformations, we will transform the double cone onto the forward lightcone. In this setting we have to show that the group $g(t)$ coincides with the dilatations. If we write $g(t)=\exp\{Mt\}$ then $g'(t)x\in\overline{V^-}$ implies that (y,Mx) is smaller zero for all $x,y\in V^+$. By means of the structure of the

Lorentz group we find that M is diagonal and hence $M=-m\mathbb{1}$, $m\in\mathbb{R}^+$. Therefore, the transformed $g(t)$ coincides with the scaled dilatations and consequently the original group coincides with the scaled Hislop–Longo transformations. □

III.5. Remarks, additions, and problems

(I) The result of Trebels deals only with double cones. Therefore, it is not possible to argue that the factor m has to be 1. This is due to the fact that the Hislop–Longo transformation $g(t-\mathrm{i}\frac{1}{2})$ maps D to real points but they are not all spacelike with respect to D. If, however, we replace the double cone by the wedge then one can argue that m must be 1.

III.5.1. Problem: Does there exist a convincing argument showing, that m must be 1? (II) In the Trebels situation, the algebra of a subdouble cone with either the same upper or lower tip fulfills the condition of −half-sided or +half-sided modular inclusion, respectively. If one is dealing with a conformally covariant theory, then the corresponding half-sided translations map, for a proper chosen (finite) group element, the algebra of the double cone onto the algebra of the backward respectively forward light cone.

(III) If the Bisognano–Wichmann property (Def. III.1.4) is fulfilled only for the subsets of the wedge, then the modular group of the wedge define geometric transformations only for this wedge. This can be extended to geometric transformations of the whole $\mathbb{R}^d$. (See Guido, Ref. 71.)

(IV) As shown by Kuckert[72] the assumptions can be changed. If one replaces the Bisognano–Wichmann property for the wedge by other symmetry conditions, with some locality property, but for the whole space, then one finds that J_W and $\Delta_W^{\mathrm{i}t}$ act local as in the Bisognano–Wichmann situation. A similar result holds for the forward light-cone, provided Ω is cyclic and separating for $\mathcal{M}(V^+)$. In these cases the assumptions are: The symmetry shall map the local net *into* the local net. The associated modular groups shall transform the local algebras in the corresponding manner.

One can replace the transformation property of the local net by transformation properties of localized operators. In this case one has to make more restrictive assumptions on the transformations and the net. For details see Ref. 72.

IV. THE PCT THEOREM AND CONNECTED QUESTIONS

The PCT theorem tells us that the product of time reversal, space reflection, and charge conjugation is always a symmetry. Reading the paper of Pauli[73] on this subject one gets the impression that a precurser of the PCT "theorem has been discovered by Schwinger.[74] But it was a mysterious transformation containing the interchange of operators. The first development of the PCT theorem in the frame of Lagrangean field theory is due to Lüders.[75] This result has triggered the clarification of the connection between spin and statistics and the role of the positive energy. (See Pauli[73] and also Lüders and Zumino.[76])

In 1957, Jost[66] gave a proof of the PCT theorem in the frame of Wightman's field theory. The beauty of this proof is the clarification of the role of the different conditions one has to impose. These are

1. Covariance of the theory under the (connected part of the) Poincaré group.
2. Positivity of the energy.
3. There are only fields, which transform with respect to finite dimensional representations of the Lorentz group. (Transformation of the index space.)
4. Locality, which means that for spacelike distances the Bose fields commute with all other fields and the Fermi fields anticommute with each other.
5. The Minkowski space has even dimensions.
6. To every field in the theory appears its conjugate complex partner.

From the spectrum condition it follows that the Wightman functions have an analytic continuation into the forward tube T_n^+

$$T_n^+=\{z_1,\ldots,z_n\in\mathbb{C}^4;\Im m(z_i-z_{i+1})\in V^+\}.$$

Using locality, Poincaré covariance of the theory, and the appearance of only finite dimensional representation of the Lorentz group in the index space, Hall and Wightman[77] could show that the analytically continued Wightman functions can be considered as functions on the complex Lorentz group. If the index space transforms under infinite dimensional representation of the Lorentz group then the Hall–Wightman theorem fails because of lack of analyticity. Examples are given by Streater[78] and by Oksak and Todorov.[79] The Hall–Wightman theorem was the starting point of Jost's investigation. If the Minkowski space has even dimensions then the complex Lorentz group contains the element $-\mathbb{1}$. This transformation is the product of time reversal and space reflection. But there is the time translation e^{iEt} with the positive energy operator. In order to keep the energy positive one has to change i into $-$i. Therefore, the time reversal has to be an antiunitary operator. If Θ is an antiunitary total reflection one obtains for a scalar field

$$\Theta\Phi(x)\Theta=\Phi^*(-x).$$

The passage to the conjugate complex is closely related to the charge conjugation. Therefore, one has to look at the product of C and PT. One remark more to the role of locality: The transition to the conjugate complex interchanges the order of an operator product. At totally spacelike points the original order can be restored. Putting things together one gets the PCT theorem for scalar fields. The general case needs in addition the handling of finite dimensional matrices which appear with fields of higher spin.

For a long time it was impossible to show the PCT theorem in the theory of local observables because one did not know the meaning of condition 3 and 6 in the setting of local observables.

A good candidate for the CPT operator is

$$\Theta=J_W U(R_W(\pi)) \tag{IV.0.1}$$

provided the origin is contained in the edge of the wedge. $R_W(\alpha)$ denotes the rotation in the two-plane perpendicular to the characteristic two-plane of the wedge, and J_W the modular conjugation of the algebra of the wedge.

If the Ansatz (IV.0.1) is correct, then the representation of the Lorentz group and the modular groups of the wedges have to fit together. Since on the vacuum sector Θ is a geometric transformation, also J_W has to act local. Moreover the transformation $(\Theta U(R_W(\pi))$ maps the algebra of the wedge onto the algebra of the opposite wedge. Therefore, the theory has to fulfill wedge duality. First we treat the question of wedge duality and afterwards that of the locality of J_W.

IV.1. The wedge duality

The problem of this subsection is: When does a Lorentz covariant theory fulfill wedge duality?

The result we present here is essentially a two-dimensional statement. In the proof we can think of sets which are cylindrical in all directions perpendicular to the characteristic two-plane of the wedge. Hence all the expressions depend only on two variables. In this situation we only have two wedges which we call the right wedge W^r and the left wedge W^l. The wedges obtained by applying a shift by a will be denoted by W^r_a and W^l_a, respectively. If we denote the double cones by K then this can be characterized by the intersection of two wedges.

$$K_{a,b}=W^r_a\cap W^l_b, \quad b-a\in W^r.$$

Let $\mathcal{M}^0_{a,b}$ be the given von Neumann algebra associated with $K_{a,b}$ fulfilling the mentioned assumption. Starting from this we obtain for the wedges the algebras:

$$\mathcal{M}^r_a=\{\bigcup_{K\subset W^r_a}\mathcal{M}^0_K\}'', \quad \mathcal{M}^l_a=\{\bigcup_{K\subset W^l_a}\mathcal{M}^0_K\}''. \tag{IV.1.1}$$

Moreover, we set

$$\mathcal{M}_a'^{r}=\{\mathcal{M}_a^r\}', \quad \mathcal{M}_a'^{l}=\{\mathcal{M}_a^l\}'. \tag{IV.1.1$'$}$$

Without loss of generality we can construct a net which might be slightly larger:

$$\mathcal{M}_{a,b}:=\mathcal{M}(K_{a,b})=\mathcal{M}_a^r\cap\mathcal{M}_b^l. \tag{IV.1.2}$$

This net fulfills again all requirements listed in the beginning. Moreover, the wedge algebra constructed with $\mathcal{M}(K)$ coincides with the wedge algebra constructed with the $\mathcal{M}^0(K)$. In what follows we only will work with the algebras $\mathcal{M}(K)$.

In Wightman field theory one is dealing with quantities $\Phi_n(x)$ localized at a point. If x belongs to the right wedge one can analytically continue the expression $U(\Lambda(t))\Phi_n(x)\Omega$ into the strip $S(-\frac{1}{2},0)$. This is due to the fact that the representation of the Lorentz group in the index space is defined for complex Lorentz transformations. The result which one obtains is an element belonging to the left wedge, namely, $U(\Lambda(t))\Phi_n(-x)\Omega$ (for entire spin). There are two problems if one wants to generalize this:

First our objects are not localized at a point but in bounded domains. Here we will find a natural generalization of the description.

The second problem consists of understanding the exchange of the left and the right wedge by the complex Lorentz transformations because of the following

IV.1.1. Remark:
If we are dealing with a von Neumann algebra $\mathcal{M}$ and a one-parametric, strongly continuous group of automorphisms α_t then one can define the analytic elements $\mathcal{M}^{anal}$ for which $\alpha_t A$ has an entire analytic extension. The set $\mathcal{M}^{anal}$ is a $*$-strong dense subalgebra of $\mathcal{M}$ and the elements $\alpha_z A,\ A\in\mathcal{M}^{anal}$ also belong to $\mathcal{M}$.

Therefore, it is not easy to understand why for an element A, localized in the right wedge, the expression $U(\Lambda(-\mathrm{i}/2))A\Omega$ can be written as $\hat{A}\Omega$ with an element $\hat{A}$ localized in the left wedge.

From Remark II.5.3.(ii) we know that $\Delta_r^{\mathrm{i}t}$ and $\Delta_l^{\mathrm{i}t}$ act on the translations as the Lorentz transformations. From this we obtain:

$$R(t)\Delta_r^{\mathrm{i}t}=U(\Lambda(t)), \quad L(t)\Delta_l'^{\mathrm{i}t}=U(\Lambda(t)). \tag{IV.1.3}$$

Here Δ_l' denotes the modular operator of $(\mathcal{M}_0^l)'$. Since $U(\Lambda)$ commutes with the modular groups and acts on the translations in the same manner as the modular groups we obtain the following commutations:

$$\begin{aligned}&[R(s),\Delta_r^{\mathrm{i}t}]=[R(s),U(\Lambda)]=[R(s),T(a)]=[R(s),J_r]=0,\\ &[L(s),\Delta_l'^{\mathrm{i}t}]=[L(s),U(\Lambda)]=[L(s),T(a)]=[L(s),J_l]=0.\end{aligned} \tag{IV.1.4}$$

The aim of this investigation is to show that $R(t)$ and $L(t)$ coincide. Since the proof of this result is rather involved we have to refer to the original paper[80] or to the complete script. Therefore, we only quote the results.

IV.1.2. Proposition:
(i) For every $A\in\mathcal{M}_{a,b}$ with $a\in W^r$ such that $U(\Lambda(t))A\Omega$ has a boundes analytic extension into the strip $S(-\frac{1}{2},0)$ with continuous boundary values there exists an element $\hat{A}\,\eta\mathcal{M}_{-b,-a}$ such that the following relation holds

$$U(\Lambda-(\mathrm{i}/2))A\Omega=\hat{A}\Omega.$$

(ii) For every $A\in\mathcal{M}_{a,b}$ with $b\in W^l$ such that $U(\Lambda(t))A\Omega$ has a bounded analytic extension into the strip $S(0,\frac{1}{2})$ with continuous boundary values there exists an element $\tilde{A}\,\eta\mathcal{M}_{-b,-a}$ such that the following relation holds

$$U(\Lambda(\mathrm{i}/2))A\Omega=\tilde{A}\Omega.$$

As mentioned before we have to establish a map from $\mathcal{M}_r$ to $\mathcal{M}_l$ and viceversa. This means there must exist sufficiently many elements which satisfy Prop. IV.1.2. In order to formulate the result we need some notation about the localization of operators.

Let $A\in\mathcal{M}$ be a local operator then we denote by K_0 the smallest double cone such that $A\in\mathcal{M}(K_0)$. By K we denote the translate of K_0 such that the center of K coincides with the origin. Let $K_0=K+x$ then we can write every localized operator in the form

$$A=A(K,x). \tag{IV.1.5}$$

With this concept we can formulate the exact result about wedge duality.

IV.1.3. Theorem:

Given a Lorentz covariant QFTLO in the vacuum sector, then the following conditions are equivalent:

(1) *The theory fulfills wedge duality.*

(2) *The set* $\{A(K,x)\}$, *such that*

(α) $K+x\subset W^r$,

(β) $U(\Lambda(t))A(K,x)\Omega$ *has a bounded analytic continuation into the strip* $S(-\frac{1}{2},0)$ *with continuous boundary values,*

(γ) $U(\Lambda(t))A^*(K,-x)\Omega$ *has a bounded analytic continuation into the strip* $S(0,\frac{1}{2})$ *with continuous boundary value,*

is $*$*-strong dense in* $\mathcal{M}_0^r$.

IV.2. The reality condition and the Bisognano–Wichmann property

In the discussion at the beginning of this section we saw that we must solve two problems before we can prove the PCT theorem. The first was the wedge duality, corresponding to the properties of the index space of Wightman fields. The second was the reality condition implying that every Wightman field has its conjugate complex partner. In analogy we pose:

IV.2.1. Reality conditon:

We say a Poincaré covariant theory of local observables in the vacuum sector, which satisfies the wedge duality, fulfills the reality condition if:

(i) *Every* $A(K,x)$ *with* $K+x\subset W^r$ *which fulfills the relation of Prop. IV.1.2 satisfies the equation*

$$\widehat{A^*}(K,P_Wx)=\{\hat{A}(K,P_Wx)\}^*. \tag{IV.2.1}$$

P_W *is the reflection in the characteristic two-plane of the wedge.*

(ii) Ω *is cyclic for the set* $\{A(K,x);K+x\subset W^r$ *which satisfy Eq. (IV.2.1)*$\}$.

With this notation we obtain:

IV.2.2. Theorem:

In a representation of a Poincaré covariant theory of local observable in the vacuum sector the modular group associated with the algebra of any wedge coincides with the corresponding Lorentz boosts iff the theory fulfills wedge duality and the above reality condition with respect to the Lorentz transformations.

Proof: If we know that $U(\Lambda(t))$ and $\Delta_W^{\mathrm{i}t}$ coincide then by Thm. IV.1.3 one has wedge duality. Moreover, the reality condition is fulfilled because for every $A(K,x)\in\mathcal{M}(W^r)$ one has

$$\widehat{A^*}(K,x)\Omega=U(\Lambda(-\mathrm{i}/2))A^*\Omega=\Delta_W^{1/2}A^*\Omega=JAJ\Omega,$$

and

$$\Delta_W^{1/2}\hat{A}(K,x)\Omega=JA^*J\Omega=\{JAJ\}^*\Omega.$$

Hence the reality condition is fulfilled.

Next assume wedge duality and the reality condition. Let $A(K,x)\in\mathcal{M}(W^r)$ fulfill the reality condition. Take an element $B\in\mathcal{M}(W^l)$ and look at the matrix elements

$$F^+(s,t)=(\Omega,B\Delta^{\mathrm{i}s}U(\Lambda(t))A(K,x)\Omega),$$

$$F^-(s,t)=(\Omega,A(K,x)U(\Lambda(-t))\Delta_r^{-\mathrm{i}s}B\Omega).$$

By choice of the elements B and $A(K,x)$ we get $F^+(s,t)=F^-(s,t)$. Moreover, wedge duality and the modular theory yields

$$F^+\left(s+\frac{i}{2},t-\frac{i}{2}\right)=F^-\left(s-\frac{i}{2},t+\frac{i}{2}\right).$$

By both coincidences and the edge of the wedge theorem, Thm. I.4.1, we obtain a bounded periodic function $F(s,t)=F(s-\mathrm{i},t+\mathrm{i})$. Since bounded entire functions are constant we find

$$F(s,-s)=\mathrm{const}=F(0,0),$$

$$(\Omega,B\Delta^{\mathrm{i}s}U(\Lambda(-s))A(K,x)\Omega)=(\Omega,BA(K,x)\Omega).$$

Since $\mathcal{M}(W^l)\Omega$ and $\{A(K,x)\Omega\}$ are dense in $\mathcal{H}$ we obtain $\Delta_r^{\mathrm{i}s}U(\Lambda(-s))=\mathbb{1}$. □

IV.3. The PCT theorem

Now we are prepared for the proof of the PCT theorem under the assumption that the wedge duality and the reality condition are fulfilled. Starting from the Ansatz Eq. (IV.0.1) one has to solve two problems:

(1) Since Θ shall be a local transformation, also J_W must be local. Since the map $A\Omega\to A^*\Omega$ is local, and since by Thm. IV.2.2 $\Delta_W^{1/2}$ and $U(\Lambda_W(-\mathrm{i}/2)$ coincide, we know that the product

$$S_W=J_W\Delta_W^{1/2}=J_WU(\Lambda_W(-\mathrm{i}/2)) \tag{IV.3.1}$$

acts local. Therefore, J_W and $U(\Lambda_W(-\mathrm{i}/2))$ must act local at the same time. The answer to this question is closely related to the next one.

(2) The operator product $J_WU(R_W(\pi))$ shall be independent of the choice of the wedge W. Using Eq. (IV.3.1) we obtain $J_W=U(\Lambda_W(-\mathrm{i}/2))S_W$ and consequently

$$J_WU(R_W(\pi))=U(\Lambda_W(-\mathrm{i}/2))U(R_W(\pi))S_W, \tag{IV.3.2}$$

where we have used the fact that $U(R_W(\pi))$ maps the algebra $\mathcal{M}(W)$ onto itself, which implies, that S_W and $U(R_W(\pi))$ commute. We will apply the expression (IV.3.2) to vectors of the form $A(K,x)\Omega$ with $K+x\subset W$. Therefore, problem (2) is solved if $U(\Lambda_W(-\mathrm{i}/2))U(R_W(\pi))A^*(K,x)\Omega$ is independent of W. (As long as $K+x\subset W$.) The product $U(\Lambda_W(-\mathrm{i}/2))U(R_W(\pi))$ is nothing else but the element $-\mathbb{1}$. Since we get to $U(\Lambda_W(-\mathrm{i}/2))U(R_W(\pi))A(K,x)\Omega$ by analytic continuation, we have to make sure that the continuation gives for different W a unique answer.

We start with the uniqueness problem because its answer is needed for the solution of the locality question. For simplicity of notation we restrict ourselves to the four-dimensional Minkowski space. In this case the Lorentz group is six-dimensional. First, with help of the Malgrange–Zerner theorem I.4.2 we will construct a function on the complex Lorentz group. The points $U(\Lambda_W(-\mathrm{i}/2))$ will be points on the boundary of the domain which we construct. Therefore, we must convince ourselves that $U(\Lambda)$ is single valued on that domain.

Let D be a double cone such that its closure does not contain the origin. We choose a wedge with $D\subset W$. Let G be the (connected) Lorentz group and set

$$N(D)=\{g\in G;D\subset gW\}. \tag{IV.3.3}$$

Since W is open, $N(D)$ is open and contains the identity of the group. From this one can choose $g_1, g_2, \ldots, g_6 \in N(D)$ and $T_1, \ldots, T_6 > 0$ such that the products $\Lambda_{g_6 W}(t_6) \cdots \Lambda_{g_1 W}(t_1)$ for $|t_i| < T_i$ are still in $N(D)$ and that the generators of $\Lambda_{g_i W}(t)$ are linearly independent.

Since by assumption the unitary groups $U(\Lambda_{g_i W}(t))$ coincide with the modular groups of $\mathcal{M}(g_i W)$ one can analytically continue the vector valued function

$$U(\Lambda_{g_6 W}(t_6)) \cdots U(\Lambda_W(t_1)) A\Omega, \quad A \in \mathcal{M}(D), \quad D \subset W \tag{IV.3.4}$$

in the variable t_i into the strip $S(-\frac{1}{2},0)$, provided one has $|t_1| < T_1, \ldots, |t_{i-1}| < T_{i-1}, |t_{i+1}| < T_{i+1}, \ldots, |t_6| < T_6$. Using the Malgrange–Zerner theorem I.4.2 one obtains a function holomorphic in all variables.

The domain of holomorphy of this function can be determined. This calculation will be done by mapping the strip $S(-\frac{1}{2},0)$ biholomorphic onto itself in such a way that the interval $|x| < T$ is mapped onto $\mathbb{R}$ and the rest of the boundary onto $-\mathrm{i}/2 + \mathbb{R}$. In these new variables the domain of holomorphy is convex and hence the function is one-valued. Hence there are no monodromy problems in these variables. Therefore, we have to show that the inverse transformation sends the boundary points to some set where the inverse map is unique. The result of these investigations are collected in.

IV.3.1. Proposition:

Let D be a double cone such that the closure of D does not contain the origin. Then for $A \in \mathcal{M}(D)$ and g such that $D \subset gW$

$$U(\Lambda_{gW}(-\mathrm{i}/2)) U(R_{gW}(\pi)) A\Omega$$

is independent of g.

Knowing this it is easy to show that J_W acts local and one obtains for $D \subset W$

$$J_W A J_W \in \mathcal{M}(P_W D).$$

This together with Prop. IV.3.1 and Eq. (IV.0.1) leads to

IV.3.2. Theorem:

Every QFTLO which fulfills wedge duality and the reality condition is PCT convariant.

IV.4. The Bisognano–Wichmann property and the construction of the Poincaré group

We saw that the PCT theorem is closely connected with the Bisognano–Wichmann property (see Def. III.1.4), i.e., the modular group of every wedge acts like the associated group of Lorentz boosts. If we assume that the theory fulfills the Bisognano–Wichmann property, then one can ask whether or not all these modular groups fit together and give rise to a representation of the Poincaré group. If the dimension of the Minkowski space is two then one has only the right and the left wedge and their translates. Since the Bisognano–Wichmann property implies that the translates of the wedge along the lightlike vectors fulfill the condition of half-sided modular inclusion, the translations are obtained by the construction of Wiesbrock[56,57] (see II.6) which together with the modular group of the wedge give rise to a representation of the Poincaré group.[55] Hence the construction procedure contains new aspects if the dimension of the Minkowski space is at least three.

A first treatment of this problem is due to Brunetti, Guido, and Longo.[81] They used the first and the second cohomology of the Poincaré group and showed that the modular groups of all wedges give rise to a representation of the covering of the Poincaré group. In a second paper Guido and Longo[82] generalized their method to charged fields and showed that in this frame the connection between spin and statistics is fulfilled.

Here we will use a construction that is based entirely on the principle of half-sided modular inclusions. It has the advantage that it gives directly a representation of the Poincaré group and not of its covering.[83] In order to avoid index manipulation we represent the result for the four-dimensional Minkowski space. The construction is in three steps. First we construct the translations by using the half-sided modular inclusions of wedges and their translates. Then we show that the algebra of the intersection of two wedges with a common lightlike vector fulfill the condition of half-sided modular inclusion with respect to the algebras of the wedges. This will allow us to construct the translational part of the stabilizer group of the common lightlike vector. Since this group connects the modular groups of different wedges we can, in the third step, construct the whole Poincaré group.

First step: *Construction of the translations*

We start our investigation by looking at a wedges $W[l,l',a]$ and its translate $W[l,l',a+\lambda l],\lambda>0$. The algebra of the second wedge fulfills the condition of $-$half-sided modular inclusion with respect to the algebra of the first wedge. Hence by Thm. II.6.2 we obtain a unitary group $U[a,\lambda l](t)$. The uniqueness Thm. II.6.5 implies that this group does not depend on a and that $U[\lambda l](t)$ can be written as $U[l](\lambda t)$. Moreover, we know that $U[l](t)$ acts local, which implies, that $U[l](\lambda t)$ does not depend on the choice of the second light ray entering in the definition of the wedge and hence of $U[l](\lambda t)$.

It remains to show that the groups $U[l](s)$ and $U[l'](s)$ commute. This follows from the uniqueness of the modular groups.

IV.4.1. Lemma:

Assume all modular groups of the wedge algebras act like their associated Lorentz groups. Then a unique continuous representation of the translation group $V(a)$ exists which fulfills spectrum condition and acts geometrically on the local algebras

$$\mathrm{Ad}\,V(a)\mathcal{M}(D)=\mathcal{M}(D+a),$$

where D denotes a double cone. [*It is assumed, that $\mathcal{M}(D)$ coincides with the intersection of the wedge algebras of all wedge containing D.*] *This representation $V(a)$ is contained in the algebra generated by the modular groups.*

Moreover, the modular groups of the wedges and the translations transform each other as if they were members of a unitary representation of the Poincaré group.

Since Lemma IV.4.1 implies that we have a representation of the Poincaré group on every characteristic two-plane, where the boosts of the wedge are the same as the modular transformations, we obtain from Sec. IV.1:

IV.4.2. Proposition:

Let a representation of a theory of local observables fulfill the above-mentioned conditions. Then this representation fulfills wedge duality, i.e.,

$$\mathcal{M}(W[l,l'])'=\mathcal{M}(W[l',l]).$$

Second step: *The stabilizer group of a light ray*

Next we want to construct the translational part of the stabilizer group of any light ray $l\in\partial V^+$. To this end we look at the family of wedges having one light ray in common,

$$\{W[l,l_2];l\ \ \text{fixed}\}. \tag{IV.4.1}$$

It is well known that the stabilizer $S(l)$ of a lightlike vector is isomorphic to the Euclidean transformation of $\mathbb{R}^2$. (See, e.g., Gelfand, Minlos, and Shapiro.[84]) The rotations are the transformations around the space direction of the light ray. In order to understand the translations let us introduce a second lightlike vector l_2 which we choose in such a way that l, t, l_2 lie in one two-plane. Let $T(l)$ be the tangent hyperplane at the forward lightcone V^+ containing the vector l. Then the affine hyperplane $l_2+T(l)$ intersects ∂V^+ in a two-dimensional set (parabola) homeomorphic to $\mathbb{R}^2$. The translations of $S(l)$ have this set as orbit.

In the concrete example $l=(1,1,0,0)$, $l_2=(1,-1,0,0)$, these translations become ($a=(a_1,a_2)\in\mathbb{R}^2$)

$$\Lambda^{l}(a)=\begin{pmatrix} 1+\frac{a^2}{2} & -\frac{a^2}{2} & a_1 & a_2 \\ \frac{a^2}{2} & 1-\frac{a^2}{2} & a_1 & a_2 \\ a_1 & -a_1 & 1 & 0 \\ a_2 & -a_2 & 0 & 1 \end{pmatrix}. \tag{IV.4.2}$$

(See also Jost, Ref. 85, Appendix.) It is easy to check that this is a representation of the two-dimensional translation group.

Starting with the above vector l_2 and define $l_2(a)=\Lambda^l(a)l_2$ one obtains between $\Lambda^l(a)$ and the Lorentz boosts of the wedge the relation

$$\Lambda^l(a-b)=\lim_{t\to\infty}\Lambda[l,l_2(a)](t)\Lambda[l,l_2(b)](-t). \tag{IV.4.3}$$

In order to show that the corresponding limits of products of modular operators exist and define a commutative group we need once more the principle of half-sided modular inclusion. The crucial result is:

IV.4.3. Theorem:

Let the theory fulfill the Bisognano–Wichmann property. Then the algebra $\mathcal{M}(W[l,l_1]\cap W[l,l_2])$ fulfills the condition of $-$half-sided modular inclusion with respect to both algebras $\mathcal{M}(W[l,l_1])$ and $\mathcal{M}(W[l,l_2])$.

Using this result and its generalization to the intersection of three wedges one can show that the product

$$\Delta[l,l_2(a)]^{\mathrm{it}}\Delta[l,l_2(b)]^{-\mathrm{it}}$$

converges for $t\to\infty$ strongly to a unitary operator $U^l(a,b)$. Furthermore, this operator depends only on the difference $a-b$, and gives rise to a unitary representation of the group $\Lambda^l(a)$ [(*see Eq.* (IV.4.2))]. By construction this group acts local. The continuity of this group representation follows from the relation between half-sided translations and the modular groups.

Third step: *construction of the rotations*

Up to now we have representations of the Lorentz boosts, the translations, and of the translational part of the stabilizer group of the light rays. In order to get a representation of the full Poincaré group one needs a representation of the rotation group.

Let x_0 be a timelike vector in the two-plane spanned by l and l', and let $\Lambda^l(a)$ be an element in the stabilizer group of l. Then $\Lambda^l(a)x_0$ is a vector on which we can apply $\Lambda^{l'}(b)$. There will be an element $b(a)$ such that $\Lambda^{l'}(b(a))\Lambda^l(a)x_0$ belongs to the two-plane containing l, l' and x_0. In this situation $s(a)$ exists such that $\Lambda[l,l'](s(a))$ maps this vector back to x_0. Therefore, the product represents a rotation

$$\Lambda[l,l'](s(a))\Lambda^{l'}(b(a))\Lambda^l(a)=R(l,a). \tag{IV.4.4}$$

Lengthy calculations show that the corresponding unitary groups

$$U(R(l,d,\varphi))=\Delta[l,l']^{\mathrm{is}(a)}U^{l'}(b(a))U^l(a) \tag{IV.4.5}$$

do not depend on l and that they depend continuously on d.

Special care has to be taken about the rotation π. This problem can be solved by showing that the rotations around a fixed axis d form a true representation of the circle group. Using this and Mackey's method of induced representation[86] one obtains a single valued representation of the whole rotation group. Putting all results together we get

IV.4.4. Theorem:
Assume the modular group of every wedge algebra $\mathcal{M}(W[l_1,l_2,a])$ *acts on every algebra of a double cone like the associated group of Lorentz boosts. Then the modular groups* $\Delta^{it}[l_1,l_2,a]$ *define a representation of the Poincaré group.*

IV.5. The approach of Buchholz and Summers

We saw that the Bisognano–Wichmann property for the modular groups implies Lorentz covariance, wedge duality, and the PCT theorem, provided the algebras of the double cones are the intersection of the wedge algebras. This implies in particular, that the modular conjugations of the wedge algebras act as reflection, i.e.,

$$J_W\mathcal{M}(D)J_W=\mathcal{M}(P_WD). \tag{IV.5.1}$$

Here P_W is the reflection in the characteristic two-plane of the wedge *W*, which leaves the apex of the wedge unchanged. If *a* is in the characteristic two-plane of *W* and $W=W(l_1,l_2,a)$ then with $x=\lambda l_1+\mu l_2+x^\perp$ one obtains

$$P_Wx=-\lambda l_1-\mu l_2+x^\perp+2a. \tag{IV.5.2}$$

If the theory fulfills Eq. (IV.5.1) for every double cone then we say it fulfills the Bisognano–Wichmann property for the modular conjugations. Since the Poincaré group is generated by the reflections (if the dimension of the Minkowski space is larger than two), it is natural to ask whether or not one can derive the Poincaré covariance also from the Bisognano–Wichmann property for modular conjugations. Using some additional assumptions this question has been answered for the translation positively by Buchholz and Summers.[87]

Since every double cone is the intersection of wedges, it is no restriction if one requires Eq. (IV.5.1) only for wedges. In a recent paper Buchholz, Dreyer, Florig, and Summers[88] have generalized this setting by requiring that the modular conjugation of every wedge algebra maps only the family of all wedge algebras onto itself. This contains a hidden version of the wedge duality. Adding to this the assumptions that the modular conjugations preserve (I) isotony and (II) stability of nonintersection, they were able to show the following: Every transformation *T* of the set of wedges onto itself, and which together with its inverse fulfills (I) and (II), is a Poincaré transformation. If, in addition, the considered set of transformations *T* is a group, which acts transitively on the set of wedges and if the Minkowski space is four dimensional, then this group contains the identity component of the Poincaré group. In a very recent paper Buchholz, Florig, and Summers[89] showed that the adjoint representation of the translations of this group, acting on the wedge algebras, is necessarily continuous.

The group representation obtained from the modular conjugations must not fulfill the spectrum condition. In order to obtain this condition one has to add additional assumptions. The authors of Ref. 88 called one of the possibilities the modular stability condition.

It is interesting to notice that the method of Buchholz, Summers, and co-workers can be transcribed to quantum field theories on de Sitter space. Whether or not this method can be generalized to other manifolds can only be answered by future calculations.

Here we will present the construction of the Poincaré group and show the continuity property of the translations. The continuity of the Lorentz transformations will only be discussed. Our construction of the Poincaré transformation differs in some points from that of Ref. 88.

In this section we define wedges slightly different from the notation in Sec. I.5. Here $W(l_1,l_2,a)$ means that the lightlike vectors l_1,l_2 belong either to $\partial V^+,\partial V^-$ or to $\partial V^-,\partial V^+$, i.e., $(l_1,l_2)<0$, and that the wedge $W(l_1,l_2,a+\rho_1l_1+\rho_2l_2)\subset W(l_1,l_2,a)$ for $\rho_1,\rho_2\geqslant 0$. This description is symmetric in both lightlike vectors and is better suited for dealing with time or space reflections.

IV.5.1. Definition:
Let $\mathcal{W}$ denote the set of all wedges. By $\mathcal{T}$ we denote the set of all transformations *T*,

$$T:\mathcal{W}\rightarrow\mathcal{W},$$

such that T^{-1} exists and T as well as T^{-1} fulfill:
(I) Isotony, i.e., $W_1\subset W_2$ implies $T(W_1)\subset T(W_2)$ and $T^{-1}(W_1)\subset T^{-1}(W_2)$.
(II) Stability of nonintersection, i.e., $W_1\cap W_2=\emptyset$ implies $T(W_1)\cap T(W_2)=\emptyset$ and $T^{-1}(W_1)\cap T^{-1}(W_2)=\emptyset$.

With these assumptions we will show:

IV.5.2. Theorem:
Let the dimension of the Minkowski space be larger than 2. *Then every transformation $T\in\mathcal{T}$ is an element of the full Poincaré group enlarged by the dilatations.*

The proof of this theorem is complicated. Therefore, we have to refer to the original paper,[88] or better to the complete script, where a shorter proof is given.

A special consequence of Thm. IV.5.2 is

IV.5.3. Theorem:
Let T_W fulfill the requirements IV.5.1, *then one has $T_W=P_W$, where P_W is the total reflection in the characteristic two-plane. This implies in particular the wedge duality*

$$T_W(W)=W'.$$

Let $\mathcal{T}_j$ be the subgroup generated by the modular conjugations of all the wedges in $\mathcal{W}$. Assume one is dealing with a QFT on a Hilbert space $\mathcal{H}$ and that there exists a vector $\Omega\in\mathcal{H}$, which is cyclic and separating for all wedge algebras $\mathcal{M}(W)$. Assume, moreover, that the modular conjugation J_W fulfills the relation

$$J_W\mathcal{M}(W_1)J_W=\mathcal{M}(T_W(W_1)),$$

$$J_WJ_{W_1}J_W=J_{T_WW_1},\quad J_W=J_{W'}.$$

Then the J_W generate an adjoint representation of the determinant $+1$ part of the Poincaré group.

It is easy to see that this representation is a central extension of the Poincaré group. Using the above relation one obtains for arbitrary W and the equation $\Pi_{i=1}^{-1}nT_{W_i}=1$

$$\prod_{i=1}^{n}J_{W_i}J_W=\prod_{i=1}^{n}J_{W_i}\ldots J_{T_{W_n}W}J_{W_n}=\cdots=J_{T_{W_1}\cdots T_{W_n}W}\prod_{i=1}^{n}J_{W_i}=J_W\prod_{i=1}^{n}J_{W_i}.$$

Therefore, $\Pi_{i=1}^{n}J_{W_i}$ belongs to the center of the group generated by the J_W's. We now restrict to the four-dimensional situation. Later we will see that the group representation is continuous. It remains to show that we are dealing with a true representation of the Poincaré group. We know from Sec. IV.4 how tedious such calculations are. Therefore, we skip this demonstration and refer to the original paper.[88]

Collecting the results we obtain

IV.5.4. Theorem:
The representation of the ''+'' *part of the Poincaré group induced by the J_W's is a true representation.*

Next we come to the continuity problem and its solution described in Ref. 89.

IV.5.5. Proposition:
Let $U(\Lambda,a)$ be the representation of the Poincaré group obtained by the products of the J_W's. Then $U(1,a)$ is strongly continuous.

The proof of this result is based on Thm. II.1.1 from which we know that the modular conjugations depend continuously on the algebras, if the algebras are continuous from inside (or outside).

The proof of the continuity of the Lorentz transformations will not be presented here. However, one can imagine how the above proof can be adapted to the situation where one looks at one-parametric subgroups $\Lambda(t)$ of the Lorentz group. One wants to compare the algebra $\mathcal{M}(\Lambda(t)W)$ with $\mathcal{M}(W)$. In order to do this one must assume, that Ω is also cyclic for the algebras $\mathcal{M}(\Lambda(t)W\cap W)$, provided t is sufficiently small. If this is the case one can look at the limit $t\searrow 0$ and argue as above.

Finally we come to the spectrum condition. As mentioned before, the representation of the translations induced by the J_W's does not have to fulfill it. In order to obtain the spectrum condition, Buchholz, Dreyer, Florig, and Summers introduced a new assumption, which they called:

IV.5.6. Modular stability condition:
The modular group of every wedge is contained in the group generated by the modular conjugations.

Since the group generated by the J_W's is the + part of the Poincaré group, it is easy to see that the modular group of the wedge coincides (up to a scale factor) with the group of the Lorentz boosts associated with the wedge. Since Ω is also cyclic for the shifted wedges one can conclude, as in Sec. IV.4, that the spectrum of the translations is contained in the closure of either V^+ or V^-. In order to obtain this result one can also use the method of Wiesbrock[59] which leads to the same conclusion.

We end this section with some

IV.5.7. Remarks:
(i) If one knows that the operators J_W fulfill all the conditions we have used in this section, and if one knows from other sources that the theory enjoys the spectrum condition, then the group generated by the J_W's must not necessarily contain the modular groups of the wedge algebras. Even in the situation where one knows that the J_W are modular conjugations and that the spectrum condition is fulfilled, a proof is missing that $\mathcal{T}_j$ contains the modular groups of the wedges.
(ii) There exist QFTLO's which do not fulfill wedge duality, or others where the Lorentz covariance is missing (also for the wedge algebras). Such theories do not fulfill the Bisognano–Wichmann property neither for the modular groups nor for the modular conjugations. Hence these criteria are a selection criterium for both, the field theory and the vacuum state. The criterium in Ref. 88 has the advantage that it also applies to certain theories without spectrum condition. If these methods apply to QFT's on curved manifolds this might be an advantage. Whether or not it is an advantage for theories on Minkowski space is a question of taste, in particular since the so-called modular stability requirement is a sufficient but not a necessary condition implying that the spectrum is contained in the forward or backward light cone.

IV.6. Remarks, additions, and problems

(I) If the local algebras are generated by Wightman fields with finite components then the result of Bisognano and Wichmann Thm. III.1.3 shows that the modular groups of the wedges coincide with the associated Lorentz boosts. On the other hand if we know the Bisognano–Wichmann property then we can derive Poincaré and PCT covariance for the local net (Secs. IV.4 and IV.3). But it is still an open problem whether or not the Bisognano–Wichmann property for a local net implies that this net is generated by Wightman fields. The existing attempts of constructing Wightman fields from local nets try to relate the field operator to the Hamilton operator (generator of the time translations H-bounds methods) Fredenhagen and Hertel.[90] It might be useful to try to find relations with respect to the modular operator of the algebra of the wedge.
(II) The construction of the Poincaré group from the modular groups of the wedges is possible if the Bisognano–Wichmann property holds. The first construction under this condition has been given by Brunetti, Guido, and Longo.[81] Their method is based on group cohomology and therefore more elegant than the method presented here. However, their method has the disadvantage that it leads to a representation of the covering group. In order to obtain a true group representation Guido and Longo[82] enlarged the group by the modular conjugations. In addition they incorporated charged fields. In this frame they proved the PCT and the spin and statistics theorem. This result implies that in the vacuum sector one has a true representation of the Poincaré group.
(III) In Tomita's modular theory one makes statements about the action of the modular group only

on the algebra and its commutant. Therefore, it is unnatural to formulate the Bisognano–Wichmann property for all local algebras $\mathcal{M}(D)$. It should only be formulated for such D which belong to W or to W'. If one does this one does not lose any information. This is a consequence of the following reason: The knowledge about the action inside W suffices to conclude that the algebras associated with the translates of a wedge along one of its defining lightlike vectors fulfill the condition of $\mp$half-sided modular inclusion with respect to $\mathcal{M}(W)$. With the help of Thm. II.6.2 one obtains the translations in the characteristic two-plane of W. Since by Thm. II.5.2 one knows the commutation between these translations and the modular group one can determine the action of this group on arbitrary $\mathcal{M}(D)$. One finds the full Bisognano–Wichmann property for the modular groups. This procedure has been worked out by Guido.[71]

Unfortunately the Bisognano–Wichmann property for the modular conjugations cannot be replaced by a local version. If we only know the action inside the wedge then we cannot compute the action of J_{W_1} on J_{W_2}. Therefore, we are not able to conclude that the products $J_{W_1}J_{W_2}$ give rise to a representation of a central extension of the Poincaré group. Hence if we assume that the modular group of the wedge algebra is contained in the group generated by the J_W's, we are not able to conclude that the modular groups fulfill the Bisognano–Wichmann property.

(IV) The Bisognano–Wichmann property for the modular groups is essential for the derivation of the CPT theorem. Since this condition is probably hard to verify in concrete examples, one has to look for conditions which imply this property. The whole Buchholz Summers program, if restricted to the Minkowski space, is of this nature. If we start from a Poincaré covariant theory, then the wedge duality and the reality condition also implies the Bisognano–Wichmann property for the modular groups. One should add other assumptions implying this property.

(V) If a Poincaré covariant QFTLO fulfills the Bisognano–Wichmann property for the modular groups then it can happen that the theory is covariant under two different representations of the Poincaré group. In this case holds:[91]

IV.6.1. Theorem:

Assume we are dealing with a local quantum field theory in the vacuum sector, which is covariant under two different vacuum representations of the Poincaré group. Let $U_0(\Lambda,a)$ be the representation generated by the modular groups of the wedge algebras and $U_1(\Lambda,a)$ the second representation. Then there exists a local gauge transformation of the Lorentz group $G(\Lambda)$ with

$$U_1(\Lambda,a)=U_0(\Lambda,a)G(\Lambda).$$

Moreover, $G(\Lambda)$ commutes with $U_0(\Lambda',a)$ for all a, Λ, Λ'. In addition $G(\Lambda)$ is a gauge transformation, i.e., it maps every local algebra onto itself.

That this situation occurs shows the following example: Take an infinite number of copies of a finite component Wightman field. Let $U(\Lambda,a)$ be the representation of the Poincaré group transforming the Wightman field. Let $G(\Lambda)$ be a representation of the Lorentz group which acts on the indices numbering the copies. Then $U(\Lambda,a)\otimes\mathbb{1}$ is the group generated by the modular groups and $U(\Lambda,a)\otimes G(\Lambda)$ is the second representation.

(VI) The reality condition together with the wedge duality implies the Bisognano–Wichmann property. Recently Guido and Wiesbrock (see Schroer and Wiesbrock[92]) have given a different condition which replaces the reality condition IV.2.1.

IV.6.2. Theorem:

Assume we are dealing with a QFTLO on the vacuum sector. Assume that for every wedge the map

$$A\Omega\to U(\Lambda_W(-\mathrm{i}/2))A^*\Omega$$

is bounded for $A\in\mathcal{M}(W)$. Here $U(\Lambda_W(t))$ denotes the group of boosts associated with W. Then the theory fulfills the Bisognano–Wichmann property.

(VII) Inspired by the result that $\mathcal{M}(W[l,l_1])\cap W[l,l_2]),l_1\neq l_2$ fulfills the condition of $-$half-sided modular inclusion with respect to both algebras $\mathcal{M}(W[l,l_1])$ and $\mathcal{M}(W[l,l_2])$ (see Thm. IV.4.3) Wiesbrock has introduced the concept of ''modular intersection.''

IV.6.3. Definition:
Let $\mathcal{M}$, $\mathcal{N}$ be two von Neumann algebras with a common cyclic and separating vector Ω. One says that $(\mathcal{M}, \mathcal{N}, \Omega)$ have the $\mp$modular intersection property if:
I. $\mathcal{M}\cap\mathcal{N}$ fulfills the condition of $\mp$half-sided modular inclusion with respect to both algebras $\mathcal{M}$ and $\mathcal{N}$.
II. There holds

$$J_{\mathcal{N}}(s-\lim_{t\to\pm\infty}\Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it})J_{\mathcal{N}}=(s-\lim_{t\to\pm\infty}\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it}).$$

In a QFTLO which fulfills the Bisognano–Wichmann property the modular intersection condition is fulfilled for the algebras of two wedges which have the first- or the second light ray in common. The condition II can be derived from Thm. IV.4.3. In particular the existence of the strong limit is guaranteed by the first condition. If we set $(s-\lim_{t\to\pm\infty}\Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it})=U$ then condition II reads $J_{\mathcal{N}}UJ_{\mathcal{N}}=U^*$.

Using a finite number of pairs fulfilling the condition of modular intersection one is able to reconstruct the algebras of all nontranslated wedges. This program has been taken up by Wiesbrock,[93,94] where he solved the problem for $\mathbb{R}^3$. Here he needs three wedges which are localized in such a way that the algebras of every pair fulfills the condition of $-$ or $+$modular intersection. Adding one shifted wedge which fulfills the condition of half-sided modular inclusion, he was able to construct the algebras of all wedges (including the translated ones) and a continuous representation of the Poincaré group which fulfills the spectrum condition.

Taking the intersection of wedge algebras on can construct the algebras for the double cones. Unfortunately one is not able to conclude that Ω is also cyclic for these algebras except one starts from a QFTLO.

V. PROPERTIES OF LOCAL ALGEBRAS

For several applications one wants to know the structure of the local algebras. The questions of interest are usually the factor property, the type of the algebra, and the action of symmetry groups. Before entering into the subject we have to collect some results of the Tomita–Takesaki theory.

V.1. Some mathematical consequences of the modular theory

The first concept is the generalization of the center of a von Neumann algerbra.

V.1.1. Definition:
Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Set $\omega(A)=(\Omega,A\Omega)$, $A\in\mathcal{M}$. The centralizer of ω consists of all elements $Z\in\mathcal{M}$ for which

$$\omega(ZA)=\omega(AZ), \quad \forall \quad A\in\mathcal{M}$$

holds.

If Z belongs to the centralizer, then the KMS condition implies

$$\sigma^t(Z)=Z, \quad t\in\mathbb{R}$$

and vice versa. In particular the center of $\mathcal{M}$ belongs to the centralizer.

It might happen that a von Neumann algebra is too large in order to possess separating states. In this case one has to generalize the concept of states. They are called weights.

V.1.2. Definition:
(a) Let $\mathcal{M}$ be a von Neumann algebra. A weight is a mapping

$$\omega:\mathcal{M}^+\to[0,\infty]$$

with the properties:
(α) $\omega(\rho A)=\rho\omega(A)$, $\rho\in\mathbb{R}^+, A\in\mathcal{M}^+$

with the multiplication rule $0.\infty = 0$.
(β) $\omega(A+B) = \omega(A) + \omega(B)$, $A,B \in \mathcal{M}^+$
(b) A weight ω is called semifinite if

$$n_\omega := \{A \in \mathcal{M}; \omega(A^*A) < \infty\}$$

is strongly dense in $\mathcal{M}$.
(c) ω is called faithful if $A \in \mathcal{M}^+$ and $\omega(A) = 0$ implies $A = 0$.
(d) A weight is called normal if for every increasing net $A_\alpha \in \mathcal{M}^+$ there holds

$$\omega(\lim_\alpha A_\alpha) = \lim_\alpha \omega(A_\alpha).$$

The set n_ω is a linear space and by the linear extension of ω this becomes a pre-Hilbert space. Moreover, n_ω is a left ideal so that one gets a representation of $\mathcal{M}$ by

$$\pi_\omega(B)A\rangle = BA\rangle.$$

If ω is a normal, faithful, semifinite weight, then one can handle the Tomita–Takesaki theory in almost the same manner as with normal faithful states. (See Haagerup, Ref. 36.) The advantage of this concept is the existence of normal, faithful, semifinite weights for every von Neumann algebra. We need weights only for the discussion of symmetries in Sec. V.4. Otherwise we use only von Neumann algebras which have normal, faithful states.

Another important aspect of the Tomita–Takesaki theory is the natural cone associated with a von Neumann algebra. It is often denoted by $\mathcal{P}^\natural$. Here we will use the notation $\mathcal{H}^+$.

V.1.3. Lemma:
Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector Ω. Let (Δ, J) be the modular operator and conjugation of $(\mathcal{M}, \Omega)$. Then the following sets coincide and are called the natural cone of $(\mathcal{M}, \Omega)$.
(i) Closure of $\Delta^{1/4}\mathcal{M}^+,\Omega$.
(ii) Closure of $\Delta^{-1/4}\mathcal{M}'^+\Omega$.
(iii) Closure of $\{Aj(A)\Omega; A \in \mathcal{M}\}$.

For the proof see Ref. 32, Prop. 2.5.26. Some of the properties of $\mathcal{H}^+$ are listed in the following:

V.1.4. Proposition:
Let $\mathcal{H}^+$ be the natural cone of $(\mathcal{M},\Omega)$. Then the following holds:
(i) $\mathcal{H}^+$ is a proper cone, i.e., $\mathcal{H}^+ \cap (-\mathcal{H}^+) = \{0\}$.
(ii) With $\mathcal{H}_r = \{\psi \in \mathcal{H}; J\psi = \psi\}$ one gets $\mathcal{H}_r = \mathcal{H}^+ - \mathcal{H}^+$.
(iii) $\mathcal{H}^+$ is a self-dual cone in $\mathcal{H}_r$, i.e., $\psi \in \mathcal{H}_r$ and $(\psi,\varphi) \geq 0 \forall \varphi \in \mathcal{H}^+$ implies $\psi \in \mathcal{H}^+$.
(iv) For every $\psi \in \mathcal{H}^+$ and $A \in \mathcal{M}$ one has $Aj(A)\psi \in \mathcal{H}^+$.
(v) $\Delta^{it}\mathcal{H}^+ = \mathcal{H}^+$ for all $t \in \mathbb{R}$.

For the proof see Ref. 32, Props. 2.5.26, 2.5.27, 2.5.28. The natural cone has some universality properties listed in the following:

V.1.5. Theorem:
Let $\mathcal{H}^+$ be the natural cone of $(\mathcal{M}, \Omega)$. Then:
(i) To every normal, positive linear functional ω on $\mathcal{M}$ exists a unique vector $\psi_\omega \in \mathcal{H}^+$ with

$$\omega(A) = (\psi_\omega, A\psi_\omega), \quad A \in \mathcal{M}.$$

(ii) The mapping $\omega \leftrightarrow \psi_\omega$ is continuous in both directions. The following estimate holds:

$$\|\psi_\omega - \psi_\rho\|^2 \leq \|\omega - \rho\| \leq \|\psi_\omega - \psi_\rho\| \|\psi_\omega + \psi_\rho\|.$$

(iii) Assume the vector $\psi \in \mathcal{H}^+$ is cyclic and separating for $\mathcal{M}$ then the natural cones

$$\mathcal{H}^+(\mathcal{M},\Omega) \quad \text{and} \quad \mathcal{H}^+(\mathcal{M},\psi)$$

coincide.

(*iv*) *Let* $\alpha \in \mathrm{Aut}\,\mathcal{M}$ *and define*

$$U(\alpha)\psi_\omega = \psi_{(\alpha^{-1*}\omega)}$$

then by linearity this map can be extended to all of $\mathcal{H}$. *This extension is a unitary operator. The set*

$$\{U(\alpha); \alpha \in \mathrm{Aut}\,\mathcal{M}\}$$

defines a unitary representation of Aut $\mathcal{M}$, *the adjoint action of which implements the automorphisms*.

For the proof see Ref. 32, Thm. 2.5.31, Prop. 2.5.30, Cor. 2.5.32. Another important result is due to Connes[95] which says that the algebras $\mathcal{M}$ and $\mathcal{M}'$ are uniquely characterized by the natural cone. First some notations:

V.1.6. Definition:

(i) A face of a cone C is a subcone $F \subset C$ with $a,b \in C, a < b$ in the order of the cone C and $b \in F$ implies $a \in F$.

(ii) The set $\mathcal{D}(\mathcal{H}^+) := \{\delta \in \mathcal{B}(\mathcal{H}); e^{t\delta}\mathcal{H}^+ = \mathcal{H}^+ \forall t \in \mathbb{R}\}$ is a Lie algebra.

(iii) A map $I:\mathcal{D}(\mathcal{H}^+) \to \mathcal{D}(\mathcal{H}^+)$ is called an orientation of $\mathcal{H}^+$ if it fulfills: $I^2 = -1$, $[I\delta_1, \delta_2] = [\delta_1, I\delta_2] = I[\delta_1, \delta_2]$ and $I(\delta^*) = -I(\delta)^*$. [To be precise, for this definition one first has to devide $\mathcal{D}(\mathcal{H}^+)$ by its center.]

(iv) Let F be a face of $\mathcal{H}^+$, then $F^\perp$ denotes the face of $\mathcal{H}^+$ which is perpendicular to F. By a result of Connes one has closure $F = F^{\perp\perp}$. P_F denotes the projection onto the Hilbert subspace generated by F. $\mathcal{H}^+$ is called facially homogeneous if $e^{t(P_F - P_{F^\perp})}\mathcal{H}^+ = \mathcal{H}^+$, $t \in \mathbb{R}$ and this for all faces F of $\mathcal{H}^+$.

The concept of orientation and homogeneity can also be formulated for arbitrary cones. The result of Connes is the following:

V.1.7. Theorem:

There is a one to one correspondence between von Neumann algebras $\mathcal{M}$ *acting on* $\mathcal{H}$ *and selfdual, orientable, and facially homogeneous cones of* $\mathcal{H}$.

Von Neumann has classified the factors by three types denoted by I, II, and III. For a long time there were only very few different type III factors known. Using canonical anticommutation relations, Powers[96] was able to construct a continuous family of different type III factors. An attempt to classify these factors were made by Araki and Woods.[97] The question of the classification has finally been settled by Connes.[98] This classification is based on the invariant S which is defined as follows:

V.1.8.Definition:

Let $\mathcal{M}$ be a von Neumann algebra and ω be a normal weight on $\mathcal{M}$. Let $E \in \mathcal{M}$ be the support of ω. Then ω is faithful on $E\mathcal{M}E$. Hence there exists a modular operator Δ_ω for this algebra. One defines:

$$S(\mathcal{M}) = \cap\{\text{spectrum } \Delta_\omega; \omega \text{ is a normal, semifinite weight on} \mathcal{M}\}.$$

If $\mathcal{M}$ is of type III, then there are the following possibilities:

V.1.9. Theorem:

Let $\mathcal{M}$ *be a type III factor, then for the Connes invariant exist the following possibilities:*

(1) $S(\mathcal{M}) = \{0,1\}$,

(2) $S(\mathcal{M}) = \underline{\{0\}} \cup \{\lambda^n; n \in \mathbb{Z}, 0 < \lambda < 1\}$,

(3) $S(\mathcal{M}) = \mathbb{R}^+$.

If $S(\mathcal{M})$ is {1} then $\mathcal{M}$ is not of type III.

V.1.10. Notation:
A factor with $S(\mathcal{M})=\{0,1\}$ is called a III_0-factor. The factors with the set (2) are called III_λ, and those with $S(\mathcal{M})=\overline{\mathbb{R}^+}$ are named III_1 factors.

Let $\mathcal{M}$ be a von Neumann algebra and ω be a normal faithful state on $\mathcal{M}$. Then it can happen, that for some $t\in\mathbb{R}$ the modular transformation σ_ω^t is inner, i.e., there exists a unitary $U\in\mathcal{M}$ with $\sigma_\omega^t(A)=UAU^*$, $A\in\mathcal{M}$. In this case one shows

$$\Delta_\omega^{it}=UJ_\omega UJ_\omega . \tag{V.1.1}$$

If σ_ω^t is inner for one normal faithful state then this is true for every such state.

Connes[98] has introduced the invariant $T(\mathcal{M})$, consisting of all $t\in\mathbb{R}$ such that σ^t is inner. It is clear that $T(\mathcal{M})$ is a subgroup of $\mathbb{R}$. For instance an algebra $\mathcal{M}$ is semifinite iff $T(\mathcal{M})=\mathbb{R}$. We do not need the full relation between $T(\mathcal{M})$ and $S(\mathcal{M})$. We are only interested in the type III_1 case. The result is the following:

V.1.11. Theorem:
A von Neumann factor is of Type III_1 iff $T(\mathcal{M})=\{0\}$. This means that all σ^t, $t\neq 0$ are outer automorphisms of $\mathcal{M}$.

In every class $III_\lambda, 0\leqslant\lambda\leqslant 1$ no classification is known except for one algebra. These are the hyperfinite factors.

V.1.12. Definition:
A factor $\mathcal{M}$ is called hyperfinite if there exists an increasing net $\mathcal{N}_\alpha\subset\mathcal{M}$ of type I algebras with

$$\mathcal{M}=\{\bigcup_\alpha\mathcal{N}_\alpha\}''.$$

The importance of this concept is the following result.[99,100]

V.1.13. Proposition:
Every of the classes III_λ contains exactly one element which is hyperfinite.

V.2. The factor problem

The locality and the spectrum conditions together with the existence of a vacuum vector imply that the global algebra is of type I. One finds that the commutant of the algebra $\mathcal{M}(\mathbb{R}^d)$ is Abelian, and that the projection E_0 onto all translational invariant vectors is an Abelian projection in $\mathcal{M}$ with central support $\mathbb{1}$. In this case the center is pointwise invariant under the translations. This has first been observed by Araki.[101] The properties of the projection E_0 is a consequence of the cluster property.

The first proof of the cluster property is due to the author.[102] A systematic study of this property was started by Doplicher, Kadison, Kastler, and Robinson[103] using the notation of asymptotic Abelian systems introduced by Doplicher, Kastler, and Robinson in Ref. 104 and independently by Ruelle.[105] This notation has been weakend by Lanford and Ruelle[106] introducing the concept of G-Abelian systems. The most general concept leading to the cluster property has been introduced by Størmer.[107] He called it large groups of automorphisms. One important consequence of the cluster property of the vacuum state is the additivity of the spectrum. The result is due to Wightman.[108]

Next we are looking at the algebra of the wedge. Here the following result is known:

V.2.1. Theorem:
Assume we are dealing with a QFTLO on the vacuum sector. Let $\mathcal{M}(W)$ be the algebra of the wedge domain. Then

$$\mathcal{Z}(\mathcal{M}(W))\subset\mathcal{Z}(\mathcal{M}(\mathbb{R}^d)),$$

where $\mathcal{Z}(\mathcal{M})$ denotes the center of $\mathcal{M}$.

This result has first been obtained by Driessler.[109] Our demonstration is taken from Ref. 60. First we show a result which has its interest of its own, and from which Thm. V.2.1 follows easily.

V.2.2. Lemma:
Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Assume $U(s) \in \mathcal{H}str(\mathcal{M})^+$ or $U(s) \in \mathcal{H}str(\mathcal{M})^-$. Then:

a. If we write $U(s) = \mathrm{e}^{\mathrm{i}Ht}$ and denote by $\mathcal{D}(H)$ the domain of definition for H then

$$\Delta^{it}\mathcal{D}(H) \subset \mathcal{D}(H).$$

b. If E_0 denotes the projection onto the eigenspace to the value 0 of H then E_0 commutes with Δ^{it}.

c. If F_1 denotes the projection onto the eigenspace to the value 1 of Δ, then one has

$$F_1 \leqslant E_0.$$

Proof: We show the lemma for $U(s) \in \mathcal{H}str(\mathcal{M})^+$. For $U(s) \in \mathcal{H}str(\mathcal{M})^-$ the arguments are essentially the same.
a. Let $\varphi, \psi \in \mathcal{D}(H)$ then we obtain from Thm. II.5.2 $(\varphi, \Delta^{\mathrm{it}} H\psi) = \mathrm{e}^{-2\pi t}(H\varphi, \Delta^{\mathrm{it}}\psi)$. Since the left side is continuous in φ it follows that $\Delta^{\mathrm{it}}\psi \in \mathcal{D}(H)$.
b. Let $H\psi = 0$ then we obtain $0 = \Delta^{\mathrm{it}}H\psi = He^{-2\pi t}\Delta^{\mathrm{it}}\psi$. From this we conclude $\Delta^{\mathrm{it}}E_0\mathcal{H} \subset E_0\mathcal{H}$. Because of the group property of Δ^{it} we get $\Delta^{\mathrm{it}}E_0\mathcal{H} = E_0\mathcal{H}$.
c. Keep s real and $s \geqslant 0$. From the assumption $\mathrm{Ad}\, U(s)\mathcal{M} \subset \mathcal{M}$ for $s \geqslant 0$ and from $\mathcal{D}(\Delta^{1/2}) = \{X\Omega; X\,\eta\mathcal{M}, \Omega \in \mathcal{D}(X) \cap \mathcal{D}(X^*)\}$ we conclude that on $\mathcal{D}(\Delta^{1/2})$ the relation $\Delta^{\mathrm{it}}U(s) = U(e^{-2\pi t}s)\Delta^{it}$ can be analytically continued in t as long as $-\frac{1}{2} \leqslant \Im m\, t \leqslant 0$. If we choose $t = -\mathrm{i}\frac{1}{4}$ then we find

$$\Delta^{1/4}U(s) = \mathrm{e}^{-Hs}\Delta^{1/4}, \quad s \geqslant 0.$$

Multiplying this equation from both sides with F_1 we find $F_1U(s)F_1 = F_1\mathrm{e}^{-Hs}F_1$. This is only possible for $E_0 \geqslant F_1$.

Next we have to show that the elements in $\mathcal{Z}(\mathcal{M})$ commute with the half-sided translations.

V.2.3.Lemma:
Let $U(t) \in \mathcal{H}str(\mathcal{M})^+$, then

$$[U(t), Z] = 0 \quad \forall Z \in \mathcal{Z}(\mathcal{M}) \quad \textit{and} \quad \forall t \in \mathbb{R}.$$

This result can also be found in Ref. 109.

Proof: Let $Z = Z^* \in \mathcal{Z}(\mathcal{M})$ and set $Z_t = \mathrm{Ad}\, U(t)Z$. For $t \geqslant 0$ the element Z_t belongs to $\mathcal{M}$ and for $t \leqslant 0$ to $\mathcal{M}'$. This implies that Z commutes with Z_t for all $t \in \mathbb{R}$. Applying $\mathrm{Ad}\, U(s)$ to the commutator we obtain $[Z_{t_1}, Z_{t_2}] = 0$. Hence $\{Z_t\}$ generates an Abelian von Neumann algebra invariant under $U(t)$. Since $U(t)$ has a positive generator it follows that $\mathrm{Ad}\, U(t)$ is inner in $\{Z_t\}''$.[110] This implies $Z_t = Z$. □

For the algebras of the double cones no similar result can be obtained. Even in the case where $\mathcal{M}(\mathbb{R}^d)$ is a factor, one can easily construct examples where $\mathcal{M}(D)$ has a nontrivial center. [See V.5.(II).] Up to now there are no conditions known, implying, that $\mathcal{M}(D)$ is a factor.

V.3. The type question

From the investigations of Kadison[61] and from Guenin and Misra[62] it is known that the local algebras cannot be of finite type. In 1967, Borchers[111] showed the following result:

V.3.1. Theorem:
(1) *Let $\mathcal{O}_1 \subset \mathcal{O}_2$ such that there exists $\mathcal{O}_3 \subset (\mathcal{O}_2 \cap \mathcal{O}_1')$. Assume E is a projection in $\mathcal{M}(\mathcal{O}_1)$, then E is equivalent to its central support in $\mathcal{M}(\mathcal{O}_2)$, $\mathrm{mod}\, \mathcal{M}(\mathcal{O}_2)$.*
(2) *If $\mathcal{O}_1 + x \subset \mathcal{O}_2$ for x in some open neighborhood of $\mathbb{R}^d$, then the central support of E in $\mathcal{M}(\mathcal{O}_2)$ belongs to the center of the global algebra.*

There is not known more under the general assumptions. If one wants to obtain better results, one has to impose additional requirements.

The situation is much better for the algebra of the wedge. This is due to the existence of half-sided translations. The first result in this direction is due to Driessler.[109] But he uses the additional assumption that the spectrum has a mass gap. Here we follow the method of Longo,[112] with a slight variation, applying Thm. V.1.11. There exists also a proof which uses the invariant $S(\mathcal{M})$ and Prop. V.1.9. (See Ref. 60.)

V.3.2. Theorem:
In a QFTLO on the vacuum Hilbert space with one vacuum vector, the algebra $\mathcal{M}(W)$ is of type III_1.

This result has used only the existence of half-sided translations. Therefore, the theorem remains true for arbitrary algebras with half-sided translation. In conformal field theory these are the algebra of the forward light cone and the algebras of the double cones.

The determination of the type of local algebras $\mathcal{M}(D)$ is burdened with some difficulties. It is known from examples, as the free massive field, that local algebras fulfill the split property[113] if specific conditions are fulfilled. This property is the following: Let $D_1 \subset D$ be such that $D_1 + x \subset D$ for x in some open neighborhood of the origin. In that case one can find a type I algebra $\mathcal{N}$ with $\mathcal{M}(D_1) \subset \mathcal{N} \subset \mathcal{M}(D)$. This implies that one cannot expect any statement about the type from purely local considerations. Some more information about the structure of $\mathcal{M}(D)$ has to be used.

This difficulty has been circumvented by Fredenhagen[114] by observing that there exists no intermediate type I algebra if the domains D_1 and D have boundary points in common. Therefore, he puts the double cone D into the corner of the wedge and tries to compare the Connes invariant S of $\mathcal{M}(D)$ and $\mathcal{M}(W)$. To do this he needs the assumption that the local algebras are generated by Wightman fields which have the Haag–Narnhofer–Stein property.[115]

Let us first explain this concept. Let $\Phi(x)$ be a Wightman field, then we say for $\Phi(x)$ exists a scaling limit if there exists a non-negative function $N(\lambda)$ defined for $\lambda > 0$ such that for all n

$$N(\lambda)^n(\Omega, \Phi(\lambda x_1) \cdots \Phi(\lambda x_n)\Omega)$$

converges for $\lambda \to 0$ to some nontrivial Wightman functional. With this concept we introduce the following

V.3.3. Requirement:
There exists a Wightman field $\Phi(x)$ such that:
(i) For every $f \in \mathcal{D}$ with supp. $f \in \mathcal{D}$ the operator $\Phi(f)$ is affiliated with $\mathcal{M}(D)$.
(ii) $\Phi(x)$ fulfills the Haag–Narnhofer–Stein scaling property.
(iii) The theory fulfills the Bisognano–Wichmann property. (If the set of Wightman fields, which fulfill (i), generate $\mathcal{M}(D)$ then (iii) is implied by the result of Bisognano and Wichmann Thm. 3.1.3.)

With this requirement Fredenhagen has shown the following result:

V.3.4. Theorem:
We are dealing with a QFTLO in the vacuum sector, such that the global algebra is a factor, and which fulfills the Requirement V.3.3. *Let W be a wedge such that zero belongs to its edge. Let $D \subset W$ be a double cone such that zero belongs to the boundary of D. Let $\mathcal{N}$ be a von Neumann algebra with*

$$\mathcal{M}(D) \subset \mathcal{N} \subset \mathcal{M}(W).$$

Then $\mathcal{N}$ is of type III_1.

For details of the proof see the complete file or the original article of Fredenhagen.[116]

More about the structure of the local algebras can be said if in addition one assumes the nuclearity condition introduced by Buchholz and Wichmann.[117]

First we must explain this concept. Let H be the generator of the time translation and Ω the vacuum vector. The map $\Theta_\beta : \mathcal{M} \to \mathcal{H}$ defined by

$$\Theta_\beta(A) = e^{-\beta H} A\Omega$$

is called nuclear if one can write it

$$\Theta_\beta(A) = \sum_n \varphi(A)\psi_n, \quad \varphi \in \mathcal{M}^*, \quad \psi_n \in \mathcal{H} \tag{V.3.1}$$

with $\Sigma_n \|\varphi_n\| \|\psi_n\| < \infty$.

The expression

$$\mathcal{N}(\Theta_\beta) := \inf\left\{ \sum_n \|\varphi_n\| \|\psi_n\| \right\},$$

where the infimum is taken over all possible representations Eq. (V.3.1). Buchholz and Wichmann suggested the nuclearity condition by comparing the situation in a bounded region with that of a thermodynamical system in a box. If one does so, one obtains some suggestion about the behavior of the norm $\mathcal{N}(\Theta_\beta)$ as function of β, the dimension of the Minkowski space and the diameter of the double cone D, when Θ_β is applied to $\mathcal{M}(D)$. In the coming investigation we only need the behavior in β. This we formulate as an assumption.

V.3.5. Condition:
We say a QFTLO fulfills the Buchholz–Wichmann property if the map $\mathcal{M}(D) \to \mathcal{H}$ defined by

$$\Theta_\beta(A) = e^{-\beta H} A\Omega, \quad A \in \mathcal{M}(D)$$

is nuclear and the nuclear norm fulfills the estimate

$$\mathcal{N}(\Theta_\beta) \leqslant M e^{(\beta_0/\beta)^n},$$

where M, β_0, n are constants which may depend on the dimension of the space and the diameter of the double cone D.

With help of this condition Buchholz, D'Antoni, and Fredenhagen[118] showed the following result:

V.3.6. Theorem:
Assume a QFTLO fulfills the Buchholz–Wichmann property, Condition V.3.5. Let $D_1 \subset D$ such that the closure of D_1 is contained in the interior of D. Then there exists a type I factor $\mathcal{P}$ with

$$\mathcal{M}(D_1) \subset \mathcal{P} \subset \mathcal{M}(D).$$

For the proof of this theorem we refer to the original paper.[118] We want to combine this result with Thm. V.3.4 and obtain:

V.3.7. Theorem:
Assume we are dealing with a QFTLO in the vacuum sector. Assume that the theory fulfills the Haag–Narnhofer–Stein assumption, Requirement V.3.3, and the Buchholz–Wichmann property, Condition III.5.5. Assume in addition that $\mathcal{M}(D)$ is continuous from inside or from outside. (The first statement means $\mathcal{M}(D) = \{\cup \mathcal{M}(D_i)\}''$ with closure $D_i \subset$ interior D_{i+1} and $\cup D_i = D$.) *Then every local algebra is isomorphic to*

$$\mathcal{M}(D) \cong \mathcal{R} \overline{\otimes} \mathcal{Z},$$

where $\mathcal{R}$ is the unique hyperfinite type III_1 factor and $\mathcal{Z}$ is the center of $\mathcal{M}(D)$.

V.4. On the implementation of symmetry groups

Assume we are describing a physical theory in terms of a C^*-algebra $\mathcal{A}$ and a symmetry group G, i.e., we have a representation of G by automorphisms of $\mathcal{A}$

$$\alpha: G \rightarrow \mathrm{Aut}(\mathcal{A}).$$

This situation is usually called a C^*-dynamical system and denoted by the triple $\{\mathcal{A}, G, \alpha\}$. For applications it is of interest to characterize those representations π of $\mathcal{A}$, for which there exists in $\mathcal{H}_\pi$ a continuous unitary representation $U(g)$ of the symmetry group which implements the automorphism:

$$U(g)\pi(x)U^*(g) = \pi(\alpha_g x). \tag{V.4.1}$$

Let α_g act strongly continuous, which means that the function $g \rightarrow \alpha_g(A)$ is a continuous function on G with values in the normed space $\mathcal{A}$. If in addition the group is locally compact, then one can integrate over the group. This led Doplicher, Kastler, and Robinson[104] to introduce the C^*-completion of the algebra of continuous $\mathcal{L}^1$ functions on G with values in $\mathcal{A}$. They called it the covariance algebra. Nowadays it is called the crossed product of $\mathcal{A}$ with G. The importance of the covariance algebra stems from the fact that there is a one to one correspondence of covariant representations of $\mathcal{A}$ and representations of the covariance algebra. For details see the book of Pedersen.[52]

If one is dealing with a C^*-dynamical system and a representation $\{\pi, \mathcal{H}\}$ of $\mathcal{A}$, then it is usually hard to decide whether or not this representation can be extended to a representation of the covariance algebra. The difficulties are twofold: If $\pi(\mathcal{A})$ has a center then the multiplicity problem may appear. Moreover, by passing to the adjoint representation of the group, one has to be aware of central extensions of the group. Both problems can be circumvented by passing to quasi-equivalent representations. The reason for the first problem is clear. The reason for the second problem is the following: If $U(g)$ is a ray representation of G on $\mathcal{H}$, then there exists a second representation $\hat{U}(g)$ which is also a ray representation, but with the complex conjugate phase factor. Therefore $U(g) \otimes \hat{U}(g)$ is a representation of the group on $\mathcal{H} \otimes \overline{\mathcal{H}}$. Replacing π by $\pi \otimes \mathbb{1}$ we obtain a covariant representation. This leads to the following notation:

V.4.1. Definition:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system and $\{\pi, \mathcal{H}\}$ be a representation of $\mathcal{A}$ then $\{\pi, \mathcal{H}\}$ is called quasicovariant, if there exists a covariant representation $\{\pi_1, U, \mathcal{H}_1\}$ such that $\{\pi, \mathcal{H}\}$ and $\{\pi_1, \mathcal{H}_1\}$ are quasi-equivalent.

Quasicovariant representations are much easier to characterize than covariant representations. The first result was obtained in Ref. 119 which was based on the assumptions of strong continuity and the locally compactness of the group. Some time later Borchers[120] observed, that it is neither necessary to assume that α_g acts strongly continuous nor that G is locally compact. To prove this the natural cone will be used, in particular Thm. V.1.5.(iv).

V.4.2. Theorem:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system. Let π be a representation of $\mathcal{A}$. Then this representation is quasicovariant iff:

(α) *The dual action α_g^* maps the folium of $\pi(\mathcal{A})$ onto itself.*

(β) *α_g^* acts strongly continuous on the folium of π. This means the function*

$$g \rightarrow \alpha_g^*(\omega)$$

is a continuous function on G with values in the folium of π, furnished with the norm topology.

The folium of a representation is the set of states, which extend to normal states of $\pi(\mathcal{A})''$.

The proof is a simple consequence of Thm. V.1.5,

This result suggests to investigate closer that part of $\mathcal{A}^*$ on which α_g^* acts strongly continuous. We introduce:

V.4.3. Definition:

By $\mathcal{A}_c^*$ we denote the set of $\phi \in \mathcal{A}^*$, ($\mathcal{A}^*$ denotes the topological dual of $\mathcal{A}$), such that for every $\epsilon > 0$ exists a neighborhood $\mathcal{U}$ of the identity of G such that

$$\|\phi \circ \alpha_g - \phi\| \leq \epsilon$$

holds for $g \in \mathcal{U}$.

Some properties of this set are described in the following:

V.4.4. Proposition:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system and assume $G(\tau)$ is a topological group, then the space $\mathcal{A}_c^*$ has the following properties:

(i) *$\mathcal{A}_c^*$ is a linear norm-closed space.*

(ii) *$\mathcal{A}_c^*$ is invariant under the action of the group, i.e., $\phi \in \mathcal{A}_c^*$ implies $\phi \circ \alpha_g \in \mathcal{A}_c^*$ for every $g \in G$.*

(iii) *With $\phi \in \mathcal{A}_c^*$ one finds also that ϕ^* and $|\phi|$ belong to $\mathcal{A}_c^*$. $\mathcal{A}_c^*$ is generated by its positive elements.*

Since this result has no connection with the Tomita–Takesaki theory, we refer for the proof to the original paper.[120]

Recall that for every positive linear functional $\omega \in \mathcal{A}^+$ exists a vector $\xi_\omega \in \mathcal{H}^+$, ($\mathcal{H}^+$ denotes the natural cone of $\mathcal{A}^{**}$) with $\omega(A) = (\xi_\omega, A\xi_\omega)$. Next we introduce some concepts:

V.4.5. Notation:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system with G being a topological group. Let $\mathcal{H}$ be the Hilbert space of the standard representation of $\mathcal{A}^{**}$ and let $\mathcal{H}^+$ be the natural cone associated with this representation then we denote

(i) $\mathcal{H}_c^+ = \{\psi_\omega ; \omega \in (\mathcal{A}_c^*)^+\}$.

(ii) $\mathcal{H}_c =$ *smallest sub-Hilbert-space of $\mathcal{H}$ containing $\mathcal{H}_c^+$.*

(iii) *Denote the canonical involution associated with the standard representation of $\mathcal{A}^{**}$ by J.*

(iv) *The algebra $\mathcal{A}^{**}$ will usually be denoted by $\mathcal{M}$. Then $\mathcal{A}^*$ and $\mathcal{M}_*$ are the same space.*

About this set we know:

V.4.6. Proposition:

With the assumptions and notations of V.4.5 one obtains

(i) *$\mathcal{H}_c^+$ is a closed cone.*

(ii) *The space $\mathcal{H}_c$ is invariant under the canonical involution J.*

(iii) *If $\mathcal{H}_c^r$ denotes the vectors $\psi \in \mathcal{H}_c$ with $J\psi = \psi$ then $\mathcal{H}_c^+$ is a self-dual cone in $\mathcal{H}_c^r$ and $\mathcal{H}_c$ is algebraically generated by $\mathcal{H}_c^+$.*

(iv) *If P_c denotes the projection onto $\mathcal{H}_c$ then for every $\psi \in \mathcal{H}^+$ one has $P_c\psi \in \mathcal{H}_c^+$.*

The proof of this proposition uses Prop. V.4.4 and the modular theory. In particular the properties of the natural cone described in Sec. V.1 are used.

In order to investigate the structure of $\mathcal{A}_c^*$ in some detail one must look at the cone $\mathcal{H}_c^+$. By this one wants to show that $\mathcal{H}_c^+$ is the natural cone of some von Neumann algebra $\mathcal{N}_c$. One has to show that the cone is facial homogeneous and oriented in the sense of Connes.[95] (See also Def. V.1.6.) If this is done, then one wants to connect the algebra $\mathcal{N}_c$ with some von Neumann subalgebra of $\mathcal{M} = \mathcal{A}''$.

In order to do this the following von Neumann algebras have to be introduced.

V.4.7. Definition:

(1) We define

$$\mathcal{M}_c^0 = \{A \in \mathcal{M}; [A, P_c] = 0\}.$$

(2) Let $A\omega(.) := \omega(A.)$ and $\omega A(.) := \omega(.A)$. Then we put

$$\mathcal{M}_m^0 = \{A \in \mathcal{M}; A\omega \in \mathcal{M}_{*,c}, \omega A \in \mathcal{M}_{*,c}, \forall \omega \in \mathcal{M}_{*,c}\}.$$

(3) Let E_c be the smallest projection in $\mathcal{M}$ with $E_c P_c = P_c$.

All these objects are invariant under α_g. First note that both sets are von Neumann algebras. The two algebras are not different. We have

V.4.8. Lemma:

(1) *The two algebras $\mathcal{M}_c^0$ and $\mathcal{M}_m^0$ coincide.*

(2) *Every element in $\mathcal{M}_c^0$ commutes with E_c.*

It turns out that the algebra $\mathcal{M}_c$ can be used for defining an orientation for the cone $\mathcal{H}_c^+$. By this $\mathcal{N}_c$ can be identified with $\mathcal{M}_c$. The precise result is

V.4.9. Theorem:

(1) *The cone $\mathcal{H}_c^+$ is facial homogeneous and oriented and is, therefore, the natural cone of a von Neumann algebra $\mathcal{N}_c$.*

(2) *The von Neumann algebra $\mathcal{N}_c$ is isomorphic to the sub-von Neumann-algebra $\mathcal{M}_c \subset \mathcal{M}_{E_c}$ where*

(α) *E_c is the smallest projection in $\mathcal{M}$ which is larger than the support projections of all states belonging to $\mathcal{M}_{*,c}$*

(β) *$\mathcal{M}_c$ is the set of operators in $\mathcal{M}_{E_c}$ which are right and left multipliers of $\mathcal{M}_{*,c}$.*

(γ) *The automorphisms α_g are automorphisms of $\mathcal{M}_c$.*

(3) *$\mathcal{M}_{*,c}$ is the predual of $\mathcal{M}_c$.*

Unfortunately the proof of this result cannot be presented here. For details see the complete text or Ref. 121.

V.5. Remarks, additions, and problems

(I) Since physical observables should be real, i.e., represented by selfadjoint operators, some physicists like to start with Jordan algebras instead of C^*- or von Neumann algebras. In this connection it is worthwhile to mention that Connes' theory of the equivalence of von Neumann algebras with cones, fulfilling some properties, extends to certain Jordan algebras, which are the analog of von Neumann algebras. This has been worked out by Iochum[122] in his thesis.

(II) It is easy to construct examples of QFTLO, where $\mathcal{M}(D)$ is not a factor. Let $\{\mathcal{M}(O),\mathcal{H},\mathbb{R}^{d+1}\}$ be a QFTLO on the $(d+1)$-dimensional Minkowski space. Define a theory on the d-dimensional space as follows. Let $\hat{D}$ be a double cone in $\mathbb{R}^d$ and D its extension to $\mathbb{R}^{d+1}$. Let $K(\hat{D})$ be the cylindrical set in $\mathbb{R}^{d+1}$, i.e., $(x^0,\ldots,x^{d-1})\in\hat{D}$ and x^d arbitrary. Then $D'\cap K(\hat{D})$ contains interior points. Choose an Abelian algebra $\mathcal{A}(\hat{D})\subset\mathcal{N}(D'\cap K(\hat{D}))$ and define $\mathcal{M}(\hat{D}) = \mathcal{M}(D)\vee\mathcal{A}(\hat{D})$. This algebra has at least $\mathcal{A}$ as center. It is clear that one can choose $\mathcal{A}(\hat{D})$ in an $\mathbb{R}^d$ invariant manner. Notice that we obtain for the wedge

$$\mathcal{M}(\hat{W})=\vee\{\mathcal{M}(D);D\subset W\}$$

because of the double cone theorem I.4.4.

Problem: Do there exist conditions implying that $\mathcal{M}(D)$ is a factor?

(III) Also for the algebras of spacelike cones one knows their type. Driessler[123] showed that the algebra of a spacelike cone $\mathcal{M}(C)$ is of type III. Borchers and Wollenberg[124] showed the following result:

V.5.1. Theorem:

Let C be a spacelike cone and e be a direction inside C. Let W be a wedge which is invariant in the e-direction. Then $\mathcal{M}(C\cap W)$ is of type III_1.

Notice if C is a cone which is causally stable, i.e., $C=C''$ then exists a larger cone $C'\supset C$ such that $C=C'\cap W$. Therefore, the algebras of such cones are of type III_1.

(IV) If one deals with special assumptions then the result of Sec. V.4 can sometimes be strengthened. If the group is the translation group of $\mathbb{R}^d$ and one is interested in those representations where the spectrum of $U(a)$ is contained in some proper cone C then one obtains a stronger result. But first we need some notation.

V.5.2. Definition:

Let $\{\mathcal{A},\mathbb{R}^d,\alpha\}$ be a C^*-dynamical system and $C\subset\mathbb{R}^d$ be a closed, convex, proper cone with interior points. Let $\hat{C}$ denote the dual cone of C. Then we denote by

(1) $\mathcal{A}_0^*(C)$ the set of elements $\varphi\in\mathcal{A}^*$ with the properties:

(α) $a\to\varphi(x\alpha_a y)$ is a continuous function on $\mathbb{R}^d$, $x,y\in\mathcal{A}$.

(β) $\varphi(x\alpha_a y)$ is the boundary value of an analytic function $W(z)$ holomorphic in the tube

$$T(\hat{C})=\{z\in\mathbb{C}^d;\Im m z\in \text{ interior of } C\}.$$

(γ) There exists a constant m such that

$$|W(z)|\leq\|\varphi\|\|x\|\|y\|e^{m\|\Im m z\|}$$

holds for $z\in T(C)$.

(δ) φ^* fulfills the same conditions as φ.

(2) $\mathcal{A}^*(C)$ is the norm-closure of $\mathcal{A}_0^*(C)$.

With this notation one obtains:

V.5.3. Theorem:

Let $\{\mathcal{A},\mathbb{R}^d,\alpha\}$ be a C^*-dynamical system and $C\subset\mathbb{R}^d$ be a closed, convex, proper cone with interior points. Then there exists a projection $E(C)$ in the center of $\mathcal{A}^{**}$ with

(1) $\varphi\in\mathcal{A}^*(C)$ *iff there holds*

$$\varphi(E(C)A)=\varphi(A),\quad \forall A\in\mathcal{A}.$$

(2) *Let $\{\mathcal{H},\pi\}$ be a representation of $\mathcal{A}$. Then one can find a continuous unitary representation $V(a)$ acting on $\mathcal{H}$, which implements α_a with spectrum $V(a)\subset C$ if and only if every vector state ω_ψ belongs to $\mathcal{A}^*(C)$.*

(3) *The representation $V(a)$ can be chosen to be in $\pi(\mathcal{A})''$.*

For details see Ref. 27.

(V) Part V.4 has some interest in connection with broken symmetries. If $\{\mathcal{A},G,\alpha\}$ is a C^*-dynamical system with G a topological group, then one is not only interested in representations where the symmetry is implemented by a continuous unitary representation of the group G, but also in representations with broken symmetries. By this we mean representations where the symmetry is no longer exact, but where there is enough symmetry left in order that it can be observed as symmetry on some observables. One possibility is to assume that there is an exact symmetry on some subalgebra. Adapting this point of view one should look for some algebra which is isomorphic to a subalgebra of $\mathcal{M}_c$, introduced in the last section. (Lagrangean field theory suggests to look at some deformed algebra. But, in the general theory it is not clear what deformation means.)

VI. TENSOR PRODUCT DECOMPOSITION OF QUANTUM FIELD THEORIES

The axioms of quantum field theory are such that they allow to describe two or more independent theories in one object. There are several mathematical procedures which permit to construct a new theory out of two or more independent theories. In all the known examples the new theory does not describe new physics. The simplest example is the direct sum, or more generally, the direct integral of theories. The inverse operation is the integral decomposition with respect to the center of the global algebra. There are effective criteria implementing that a theory is indecomposable with respect to the direct sum operation. This is the cluster decomposition property or equivalently the uniqueness of the vacuum vector.[102,103]

More complicated is the direct product of theories. Starting with two theories $\{\mathcal{M}_i(\mathcal{O}),U_i(\Lambda,x),\mathcal{H}_i,\Omega_i\}$, $i=1,2$ one can define a new theory on $\mathcal{H}_1\overline{\otimes}\mathcal{H}_2$ by $\mathcal{M}(\mathcal{O})=\mathcal{M}_1(\mathcal{O})\overline{\otimes}\mathcal{M}_2(\mathcal{O})$, $U(\Lambda,x)=U_1(\Lambda,x)\otimes U_2(\Lambda,x)$ and $\Omega=\Omega_1\otimes\Omega_2$. The new theory $\{\mathcal{M}(\mathcal{O}),U(\Lambda,x),\mathcal{H},\Omega\}$ fulfills again all axioms of local quantum field theory. In order to discover the direct product structure one has to look at the subtheory $\{\mathcal{M}_1(\mathcal{O})\otimes\mathbb{1},U(\Lambda,x),\mathcal{H},\Omega\}$ which fulfills the assumptions of the theory of local observables except the cyclicity assumption for the vacuum vector. In this section we want to develop the theory for the converse operation, i.e., decomposition of tensor products. Besides the usual assumptions we require that the global algebra is a factor, and that the theory satisfies the Bisognano–Wichmann property.

VI.0.1. Remark:

(1) As a consequence of the Bisognano–Wichmann property one concludes that the theory fulfills the wedge duality, i.e., for every wedge the relation

$$\mathcal{M}(W)' = \mathcal{M}(W')$$

holds, where W' denotes the opposite wedge of W. For the proof see Prop. IV.4.2.
(2) If one identifies the algebra of the double cone D with

$$\mathcal{M}(D) = \cap\{\mathcal{M}(W); D \subset W\}. \tag{VI.0.1}$$

then the general duality property

$$\mathcal{M}(D)' = \mathcal{M}(D')$$

holds, where D' denotes the (interior) of the spacelike complement of D.

VI.1. On modular covariant subalgebras

In order to understand the problem let us start with the assumption that our theory is a tensor product.

$$\{\mathcal{M}_1(\mathcal{O}) \overline{\otimes} \mathcal{M}_2(\mathcal{O}), U_1(x) \otimes U_2(x), \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2, \Omega_1 \otimes \Omega_2\}.$$

First we look at one algebra $\mathcal{M}$ for a suitable chosen domain. Then we have $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$. Since Ω is a product state we know that also the modular group splits, i.e.,

$$\Delta^{\mathrm{i}t} = \Delta_1^{\mathrm{i}t} \otimes \Delta_2^{\mathrm{i}t}.$$

If this is the case then $\mathcal{M}_1 \otimes \mathbb{1}$ is a subalgebra of $\mathcal{M}$ which is mapped by σ^t onto itself

$$\sigma^t(\mathcal{M}_1 \otimes \mathbb{1}) = \mathcal{M}_1 \otimes \mathbb{1}.$$

Subalgebras which are mapped by σ^t onto itself are ''modular covariant subalgebras.''

We start our investigation by introducing modular covariant subalgebras and describing their relations to normal and faithful conditional expectations. In addition we describe Takesaki's result on the structure of modular covariant subalgebras.[125]

Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and let the vector $\Omega \in \mathcal{H}$ be cyclic and separating for $\mathcal{M}$. Then we denote by Δ, J the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \Omega)$.

VI.1.1. Definition:
A von Neuman subalgebra $\mathcal{N} \subset \mathcal{M} (\mathbb{1} \in \mathcal{N})$ is called modular covariant if it fulfills the equation

$$\Delta^{\mathrm{i}t} \mathcal{N} \Delta^{-\mathrm{i}t} = \mathcal{N}, \quad \forall t \in \mathbb{R}.$$

The set of modular covariant subalgebras of $\mathcal{M}$ will be denoted by $\mathcal{M}cs(\mathcal{M})$.

Notice that the vector Ω is separating for $\mathcal{N}$ but not cyclic, because cyclicity implies $\mathcal{N} = \mathcal{M}$. (See, e.g., Kadison and Ringrose, Ref. 33, Thm. 9.2.36.)
The symbol $[\mathcal{N}\Omega]$ denotes the projection onto the Hilbert subspace generated by $\mathcal{N}\Omega$.

Modular covariant subalgebras have the following well known and easy to verify properties. (See Refs. 125, 126, 127, and 91.)

VI.1.2. Lemma: Let $\mathcal{N} \in \mathcal{M}cs(\mathcal{M})$. Let $\mathcal{H}_{\mathcal{N}}$ be the closure of $\mathcal{N}\Omega$ and denote by $E_{\mathcal{N}}$ the projection onto $\mathcal{H}_{\mathcal{N}}$. By $\hat{\mathcal{N}}$ we denote the restriction of $\mathcal{N}$ to $\mathcal{H}_{\mathcal{N}}$. Then:

1. *$E_{\mathcal{N}}$ commutes with $\Delta^{\mathrm{i}t}$ and J. The restriction of Δ and J to $\mathcal{H}_{\mathcal{N}}$ will be denoted by $\hat{\Delta}$ and $\hat{J}$.*
2. *$\hat{\Delta}$ and $\hat{J}$ are the modular group and modular conjugation of $(\hat{\mathcal{N}}, \Omega)$.*
3. *The commutant of $\hat{\mathcal{N}}$ in $\mathcal{H}_{\mathcal{N}}$ coincides with $\hat{J}\hat{\mathcal{N}}\hat{J}$.*
4. *The map $\mathcal{N} \to \hat{\mathcal{N}}$ is an isomorphism of von Neumann algebras.*
5. *$A \in \mathcal{M}$ and $[A, E_{\mathcal{N}}] = 0$ implies $A \in \mathcal{N}$.*
6. *$A \in \mathcal{M}$ and $A\Omega \in \mathcal{H}_{\mathcal{N}}$ implies $A \in \mathcal{N}$.*

For the proof see the paper of Takesaki[125] or the complete script. The results of the last lemma have been strengthened.

VI.1.3. Theorem: (Takesaki)

With the assumptions and notations of Lemma VI. 1.2 *we obtain:*

(1) *For $A \in \mathcal{M}$ one has $EAE \in \hat{\mathcal{N}}$.*

(2) *There exists a normal faithful conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{N}$.*

(3) *$\mathcal{E}$ commutes with the modular action:*

$$\mathcal{E}(\mathrm{Ad}\,\Delta^{\mathrm{i}t}A) = \mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{E}(A), \quad A \subset \mathcal{M}.$$

(4) *There exists also a conditional expectation $\mathcal{E}'$ from $\mathcal{M}'$ to $J\mathcal{E}(\mathcal{M})J$ defined by*

$$\mathcal{E}'(A') = J\mathcal{E}(JA'J)J, \quad A' \in \mathcal{M}'.$$

(5) *Let E be a projection with $E\Omega = \Omega$. If there is a von Neumann algebra $\mathcal{N} \subset \mathcal{M}$ with $E \in \mathcal{N}'$ and the central support of E in $\mathcal{N}'$ is $\mathbb{1}$ and in addition one has $E\mathcal{M}E = \mathcal{N}E$ then $\mathcal{N}$ is a modular covariant subalgebra of $\mathcal{M}$.*

VI.2. Conditional expectations and half-sided translations

If $\mathcal{M}$ is a von Neumann algebra with cyclic and separating vector then we call the anti-linear operator $S_{\mathcal{M}} := J_{\mathcal{M}} \Delta_{\mathcal{M}}^{1/2}$ the Tomita conjugation of $(\mathcal{M}, \Omega)$. In this section we will deal with operators of the same kind, i.e., operators S fulfilling:

(*i*) S is a densely defined closed antilinear operator with domain of definition $\mathcal{D}(S)$.

(*ii*) $S^2 = \mathbb{1}$ on $\mathcal{D}(S)$.

(*iii*) $\Omega \in \mathcal{D}(S)$ and $S\Omega = \Omega$.

We will call such operators generalized Tomita conjugations.

Since S is closed it has a polar decomposition $S = J\Delta^{1/2}$. Then Δ is invertible and J is a conjugation, i.e.,

$$J\Delta J = \Delta^{-1}, \quad J = J^* = J^{-1}. \qquad \text{(VI.2.1)}$$

These properties follow from the condition $S^2 = \mathbb{1}$. (See, e.g., Bratteli and Robinson, Ref. 32, Prop. 2.5.11.)

We often deal with the situation that we have a generalized Tomita conjugation S and a Tomita conjugation $S_{\mathcal{M}}$ which is an extension of S. From Eq. (II.1.3) we know $(1 + \Delta_{\mathcal{M}})^{-1} \geqslant (1 + \Delta)^{-1}$. This implies that the operator-valued function $C(t) := \Delta_{\mathcal{M}}^{-\mathrm{i}t} \Delta^{\mathrm{i}t}$ has a bounded analytic extension into the strip $S(0, \frac{1}{2})$. We are interested in determining the value of this function at the upper boundary. We obtain as in Sec. II.3

VI.2.1. Lemma:

Let S be a generalized Tomita conjugation and $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$ such that the latter is an extension of S. Define $C(t) := \Delta_{\mathcal{M}}^{-\mathrm{i}t} \Delta^{\mathrm{i}t}$. Then $C(t)$ has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$ and at the upper boundary one has

$$C(t + \mathrm{i}/2) = J_{\mathcal{M}} C(t) J.$$

Moreover, the following estimate holds:

$$\|C(\tau)\| \leqslant 1.$$

We saw in Sec. III.2 that the elements in $\mathcal{C}har(\mathcal{M})$ are in one to one correspondence with the von Neumann subalgebras belonging to $\mathcal{S}ub(\mathcal{M})$. Therefore, it is interesting to know which condition of Lemma II.3.2 is the crucial one. It turns out that the conditions (1)–(6) can easily be satisfied, but that condition (7) is the essential one. In order to overcome the lack of condition 7 of Lemma II.3.2 we will use a property similar to that of half-sided modular inclusions.

VI.2.2. Theorem:
Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω and let $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$. Let S be a generalized Tomita conjugation and assume $S_{\mathcal{M}}$ is an extension of S. Assume in addition that S is an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t} S \Delta_{\mathcal{M}}^{-\mathrm{i}t}$ for $t \leqslant 0$. Then:

1. *There exists a unitary group $U(t)$ with*
   $(\alpha) U(t)\Omega = \Omega$ *for all* $t \in \mathbb{R}$.
   $(\beta) U(t)$ *has a non-negative generator.*
2. *Between the modular group of $\mathcal{M}$ and $U(t)$ exist the relations*

$$\Delta_{\mathcal{M}}^{\mathrm{i}t} U(s) \Delta_{\mathcal{M}}^{-\mathrm{i}t} = U(\mathrm{e}^{-2\pi t} s), \quad J_{\mathcal{M}} U(t) J_{\mathcal{M}} = U(-t).$$

3. *Define*

$$S_t = \Delta_{\mathcal{M}}^{\mathrm{i}t} S \Delta_{\mathcal{M}}^{-\mathrm{i}t}$$

which is monotonously increasing with t and set

$$S_\infty = \lim_{t\to\infty} S_t .$$

Then there holds for $s > 0$

$$U(s) S_\infty U(-s) = S_{-(1/2\pi)\log s} .$$

Notice: There exists a variant of this theorem which is obtained by replacing everywhere t by $-t$.

The statement of the theorem needs some explanation. By assumption the family $\Delta_{\mathcal{M}}^{\mathrm{i}t} S \Delta_{\mathcal{M}}^{-\mathrm{i}t}$ is increasing with t. Hence the projections onto the graphs are an increasing family of projections which converges strongly. Since all these projections are majorized by the projection onto the graph of $S_{\mathcal{M}}$ the limit is smaller or equal to the majorant.

The proof of this theorem is a variation of the proof of Wiesbrock's theorem on half-sided modular inclusions presented in Sec. II.4 but unfortunately we cannot present it here. For details see the complete script.

From Thm. VI.2.2 one can draw several conclusions. We start with the following result:

VI.2.3. Corollary:
Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω and let $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$. Let S be a generalized Tomita conjugation and assume $S_{\mathcal{M}}$ is an extension of S. Assume also that S is an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t} S \Delta_{\mathcal{M}}^{-\mathrm{i}t}$ for $t \leqslant 0$. If we have in addition

$$S_{\mathcal{M}} = \lim_{t\to\infty} S_t ,$$

then S is the Tomita conjugation of a von Neumann algebra $\mathcal{N}$ which has Ω as cyclic and separating vector. Moreover, one has

$$\mathcal{N} = U(1) \mathcal{M} U(-1).$$

VI.2.4. Remark:
Unfortunately I could not show that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, although it is suggested by the fact that $S_{\mathcal{M}}$ is an extension of $S_{\mathcal{N}}$. Up to now one needs additional information in order to conclude that $\mathcal{N}$ is a subalgebra of $\mathcal{M}$.

Proof of the Corollary: With $S_\infty = \lim_{t\to\infty} S_t$ we know from Thm. VI.2.2 the relation $S = U(1) S_\infty U(-1)$. With $S_\infty = S_{\mathcal{M}}$ it follows $S = U(1) S_{\mathcal{M}} U(-1)$. Since $\mathcal{M}\Omega$ is a core for $S_{\mathcal{M}}$ it follows with $\mathcal{N} = U(1)\mathcal{M}U(-1)$ that $\mathcal{N}\Omega$ is a core for S. Hence the corollary is proved. □

In connection with conditional expectations one can conclude that the algebra $\mathcal{N}$, described in Corollary VI.2.3, is a subalgebra of $\mathcal{M}$.

A half-sided translation associated with $\mathcal{M}$ is a one-parametric unitary group $V(t)$ fulfilling:
(i) $V(t)\Omega=\Omega$ for all $t\in\mathbb{R}$.
(ii) $V(t)$ has a non-negative generator.
(iii) $V(t)\mathcal{M}V(-t)\subset\mathcal{M}$ for $t\geqslant 0$ (or for $t\leqslant 0$).
With these concepts we show:

VI.2.5. Theorem:
Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω. Assume $\mathcal{N}$ is a modular covariant subalgebra of $\mathcal{M}$ and $\mathcal{E}$ the associated conditional expectation. (See Thm. VI.1.3.) Denote by $\hat{\mathcal{N}}$ resp. $\hat{\mathcal{E}}$ the restriction of $\mathcal{N}$ resp. $\mathcal{E}$ to the cyclic subspace of $\mathcal{N}$. Assume $V(t)$ is a +half-sided translation for $\mathcal{M}$. Then:
(*i*) *$\mathcal{E}(V(t)\mathcal{M}V(-t))$ is dense in the von Neumann algebra $\{\mathcal{E}(V(t)\mathcal{M}V(-t))\}''$.*
(*ii*) *There exists a +half-sided translation for $\hat{\mathcal{N}}=\hat{\mathcal{E}}(\mathcal{M})$ with*

$$U(t)\hat{\mathcal{N}}U(-t)=\{\hat{\mathcal{E}}(V(t))\mathcal{M}(V(-t))\}''.$$

Since $V(t)$ has a non-negative generator we conclude by a Reeh–Schlieder type argument, that $EV(t)\mathcal{M}\Omega$ is dense in $E\mathcal{H}$. Let $S_{-1/2\pi\log t}$ be the map $EV(t)AV(-t)\Omega\to EV(t)A^*V(-t)\Omega$. It is not difficult to show that this map is preclosed. The closure, denoted by the same symbol, fulfills the conditions of Thm. VI.2.2. Hence one gets a group $U(t)$ with

$$S_t=U(\mathrm{e}^{2\pi t})S_{\hat{\mathcal{N}}}U(-\mathrm{e}^{2\pi t}).$$

The sets $EV(\mathrm{e}^{2\pi t})\mathcal{M}\Omega$ and $U(\mathrm{e}^{2\pi t})\hat{\mathcal{N}}\Omega$ are both a core for S_t which implies that $EV(\mathrm{e}^{2\pi t})\mathcal{M}\Omega$ is dense in $U(\mathrm{e}^{2\pi t})\hat{\mathcal{N}}\Omega$ in the graph topology of S_t. Since the graph topology of S_t is stronger than the Hilbert space topology we get the density in the Hilbert space topology. Since Ω is separating and since $EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E$ is convex we conclude that $EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E$ is strongly dense in $U(\mathrm{e}^{2\pi t})\hat{\mathcal{N}}U(-\mathrm{e}^{2\pi t})$. Hence the theorem is proved. For details see Ref. 128 or the complete script.

VI.3. Construction of subtheories

If we start with a wedge W and assume the algebra $\mathcal{M}(W)$ has a modular covariant subalgebra $\mathcal{N}(W)$. Let $\mathcal{E}_W$ be the associated conditional expectation and E_W the projection onto $[\mathcal{N}(W)\Omega]$. If we now change the wedge to $\Lambda W+x$ then of course $U(\Lambda,x)\mathcal{N}(W)U(\Lambda,x)^*$ is a modular covariant subalgebra of $\mathcal{M}(\Lambda W+x)$. But in order to obtain a decomposition of the global field theory the projections E_W and $E_{\Lambda W+x}$ have to coincide. If this is the case then we also need conditional expectations for the algebras $\mathcal{M}(D)$ associated with double cones. In order to be able to construct such conditional expectations the algebras must be closely related to the algebras of wedges. Therefore, we set

$$\mathcal{M}(D)=\cap\{\mathcal{M}_{\Lambda W+x};D\subset\Lambda W+x\}.$$

Now we can define what we mean by the coherence property.

VI.3.1. Definition:
Assume we deal with a quantum field theory in the vacuum sector. Assume with every double cone D and every wedge W is associated a modular covariant subalgebra $\mathcal{N}(D)\subset\mathcal{M}(D)$ and $\mathcal{N}(W)\subset\mathcal{M}(W)$. Then we call this family coherent if the projections E_D and E_W coincide for all double cones D and for all wedges W.

Unfortunately it is not always possible to transport the conditional expectation from one wedge to all others in a coherent way. Half-sided translations can be used only if the positive linear maps $L_t(A):\mathcal{M}\to\hat{\mathcal{N}}$ defined by

$$L_t(A)=U(-t)EV(t)AV(-t)EU(t)$$

are trivial. These half-sided translations of $\mathcal{M}(W)$ would be necessary in order to transport the conditional expectations to the shifted wedges or to pass to other wedges with one light ray in common. (See Sec. IV.4.)

In case one knows that the translations in the characteristic two-plane of the wedge W commute with E_W one can conclude more:

VI.3.2. Lemma:

Let the dimension of the Minkowski space be larger than 2. Let $\mathcal{N}(W)$ be a modular covariant subalgebra of $\mathcal{M}(W)$. Assume E_W commutes with the translations in the characteristic two-plane of W. Then E_W commutes with all translations.

This is an easy consequence of the spectrum condition. Assume we have a coherent family of modular covariant subalgebras for all wedges.

It remains to construct a modular covariant subalgebra for every double cone.

VI.3.3. Lemma:

Let $\mathcal{N}(W)$ be a coherent family of modular covariant subalgebras of $\mathcal{M}(W)$. Define for any double cone

$$\mathcal{N}(D)=\cap\{\mathcal{N}(W);D\subset W\}.$$

Then $\mathcal{N}(D)$ is a modular covariant subalgebra of

$$\mathcal{M}(D)=\cap\{\mathcal{M}(W);D\subset W\}.$$

Moreover, one has

$$[\mathcal{N}(D)\Omega]=[\mathcal{N}(W)\Omega].$$

One knows $\mathcal{N}(W)=\mathcal{M}(W)\cup\{E,\mathbb{1}\}'$. In analogy one defines

$$\mathcal{N}(D)=\mathcal{M}(D)\cap\{E,\mathbb{1}\}'. \tag{VI.3.1}$$

Because of $\mathcal{N}(D)\Omega=E\mathcal{M}(D)\Omega$ we get that Ω is cyclic for $\hat{\mathcal{N}}(D)$ in $E\mathcal{H}$. Using the fact that Ω is also separating for $\mathcal{N}(D)$ one finds that Thm. VI.1.3 is applicable, which shows that $\mathcal{N}(D)$ is a modular covariant subalgebra of $\mathcal{M}(D)$.

We saw that the coherence property is not automatic. Therefore, we have to assume this in the future. Using the results of Sec. IV.4 one finds:

VI.3.4. Lemma:

Let $\{\mathcal{M}(D),U(\Lambda,x),\Omega\}$ be a theory of local observables fulfilling the Bisognano–Wichmann property. Let $\{\mathcal{N}(W),\mathcal{N}(D)\}$ be a coherent family of modular covariant subalgebras and $E=E_W$ be the associated projection. Then $E\mathcal{H}$ is invariant under the Poincaré transformations $U(\Lambda,x)$. Moreover, for every wedge the restrictions $\hat{\Delta}_W^{it}$ and $\hat{U}(\Lambda_W(t),0)$ coincide. Here $\Lambda_W(t)$ denotes the Lorentz boosts which map W onto itself.

We collect the main results of this section in the following

VI.3.5. Theorem:

Let $\{\mathcal{M}(D),U(\Lambda,x),\mathcal{H},\Omega\}$ be a theory of local observables fulfilling the assumptions of the introduction. Assume there exists a coherent family of modular covariant subalgebras $\mathcal{N}(W)$ of $\mathcal{M}(W)$. Then a local quantum field theory $\{\hat{\mathcal{N}}(D),\hat{U}(\Lambda,x),E\mathcal{H},\Omega\}$ exists which fulfills the axioms listed in the introduction. In particular one has for every wedge

$$\hat{\mathcal{N}}(W)=\vee\{\hat{\mathcal{N}}(D);D\subset W\}.$$

For details of the proof see the complete script.

VI.4. Decomposition of the global algebra

The investigations of this subsection are based on a result of Takesaki.[125] Notice if $\mathcal{N}$ is a modular covariant subalgebra of $\mathcal{M}$, then this is also true for $\mathcal{N}^c := \mathcal{N}' \cap \mathcal{M}$.

The existence of the two conditional expectations $\mathcal{E}$ and $\mathcal{E}^c$ has some important consequences.

VI.4.1. Theorem:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Assume $\mathcal{N} \in \mathcal{M}cs(\mathcal{M})$ is a von Neumann subfactor. Let $\mathcal{N}^c$ be the relative commutant of $\mathcal{N}$ in $\mathcal{M}$ and let $\mathcal{R} = \mathcal{N} \vee \mathcal{N}^c$ be the von Neumann algebra generated by $\mathcal{N}$ and $\mathcal{N}^c$. Then the map

$$\pi: \sum A_i \otimes B_i \in \mathcal{N} \otimes \mathcal{N}^c \rightarrow \sum A_i B_i \in \mathcal{R} \subset \mathcal{M}$$

extends to an isomorphism of $\mathcal{N} \overline{\otimes} \mathcal{N}^c$ onto $\mathcal{R} = \mathcal{N} \vee \mathcal{N}^c$. Moreover the vacuum state $(\Omega, . \Omega)$ is a product state on $\mathcal{R}$, i.e., $A \in \mathcal{N}$ and $B \in \mathcal{N}^c$ implies

$$(\Omega, AB\Omega) = (\Omega, A\Omega)(\Omega, B\Omega).$$

In order to apply Takesaki's result on tensor products we have to know that the modular covariant subalgebra $\mathcal{N}(W)$ of $\mathcal{M}(W)$ is a factor, which will be shown under the assumption that $\mathcal{M}(W)$ itself is a factor. This is known to be the case if the global algebra is a factor. Since the factor property for $\mathcal{M}(D)$ is not known we are not able to show that $\mathcal{N}(D)$ is a factor. Hence we cannot use Takesaki's result. Here we will use a characterization of tensor products due to Ge and Kadison.[129]

For the factor property of $\mathcal{N}(W)$ we use Lemma V.2.2: Let $U(t)$ be a half-sided translation of the von Neumann algebra $\mathcal{M}$. Denote by E_0 the projection onto the $U(t)$ invariant vectors and by F_1 the projection onto the eigenvectors of $\Delta_{\mathcal{M}}$ to the eigenvalue 1. Then one has

$$F_1 \leqslant E_0.$$

From this we conclude:

VI.4.2. Proposition:

Let $\{\mathcal{M}(D), U(\Lambda, x), \mathcal{H}, \Omega\}$ be a theory of local observables. Assume the global algebra is a factor and hence $\mathcal{M}(W)$ is a factor. Then every modular covariant subalgebra of $\mathcal{M}(W)$ is a factor.

Proof: Let $\mathcal{N}(W)$ be a modular covariant subalgebra of $\mathcal{M}(W)$ and let Z be in the center of $\mathcal{N}(W)$. Then $\hat{Z}$ is in the center of $\hat{\mathcal{N}}(W)$ and hence it commutes with $\hat{\Delta}_W^{it}$. Since the map $\mathcal{N}(W) \rightarrow \hat{\mathcal{N}}(W)$ is an isomorphism we find that Z commutes with Δ_W^{it}. This implies $Z\Omega \in F_1 \mathcal{H} \subset E_0 \mathcal{H}$. As the group generated by half-sided translations for $\mathcal{M}(W)$ contains the time translation it follows $E_0 \mathcal{H} = \mathbb{C}\Omega$. Hence $Z\Omega = z\Omega, z \in \mathbb{C}$ and the separability of Ω implies $Z = z\mathbb{1}$. This shows the proposition. □

Knowing that $\mathcal{N}(W)$ is a factor, we can use Takesaki's result for the construction of tensor products. But first we have to look at the relative commutants.

VI.4.3. Lemma:

Assume $\{\mathcal{N}(W)\}$ is a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$. Let $\mathcal{N}^c(W)$ be the relative commutant of $\mathcal{N}(W)$ in $\mathcal{M}(W)$. Define $\mathcal{N}^p(W) = \mathcal{N}(W) \vee \mathcal{N}^c(W)$. Then $\{\mathcal{N}^c(W)\}$ and $\{\mathcal{N}^p(W)\}$ are both coherent families of subalgebras of $\{\mathcal{M}(W)\}$.

The proof of this lemma uses the covariance of the two families $\{\mathcal{M}(W)\}$ and $\{\mathcal{N}(W)\}$. In addition one has to look at the algebras $\mathcal{M}(W(l,l_1) \cap W(l,l_2))$ and $\mathcal{N}(W(l,l_1) \cap W(l,l_2))$, where the first vectors coincide in order to show that $\mathcal{N}^c(W(l,l_1) \cap W(l,l_2))$ fulfills the condition of half-sided modular inclusion with respect to the algebras $\mathcal{N}^c(W(l,l_1))$ and $\mathcal{N}^c(W(l,l_2))$. From this one concludes the coherence property.

VI.4.4. Remark:
The relative commutant of $\mathcal{N}^p(W)$ is trivial, because $(\mathcal{N}^p(W))^c$ belongs to the center of the factor $\mathcal{N}^p(W)$ (see Prop. VI.4.1).

Since we do not know whether or not $\mathcal{M}(D)$ and $\mathcal{N}(D)$ are factors, we will define $\mathcal{N}^c(D)$ and $\mathcal{N}^p(D)$ differently.

VI.4.5. Definition:
With the assumptions as before we set for double cones

$$\mathcal{N}^c(D) = \cap\{\mathcal{N}^c(W); D\subset W\},$$
$$\mathcal{N}^p(D) = \cap\{\mathcal{N}^p(W); D\subset W\}, \tag{VI.4.1}$$

Since these definitions are similar to those in Lemma VI.3.3, the conclusion of that lemma holds for $\mathcal{N}^c(D)$ and $\mathcal{N}^p(D)$ with the obvious changes.

Next we have to look at conditions which imply that $\mathcal{M}(D)$ is isomorphic to a tensor product. For the proof of such condition we need a result of Ge and Kadison which is based on the tensor slice mapping introduced by Tomijama.[130] First we have to explain this concept.

Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Let ω and ρ be normal linear functionals on $\mathcal{R}$ and $\mathcal{S}$, respectively. Then their product $\omega\otimes\rho$ defines a linear functional on $\mathcal{R}\overline{\otimes}\mathcal{S}$ which is defined on $\mathcal{H}\overline{\otimes}\mathcal{K}$. Keeping ω fixed and taking $\psi,\chi\in\mathcal{K}$ and choosing $T\in\mathcal{R}\overline{\otimes}\mathcal{S}$ then the expression $\omega\otimes\rho_{\psi,\chi}(T)$ defines a sesquilinear form on $\mathcal{K}$. This form is continuous and defines by the Riesz representation theorem a linear operator $\Psi_\omega(T)$. Since the commutant of $\mathcal{R}\overline{\otimes}\mathcal{S}$ is $\mathcal{R}'\overline{\otimes}\mathcal{S}'$ it is easy to see that $\Psi_\omega(T)$ belongs to $\mathcal{S}$. This is the tensor slice mapping introduced by Tomijama. In the same manner there exists a mapping $\Psi_\rho:\mathcal{R}\overline{\otimes}\mathcal{S}\rightarrow\mathcal{R}$.

With this concept the following result of Ge and Kadison[129] holds, which we quote without proof:

VI.4.6. Proposition:
Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{R}\overline{\otimes}\mathcal{S}$, then $\mathcal{M}$ splits, i.e., $\mathcal{M}=\mathcal{R}_1\overline{\otimes}\mathcal{S}_1$ with $\mathcal{R}_1\subset\mathcal{R}$ and $\mathcal{S}_1\subset\mathcal{S}$ exactly if every tensor slice mapping sends $\mathcal{M}$ into $\mathcal{M}$.

Using this result we obtain:

VI.4.7. Proposition:
Let $\mathcal{N}(D)$ be defined as in Lemma VI.3.3 and $\mathcal{N}^c(D)$, $\mathcal{N}^p(D)$ as in Eq. (VI.4.1) then one has

$$\mathcal{N}^p(D)\cong\mathcal{N}(D)\overline{\otimes}\mathcal{N}^c(D).$$

The proof uses the fact that $\mathcal{M}(D)\subset\mathcal{M}(W)$ and that for the latter algebra we know the tensor product structure.

Collecting the results of this section we obtain:

VI.4.8. Theorem:
Let $\{\mathcal{M}(\mathcal{O}),U(\Lambda,x),\mathcal{H},\Omega\}$ be a theory of local observables fulfilling the assumptions listed in the introduction. Assume that $\{\mathcal{N}(W)\}$ is a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$. Let $\mathcal{N}^c(W)$ be the relative commutant of $\mathcal{N}(W)$ in $\mathcal{M}(W)$ and $\mathcal{N}^p(W)=\mathcal{N}(W)\vee\mathcal{N}^c(W)$. Then:

(1) *There exists on $\mathcal{H}$ a subtheory of local observables*

$$\{\mathcal{N}^p(D),\mathcal{N}^p(W),U(\Lambda,x)\}$$

covariant under the existing unitary group $U(\Lambda,x)$. Moreover, $\{\mathcal{N}^p(D),\mathcal{N}^p(W)\}$ are modular covariant subalgebras of $\{\mathcal{M}(D),\mathcal{M}(W)\}$ such that $\mathcal{N}^p(W)$ has a trivial relative commutant in $\mathcal{M}(W)$. If E^p denotes the projection onto $[\mathcal{N}^p(W)\Omega]$ then E^p commutes with $\mathcal{N}^p(D)$, $\mathcal{N}^p(W)$ and the group representation $U(\Lambda,x)$. Moreover, Ω is cyclic for $\mathcal{N}^p(D)$ in $E^p\mathcal{H}$. If we denote the restriction of $\mathcal{N}^p(D)$ and $U(\Lambda,x)$ by $\hat{\mathcal{N}}^p(D)$ and $\hat{U}(\Lambda,x)$, respectively, then

$$\{\hat{\mathcal{N}}^p(D),\hat{U}(\Lambda,x),E^p\mathcal{H},\Omega\}$$

defines a theory of local observables satisfying the axioms listed in the introduction.
(2) *There exist two coherent families* $\{\mathcal{N}(D),\mathcal{N}(W)\}$ *and* $\{\mathcal{N}^c(D),\mathcal{N}^c(W)\}$ *of modular covariant subalgebras of* $\{\mathcal{M}(D),\mathcal{M}(W)\}$. *If E and* E^c *are the projections onto* $[\mathcal{N}(W)]$ *and* $[\mathcal{N}^c(W)]$, *respectively, then these projections commute with* $U(\Lambda,x)$ *and E with* $\mathcal{N}(D)$ *and* E^c *with* $\mathcal{N}^c(D)$. *With this we obtain:*

$$\{\hat{\mathcal{N}}^p(D),\hat{U}(\Lambda,x),E^p\mathcal{H},\Omega\}\cong\{\hat{\mathcal{N}}^0(D)\overline{\otimes}\hat{\mathcal{N}}^c(D),\hat{U}^0(\Lambda,x)\otimes\hat{U}^c(\Lambda,x),E\mathcal{H}\overline{\otimes}E^c\mathcal{H},\Omega^0\otimes\Omega^c\}.$$

In this formula $\hat{X}^0$ *denotes the restriction to* $E\mathcal{H}$ *and* $\hat{X}^c$ *the restriction to* $E^c\mathcal{H}$.

VI.5. The hidden charge problem

If we look at the modular covariant subalgebras $\mathcal{N}(W)$ of $\mathcal{M}(W)$, then it can happen that the relative commutant $\mathcal{N}^c(W)$ of $\mathcal{N}(W)$ in $\mathcal{M}(W)$ is trivial, i.e., $\mathcal{N}^c(W)=\mathbb{C}\mathbb{1}$. This is called the hidden charge problem because of the following reason: If we start with a theory of local observables $\{\mathcal{N}(\mathcal{O}),U(\Lambda,x),\mathcal{H},\Omega\}$ such that the theory has charged sectors which are connected by localized Bose fields, then we can add these Bose fields and obtain a field algebra $\{\mathcal{F}(\mathcal{O}),\hat{U}(\Lambda,x),\hat{\mathcal{H}},\Omega\}$ which also fulfills the assumptions of the theory of local observables. Knowing only the latter theory one would like to discover the local net $\{\mathcal{N}(\mathcal{O}),U(\Lambda,x),\mathcal{H},\Omega\}$ and the structure of the charged fields. The simplest case has been discussed in Ref. 131, namely, that the charged fields are covariant under the action of a compact Abelian group. In this case one has unitary operators in $\mathcal{M}(W)$ which define automorphisms of $\mathcal{N}(W)$. This is no longer true in the general situation. The next, more complicated case is described by Doplicher, Haag, and Roberts.[132,133] Here, or more general in the situation described by Buchholz and Fredenhagen,[134] the commutant of $\mathcal{N}(W)\vee\mathcal{N}(W')$ is generated by minimal projections. In general one has to cope with the situation where the commutant of $\mathcal{N}(W)\vee\mathcal{N}(W')$ is not generated by minimal projections. In both cases, the tensor product decomposition and the hidden charge situation, one has to look at subtheories. Therefore, both problems are mingled and one has to disentangle and to solve them.

Let $\{\mathcal{N}(W)\}$ be a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$ and assume that the relative commutant $\mathcal{N}^c(W)$ of $\mathcal{N}(W)$ in $\mathcal{M}(W)$ is trivial. Let E be the projection onto $[\mathcal{N}(W)\Omega]$. We introduce:

VI.5.1. Definition:
(1) $\mathcal{G}$ denotes the set of wedges, double cones, and spacelike complements of double cones.
(2) For $G\in\mathcal{G}$ we define

$$\mathcal{M}_1(G)=\mathcal{M}(G)\vee\{\mathbb{1},E\}''.$$

(3) $\mathcal{N}_1^c(G)$ denotes the relative commutant of $\mathcal{N}(G)$ with respect to $\mathcal{M}_1(G)$. Since by Remark VI.0.1 duality holds inside $\mathcal{G}$ one has

$$\mathcal{M}_1(G)=\mathcal{N}(G').$$

(4) $\mathcal{N}_\infty$ denotes the von Neumann algebra generated by all $\mathcal{N}(G)$.

The following properties of $\mathcal{M}_1(G)$ are easy to derive.

VI.5.2. Lemma:
Let $\mathcal{M}_1(G)$ *be the algebra defined in VI.5.1. Then:*
(1) *For every wedge the algebra* $\mathcal{M}_1(W)$ *is a factor.*
(2) *For the relative commutant of* $\mathcal{M}(G)$ *in* $\mathcal{M}_1(G)$ *one has*

$$\mathcal{M}_1(G)\cap\mathcal{M}(G)'=\mathcal{M}(G')\cap\mathcal{N}(G')'=\mathcal{N}^c(G').$$

Hence for every wedge $\mathcal{M}_1(W)\cap\mathcal{M}(W)'$ *is trivial.*
(3) *For the relative commutant* $\mathcal{N}_1^c(G)$ *one has*

$$\mathcal{N}_1^c(G)=\mathcal{M}_1(G)\cap\mathcal{M}_1(G')=\mathcal{N}_1^c(G').$$

(4) $\mathrm{Ad}\,\Delta_G^{\mathrm{i}t}\mathcal{M}_1(G)=\mathcal{M}_1(G)$ *and hence*

$\mathrm{Ad}\,\Delta_G^{\mathrm{i}t}\mathcal{N}_1^c(G)=\mathcal{N}_1^c(G)$

Our first goal is to look at partial isometries in $\mathcal{M}(W)$.

VI.5.3. Definition:

Let $\mathcal{N}(W)$ be a modular covariant subalgebra of $\mathcal{M}(W)$. We set:
(i) $\mathcal{J}(W)=\{V\in\mathcal{M}(W);V$ partial isometry with $V^*V=\mathbb{1},VV^*=R(V)\}$.
(ii) $\mathcal{P}(W)=\{VEV^*=:F;V\in\mathcal{J}(W)\}$, where $E=[\mathcal{N}(W)\Omega]=[\mathcal{N}(W')\Omega]$.
(iii) By $\mathcal{U}(W)$ we denote the set of unitaries in $\mathcal{M}(W)$.

With this notation we show:

VI.5.4. Lemma:

(1) *Let* $F\in\mathcal{P}(W)$ *and P be a projection in* $\mathcal{M}_1(W)$ *with* $P\leqslant F$. *Then:*
(α) $P\in\mathcal{P}(W)$, *i.e., there exists an element* $V_1\in\mathcal{J}(W)$ *with* $P=V_1EV_1^*$.
(β) *There exists an element* $W\in\mathcal{J}(W)\cap\mathcal{N}(W)$ *with* $V_1=VW$ *where V is defined by* $F=VEV^*$.
(γ) *If* $F=P$ *then W is unitary.*
(2) *Let* $F_1=V_1EV_1^*$, $F_2=V_2EV_2^*$ *be in* $\mathcal{P}(W)$. *Assume* $(V_1V_1^*)(V_2V_2^*)=0$. *Then exists an element* $V\in\mathcal{J}(W)$ *with* $VEV^*=F_1+F_2$.
(3) *Let* $F\in\mathcal{P}(W)$ *then exists a unitary element* $U\in\mathcal{U}(W)$ *with* $F\leqslant UEU^*$.

The proof of this lemma is based on the fact that $\mathcal{N}(W)$ is a factor of type III. Hence for every projection H in $\mathcal{N}(W)$ exists a partial isometry in $\mathcal{N}(W)$ with support $\mathbb{1}$ and range H.

By the result of the last lemma it is sufficient to look at unitary elements in $\mathcal{J}(W)$, i.e., at elements of $\mathcal{U}(W)$. Now we introduce the sectors associated with elements $V\in\mathcal{J}(W)$.

VI.5.5. Definition:

Let $\{\mathcal{N}(W)\}$ be a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$.
(1) For $V\in\mathcal{J}(W)$ we set

$$S(V)=[\mathcal{N}(W)VE\mathcal{H}].$$

(2) $\mathcal{N}'_\infty=\cap_D\mathcal{N}(D)'\cap_W\mathcal{N}(W)'$

Notice that the projection $S(V)$ does not only belong to $\mathcal{N}(W)'$ but also to $\mathcal{N}(W')'$. Since the Hilbert space $E\mathcal{H}$ is invariant under $\mathcal{N}(W')$ we observe

VI.5.6 Theorem:

Let $\{\mathcal{N}(W)\}$ *be a coherent family of modular covariant subalgebras of* $\{\mathcal{M}(W)\}$. *Then for every* $V\in\mathcal{J}(W)$ *the projection* $S(V)$ *belongs to* $\mathcal{N}'_\infty$.

The proof of this theorem consists of three parts. First assume V belongs to $\mathcal{J}(W+a)$ where a belongs to the interior of the wedge W, then the statement is true because of the spectrum condition. Next we have to show that $S(\mathrm{Ad}\,U(\lambda a)V)$ depends weakly continuous on λ. The third part consists of showing that the statement remains true if one takes limits of elements described in the first part.

Little is known about the structure of $\mathcal{N}_1^c$. A special situation appears if one has $S(V)=VEV^*$. In this case we obtain

VI.5.7. Proposition:

Assume $V\in\mathcal{J}(W)$ *is such that* $S(V)=VEV^*$. *Then it fulfills the following properties:*
(*i*) *V is unitary.*
(*ii*) $S(V)$ *is a minimal in* $\mathcal{N}_1^c(W)$.
(*iii*) V^* *induces an isomorphism of* $\mathcal{N}$, *i.e.,*

$$V^*\mathcal{N}(W)V=\mathcal{N}(W).$$

This proposition follows from the fact that E is minimal in $\mathcal{N}_1^c(W)$.
Finally we are interested in the structure of the set of V's such that $S(V_1)=S(V_2)$ holds. We obtain a result only if $S(V_1)$ is a minimal projection in $\mathcal{N}_1^c(W)$.

VI.5.8. Theorem:
Assume $V_1, V_2 \in \mathcal{U}(W)$ such that $S(V_1)=S(V_2)\neq E$ holds. If in addition $S(V_1)$ is a minimal projection in $\mathcal{N}_1^c(W)$ then there exist two unitary operators $W_1, W_2 \in \mathcal{M}(W)$ with

$$V_2 = W_1 V_1 W_2 .$$

From this result we learn that the ''minimal sectors'' $S(V)$ are characterized by the left–right co-sets $\mathcal{U}(\mathcal{N}(W))V\mathcal{U}(\mathcal{N}(W))$. Hence one can multiply minimal sectors and decompose the product into sectors. Unfortunately it is not known whether or not the algebra $\mathcal{N}_1^c(W)$ is of type I.

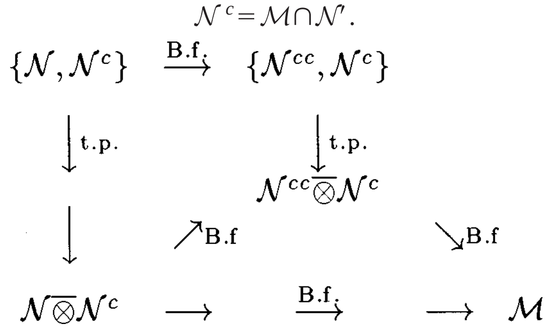
VI.6. Structure of decomposable theories

In this section it will always be assumed that $\{\mathcal{N}(W)\}$ is a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$.

Having solved the decomposition problem for tensor products and the hidden charge problem we shall have a look at the situations which might occur.
1. The simplest case is that, where $\mathcal{N}(W)$ and $\mathcal{N}^c(W)$ together generate $\mathcal{M}(W)$. In this situation the theory is the tensor product of two ''simpler'' theories.
2. The other extreme is the case where $\mathcal{N}^c(W)$ consists of multiples of the identity. This is the pure hidden charge situation.
3. If $\mathcal{N}^c(W)$ is not trivial then $\mathcal{N}(W)$ and $\mathcal{N}^{cc}(W)$ are not necessarily the same. Since the relative commutant of $\mathcal{N}(W)$ in $\mathcal{N}^{cc}(W)$ is trivial, the passage from $\mathcal{N}(W)$ to $\mathcal{N}^{cc}(W)$ is again a hidden charge problem. If we have solved this problem, then there are again two possibilities:
3.a. $\mathcal{N}^c(W)$ and $\mathcal{N}^{cc}(W)$ generate the whole algebra $\mathcal{M}(W)$. This is the same as situation 1.
3.b. $\mathcal{N}^c(W)$ and $\mathcal{N}^{cc}(W)$ generate only a subalgebra $\mathcal{N}^p(W)=\mathcal{N}^c(W)\overline{\otimes}\mathcal{N}^{cc}(W)$. In order to get to $\mathcal{M}(W)$ one has to solve the hidden charge problem for the algebra $\mathcal{N}^p(W)$.
4. Starting from $\mathcal{N}(W)$ and $\mathcal{N}^c(W)$ then it can happen that $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)=\mathcal{N}^p(W)$ is not the whole algebra $\mathcal{M}(W)$. In this situation one has to solve the hidden charge problem for $\mathcal{N}^p(W)$.

The discussion of the cases 1–4 can be summarized in the following diagram:

$$\mathcal{N}^c = \mathcal{M}\cap\mathcal{N}' .$$



t.p. stands for the construction of the tensor product.
B.f. stands for the construction of the Bose field.

If we have reached the algebra $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ then one has to solve a hidden charge problem in order to get to $\mathcal{M}(W)$. But the algebra $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ is a subalgebra of $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^c(W)$. If these algebras are different then the relative commutant of $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ in $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^c(W)$ consists again of the multiples of the identity. Hence the passage from $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ to $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^c(W)$ is a hidden charge problem.

It remains to explain why the algebra $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^{c}(W)$ does not need to coincide with $\mathcal{M}(W)$, although we have solved a hidden charge problem in order to pass from $\mathcal{N}(W)$ to $\mathcal{N}^{cc}(W)$. It might happen that both theories constructed from $\mathcal{N}^{cc}(W)$ and $\mathcal{N}^{c}(W)$ have sectors associated with Fermi fields. Let us denote these theories by $\{\mathcal{F}^{cc}(\mathcal{O})\}$ and $\{\mathcal{F}^{c}(\mathcal{O})\}$. Now let us take the tensor product $\{\mathcal{F}^{cc}(\mathcal{O})\overline{\otimes}\mathcal{F}^{c}(\mathcal{O})\}$. In this situation the theory $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^{c}(W)$ has Bose as well as Fermi sectors because the tensor product of two Fermi fields is a Bose field. If we restrict the theory to all Bose sectors, then there are sectors which are Bose sectors but not tensor products of Bose sectors. Therefore, $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^{c}(W)$ do not need to coincide with $\mathcal{M}(W)$.

VI.7. Remarks, additions, and problems

(i) The decomposition theory is based on the existence of modular covariant subalgebras $\mathcal{N}(W)\in\mathcal{M}(W)$. Therefore, the structure of this set $\mathcal{M}cs(\mathcal{M})$ defined in VI.1.2 is of interest. In particular one would like to know whether or not two different modular covariant subalgebras must have a nontrivial intersection.

(ii) The main problem of the decomposition theory is the construction of coherent families of modular covariant subalgebras. In Sec. VI.2 we have investigated the relation of half-sided translations to modular covariant subalgebras. Theorem VI.2.5 indicates that the family of modular covariant subalgebras obtained from one such subalgebra by means of Poincaré transformations is often coherent. But conditions are missing implementing that this is the case.

(iii) If $\mathcal{N}^{c}(W)$ is trivial then only little is known about the algebra $\mathcal{N}_1^{c}(W)$. In the usual theory of superselection sectors $(d=4)$ one finds that $S(V)\mathcal{N}_1^{c}(W)$ is of type I. Is this true in the general case of hidden charges? If this holds then with help of the method of Doplicher and Roberts[135] one should be able to construct the compact gauge group. However, if $S(V)\mathcal{N}_1^{c}(W)$ is of type II or III then this implies that the gauge group cannot be compact.

(iv) Nothing has been said about the statistics of sectors. It would be nice if one could repeat the arguments of Doplicher, Haag, and Roberts in the scheme presented here.

(v) During the investigation of the hidden charge problem we have envisaged the possibility of a continuous family of charged sectors. Can one construct such an example, eventually with help of Guichardet's continuous tensor product?[136] During the construction one has to face the problem that the field algebra shall be countably decomposable. The opposite possibility is the case where the center of $\mathcal{N}_1^{c}(W)$ is purely atomic. To answer these questions further investigations are needed.

(vi) Although we derived the structure of the superselection sectors only for Bose fields, it should be possible to do the same also for Bose and Fermi fields. In this case $\mathcal{F}(\mathcal{O})$ is a graded algebra which can be handled with small modifications as the pure Bose case.

(vii) The content of Sec. VI has partly been explained in Ref. 128. The structure of subtheories of QFTLO has also been investigated by Davidson in his thesis.[137]

VII. PROBLEMS FOR THE FUTURE

At the end of every section we have mentioned some problems. Nevertheless, there are some questions which should be discussed because they are, in my opinion, of importance for the future development of QFTLO.

VII.1. About the restriction to lower dimensions

Axiomatic approach to QFTLO has, compared to the Lagrangean setting, the disadvantage, that there exist mathematical operations, which allow to construct new theories out of two or more given ones. These new theories do not contain any new physics. Examples of such operations are the direct sum, direct product, and additions of charged Bose fields to the observables. Therefore, one is interested in characterizing theories which are indecomposable with respect to such operations. However, there is one operation which is of different nature. This is the restriction to lower dimensions. For Wightman fields it is known[138] that the field operators are C^∞-functions in spacelike directions with values in the space of operator valued distributions (in the time direction). Hence one can restrict Wightman fields to lower dimensions, as long as the lower dimensional space contains the time direction. The restriction in x space corresponds to integration in momentum space. Therefore, if the original theory has an isolated mass, then such information gets lost by this operation. Hence also this operation is unwanted.

In QFTLO exists a similar operation. Assume $\{\mathcal{M}(\mathcal{O}),\mathbb{R}^{d+1},\alpha\}$ is a given theory, then one can construct a theory on $\mathbb{R}^d$ as follows: Let $\hat{D}$ be a double cone in $\mathbb{R}^d$, then this is the intersection of a double cone $D(\hat{D})$ in $\mathbb{R}^{d+1}$ with $\mathcal{R}^d$. On the other hand denote by $K(\hat{D})$ the cylindrical set obtained by choosing the first d variables in $\hat{D}$ and the last variable arbitrary. $\hat{D}$ is again the intersection of $K(\hat{D})$ with $\mathbb{R}^d$. Now we choose $\mathcal{N}(\hat{D})$ such that

$$\mathcal{M}(D(\hat{D}))\subset\mathcal{N}(\hat{D})\subset\mathcal{M}(K(\hat{D}))$$

holds. Then $\{\mathcal{N}(\hat{D}),\mathbb{R}^d,\alpha\}$ defines a QFTLO provided we choose that $\mathcal{N}(\hat{D})$ fulfills covariance (in $\mathbb{R}^d$) and isotony, but these conditions are easily fulfilled. Therefore, there exist many different restrictions. Notice that for the wedge algebras all these different restrictions coincide and are equal to $\mathcal{M}(W)$. This follows from the double cone theorem, Thm. I.4.4.

Since the restriction leads to unwanted effects one would like to reconstruct the original theory. I hope, that with help of Tomitas modular theory this will be possible one day. Let us look at examples, in order to see, that my hope is not completely unjustified.

VII.1.1. Example: Take a conformal QFT in two dimensions. Choose a fixed timelike direction and restrict the theory to this line. As algebra of an interval take the algebra of the associated double cone, i.e., if (a,b), $a<b$ is the interval then we associate to it the algebra of the double cone $(a+V^+)\cap(b-V^+)$ where V^+ denotes the forward light-cone. By this we obtain a theory on the line.

The algebra $\mathcal{M}(V^++a)$ with *a not on the line* fulfills the condition of half-sided modular inclusion with respect to the algebra of $\mathbb{R}^+$. This algebra is not associated with any set of $\mathbb{R}^1$. Moreover, the associated translation commutes with the translation along the time axis. From the two-dimensional group of translations it should be possible to reconstruct the original theory on $\mathbb{R}^2$.

VII.1.2. Example: Take a standard QFTLO in three dimensions and restrict it to two dimensions. Then one should be able to recover the original theory since the algebra $\mathcal{M}(W(l_1,l_2)\cap W(l_1,l_3))$ fulfills the condition of half-sided modular inclusion with respect to the wedge algebra. This algebra is not associated to a subset of $\mathbb{R}^2$. But the corresponding half-sided translations allow to reconstruct the translational part of the stabilizer group of l_1. Also here one should be able to reconstruct the original theory on $\mathbb{R}^3$.

In order to be able to reconstruct the original theory one has to understand the spaces of $\pm$half-sided translations (and the spaces of half-sided modular inclusions) for the algebras of the wedge domains. In conformal field theories one has to look also at the algebra of the forward light-cone.

When we constructed the Poincaré group from the modular groups of the wedges (Sec. IV.4) we were able to show that certain half-sided translations commute. One has to understand better the principle behind this phenomenon.

Looking at the example of the forward light-cone in conformal field theory one sees, that the algebras of any subdomain S fulfilling $S+V^+=S$ belong to $\mathcal{H}smi(\mathcal{M}(V^+))^-$. Hence there exists a half-sided translation associated with it. For $a\in S$ one has the half-sided translation of $\mathcal{M}(V^+ +a)$ with its generator denoted by H_a. It should be possible to express the generator of the group associated with $\mathcal{M}(S)$ in terms of the family $\{H_a\}$.

The spaces $\mathcal{H}smi(\mathcal{M})^-$ and $\mathcal{H}smi(\mathcal{M})^+$ have certain order and convexity properties. These are explained in Ref. 139. Moreover, one can introduce an equivalence relation in $\mathcal{H}smi(\mathcal{M})^-$ [and also in $\mathcal{H}smi(\mathcal{M})^+$] as follows:

VII.1.3. Definition: Let $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{H}smi(\mathcal{M})^-$ and $U_i(t)$, $i=1, 2$ their associated translations. Then $\mathrm{Ad}\, U_i(t-1)\mathcal{N}_i$ will be denoted by $\mathcal{N}_i(t)$. We call $\mathcal{N}_1$ and $\mathcal{N}_2$ equivalent

$$\mathcal{N}_1 \sim \mathcal{N}_2$$

if there exist two nonzero positive numbers λ_1, λ_2 with

$$\mathcal{N}_1(\lambda_1) \subset \mathcal{N}_2 \subset \mathcal{N}_1(\lambda_2).$$

Because of the decreasing monotony of $\mathcal{N}_1(\lambda)$ one must have $\lambda_2 \leqslant \lambda_1$.
It is interesting to notice that this order structure survives if one passes to the space of equivalence classes. This discussion shows that $\mathcal{H}smi(\mathcal{M})^-$ has a rich structure, but up to now it is not clear how to get to the geometric structure on which the algebra $\mathcal{M}$ is based.

In the example of the wedge one has to construct the algebra $\mathcal{M}(W(l_1,l_3))$ from the knowledge of the algebra $\mathcal{M}(W(l_1,l_2) \cap W(l_1,l_3))$. This is possible since the half-sided translation connecting $\mathcal{M}(W(l_1,l_3))$ with $\mathcal{M}(W(l_1,l_2) \cap W(l_1,l_3))$ is also a half-sided translation of the latter algebra. Knowing this translation one can reconstruct $\mathcal{M}(W(l_1,l_3))$. The only problem here is the normalization of the group. If $U(t) \in \mathcal{H}str(\mathcal{M})^+$ and $\lambda > 0$, then $U(\lambda t) \in \mathcal{H}str(\mathcal{M})^+$. Therefore, λ has to be fixed for the correct application.

VII.2. Vacuum states on the hyperfinite III_1 algebra

As discussed in Thm. V.3.7 the Buchholz–Wichmann nuclearity property Cond. V.3.5 implies that the local algebras are hyperfinite III_1 algebras. Therefore, the algebras belonging to wedges are also hyperfinite and of type III_1. By a result of Haagerup[100] there exists (up to unitary equivalence) only one hyperfinite III_1 factor. Therefore, it is tempting to ask whether or not the vacuum state of a QFTLO can be characterized by algebraic means. What I have in mind is the structure of the set of half-sided translations, or equivalently half-sided modular inclusions connected with the vacuum state of the given theory. The situation shall be explained by examples.

VII.2.1. Example: The QFTLO on the line.

Here the wedge algebra is associated with the half-line $\mathbb{R}^+ = \{(0,\infty)\}$. If we look at the algebra associated with the set $(1,\infty)$, then this fulfills the condition of $-$half-sided modular inclusion and the algebra belonging to $(0,1)$ fulfills the condition of $+$half-sided modular inclusion. In this situation $\mathcal{M}((0,1))$ is the relative commutant of $\mathcal{M}((1,\infty))$ in $\mathcal{M}(\mathbb{R}^+)$ and the corresponding half-sided translations together with the modular group of $\mathcal{M}(\mathbb{R}^+)$ generate the Möbius group.

VII.2.2. Example: QFTLO on the d-dimensional Minkowski space, $d > 1$.

For $d=2$ one has for the algebra of the wedge two half-sided translations with opposite sign. These are the translations along the two lightlike directions. In this case the two translations commute and the two translations together with the modular group of the wedge algebra generate the two-dimensional Poincaré group. In higher dimension we will restrict to theories fulfilling the Bisognano–Wichmann property. In this situation we know from Thm. IV.4.3 that the algebra $\mathcal{M}(W[l,l_1]) \cap W[l,l_2])$ fulfills the condition of $-$half-sided modular inclusion with respect to the algebras $\mathcal{M}(W[l,l_1])$ and $\mathcal{M}(W[l,l_2])$. In this situation we obtain for $\mathcal{M}(W[l,l_1])$ a family of half-sided modular inclusions labeled by the direction of l_2. A precise characterization of this situation is still missing. This is due to the fact that one is looking for Loerentz transformations and not for the group generated by the half-sided translations.

VII.2.3. Example: Conformal field theories in higher dimension.

In this situation the set of half-sided modular inclusions is much larger. This is due to the fact that one has timelike commutativity. Let G be a set with $G + V^+ = G$ then it is easy to see that $\mathcal{M}(G \cap W)$ fulfills the condition of $-$half-sided modular inclusion with respect to the algebra $\mathcal{M}(W)$. But the importance of the associated half-sided translations is not known.

VII.2.4. Example: QFT on the two dimensional de Sitter space.

The two-dimensional de Sitter space is isomorphic to the one-sheeted hyperboloid in the three-dimensional Minkowski space. A wedge in this space is the intersection of the wedge in the ambient space with the hyperboloid. It turns out, that also in this situation the translations along the the lightlike directions are half-sided translations. But the situation is different as well from the field theory on the two-dimensional Minkowski space as from the field theory on the line. Since the ''shifted wedges'' of the de Sitter space can have an empty intersection it follows that the vacuum vector is not cyclic for the corresponding algebras. This implies that the two translations do not commute. Hence the situation is different from the Minkowski space theory. The situation is probably different from that of the line, because it is unlikely, that the different subalgebras fulfilling the condition of $\pm$half-sided modular inclusion are relative commutants of each other. (For details on QFT on de Sitter space, see, e.g., Ref. 140.)

VII.2.5. Problems:

(1) Can one characterize those states on a hyperfinite III_1 factor which permit one or more $\pm$half-sided modular inclusions?

(2) If a state permits at least one half-sided modular inclusion, what are the different families of such inclusions which can appear?

(3) Can one discriminate different theories of local observables by means of the set of half-sided modular inclusions?

VII.3. Can one interpret the local modular groups as local dynamics?

For many questions in quantum physics it is advantageous to have a local dynamics. This is in particular the case if one is interested in defining Gibbs states of a system. If one starts from the usual quantum theory one chooses as subsystems the particle in a box with reflecting walls or periodic boundary conditions. This defines a quantum system and the corresponding Hamiltonian is considered as the local one. In Lagrangean quantum field theory the energy is usually given as an integral over a Hamiltonian density. In this situation one takes as local energy the integral of the energy density over the region one is interested in. Sometimes one has to take for the integration a smooth test function which is one in the domain of interest and which tends to zero in a small neighborhood of that region. In the theory of local observables a definition of a local dynamics or an energy density is up to now only possible if the theory fulfills the nuclearity condition of Buchholz and Wichmann.[117] For the construction of a local dynamics, see, e.g., Buchholz and Junglas[141] and for the energy density see Buchholz, Doplicher, and Longo.[142] Since for a general QFTLO there exists no concept which could be used as local dynamics, it is tempting to interprete the properly scaled modular groups of local regions as local dynamics.

First we have to explain what we want to understand by a local dynamics. Let us fix a vector x_0 in the forward light cone V^+ with $x_0^2=1$. The double cones $D_R^{x_0}$ are defined by

$$D_R^{x_0}=\{Rx_0-V^+\}\cap\{-Rx_0+V^+\}. \tag{VII.3.1}$$

Let $U_R(t)$ be a family of unitary groups depending continuously on R such that the group $\mathrm{Ad}\ U_R(t)$ belongs to the automorphisms of $\mathcal{M}(D_R^{x_0})$. Then we say that these groups define a local dynamics if for every bounded set $\mathcal{O}$ the expression

$$U_R(t)A\Omega, \quad A\in\mathcal{M}(\mathcal{O})$$

converges for $R\to\infty$ to $T(tx_0)A\Omega$ in the topology of the Hilbert space and this uniformly on every compact of the t axis.

That the modular groups might be a good candidate is indicated by the following two examples.

VII.3.1. Example: For a fixed double cone we choose $D=\{x;|x^0|+\|\vec{x}\|<1\}$ and the running double cone will be replaced by a running family of wedges $W_R:=W-Rx^1$ with $R>1$ and x^1 is a fixed vector perpendicular to the time direction x^0 with $(x^1)^2=-1$. If we denote the modular group of W_R by Δ_R^{it} then we choose as local dynamics

$$U_R(t)=\Delta_R^{-i(t/2\pi R)}.$$

Because of $\Delta_R^{-i(t/2\pi R)}=T(-Rx^1)\Delta_0^{-i(t/2\pi R)}T(Rx^1)$ this becomes with Remark II.5.3 $=T((\Lambda_W(-t/2\pi R)-1)Rx^1)\Delta_0^{-i(t/2\pi R)}$, where $T(x)$ denotes the representation of the translations. With Eq. (I.5.3) we find:

$$\left(\Lambda_W\left(-\frac{t}{2\pi R}\right)-1\right)Rx:^1=x^0R\sinh\frac{t}{R}+x^1R\left(\cosh\left(-\frac{t}{R}\right)-1\right)=x^0t+\mathcal{O}\left(\frac{1}{R}\right).$$

This implies

$$U_R(t)A\Omega=T(tx^0+\mathcal{O}(1/R))\Delta_0^{-i(t/2\pi R)}A\Omega.$$

Since Δ_0^{it} is strongly continuous we obtain by the unitarity of the operators

$$s-\lim_{R\to\infty}U_R(t)A\Omega=T(t)A\Omega,\quad A\in\mathcal{M}(D).$$

VII.3.2. Example: As a second example we look at conformal field theory, where the modular groups of the double cones are known (Thm. III.2.2). We choose as running domains the double cones of radius R and choose

$$U_R(t)=\Delta_R^{-i(t/\pi R)}.$$

With the notation of Thm. III.2.2 this corresponds to the transformation

$$x^{\pm}\left(-\frac{t}{\pi R}\right)=R\,\frac{-(1-x^{\pm}/R)+e^{2t/R}(1+x^{\pm}/R)}{(1-x^{\pm}/R)+e^{2t/R}(1+x^{\pm}/R)}.$$

For small $x^{\pm}$ and large R we obtain

$$x^{\pm}\left(-\frac{t}{\pi R}\right)=x^{\pm}+t+\mathcal{O}\left(\frac{1}{R}\right).$$

Since the representation of the conformal group is continuous it follows, also in this example, that $U(t)$ converges for large R to the time translation.

There is one essential difference between the two examples, namely, the scaling of the corresponding modular groups differs by the factor 2. I think that one has to understand the origin of the difference in the scaling factors before one is able to prove that $\Delta_R^{-i(t/\pi R)}$ converges to the time translation also in the general case.

VII.4. Modular theory in charged sectors

Almost all the results described in this review are based on the fact that cyclic and separating vector Ω for the local algebras is at the same time the only vector which is invariant under the representation of the Poincaré group. We do not have this situation in the charged sectors. But if we take a vector ψ which has compact energy contribution and if l is one of the lightlike vectors defining the wedge $W(l,l')$, then $U(\lambda l)$, $\lambda\in\mathbb{R}$ is again a group with positive generator which maps $\mathcal{M}(W(l,l')$ into itself. Moreover the vector $U(\lambda l)\psi$ is again a vector which is cyclic and separating for $\mathcal{M}(W)$. In addition the modular group of $U(\lambda l)\psi$ can be computed from that of ψ with help of the cocycle Radon Nikodym derivative $[DU(\lambda l)\psi:D\psi]_t$.[143,144] If we denote the Radon Nikodym derivative for a moment by u_t, then the cocycle relation means

$$u_{s+t}=u_s\sigma_\psi^s(u_t).\tag{VII.4.1}$$

The action of the modular group belonging to $U(\lambda l)\psi$ can be computed with help of the formula

$$\sigma^t_{U(\lambda l)\psi}(A) = [DU(\lambda l)\psi : D\psi]_t \sigma^t_\psi(A)[DU(\lambda l)\psi : D\psi]^*_t, \quad A \in \mathcal{M}(W). \tag{VII.4.2}$$

VII.4.1. Problems:
(i) We know that the group $U(\lambda l)$ has an analytic continuation into the upper complex half-plane. What does this imply for the Radon Nikodym derivative $[DU(\lambda l)\psi : D\psi]_t$? Note that for complex λ the vector $U(\lambda l)\psi$ is again cyclic and separating for $\mathcal{M}(\mathcal{O})$, which implies that the Radon Nikodym derivative is also defined for those values of λ.
(ii) Does there exist any relation between $\Delta^{it}_\psi, [DU(\lambda l)\psi : D\psi]_t$ and $U(\lambda l)$ besides the known standard ones?

APPENDIX A: BIBLIOGRAPHY ON THE ALGEBRAIC THEORY OF SUPERSELECTION SECTORS IN LOW DIMENSIONS

In the last decade, the algebraic theory of superselection sectors was supplemented by a vast reservoir of examples originating in two-dimensional conformal quantum field theory. As is well known, in low dimensions the possibility of braid group statistics is a new feature beyond the original DHR analysis, which is however easily incorporated into the original framework. The following is a list of prominent references in the algebraic theory of superselection sectors in low dimensions.

The DHR theory was adapted to the case of braid group statistics in Refs. 145 and 146. The local von Neumann algebras for specific models based on non-Abelian current algebras were constructed and analyzed in Refs. 147–149. Modular theory was applied to a general study of global properties of chiral nets concerning Haag duality, conformal covariance, spin-statistics theorem and CPT theorem in Refs. 150 and 151. Models with a breakdown of Haag duality and the construction of the associated dual net were discussed in Refs. 152–154. Sufficient conditions to reconstruct, using modular theory,[55] a chiral net with conformal symmetry and spectrum condition from a single half-sided modular inclusion of von Neumann algebras were formulated in Refs. 56 and 57. For models with Haag duality in two dimensions it was shown that the split property for wedges (presumably related to a mass gap) excludes the existence of localized superselection sectors at all,[155] while solitonic sectors will generically emerge. Properties of the latter were studied in Refs. 156–158.

The issue of charged fields which create superselection sectors from the vacuum, and of an underlying symmetry principle, was addressed from various sides. A reconstruction theorem comparable to the result by Doplicher and Roberts[159] cannot be achieved since non-Abelian braid group statistics poses an obvious obstruction. In the Abelian case, an anyonic field algebra was constructed in Ref. 160. The reduced field bundle (RFB) of intertwining nonlocal fields was introduced as a general construction in Ref. 145, and conformal covariance properties of these algebras were analyzed in Ref. 161. Pointlike exchange fields associated with the RFB were constructed in Ref. 162, and the weak C^* Hopf symmetry of the RFB was discovered in Refs. 163 and 164. Other, ultimately unsatisfactory, symmetry concepts were discussed in Refs. 165 and 166. A theory of sector induction and restriction between a theory and a subtheory equipped with a global conditional expectation was initiated in Ref. 167 and was further elaborated with a view on specific chiral models in Ref. 168.

APPENDIX B: REFERENCES FOR APPLICATIONS OF TOMITA–TAKESAKI THEORY IN QUANTUM FIELD THEORY ON CURVED SPACETIME

Listed below are references containing applications of Tomita–Takesaki theory to quantum field theory on curved space–time.

On a generic curved space–time, there are in general no symmetries (space–time isometries) present, and hence there is no natural candidate for a vacuum state. Likewise, in a generic curved space–time, it is in general not clear which space–time regions, if any, play a similar role as the wedge regions in Minkowski space–time in the sense that the modular objects corresponding to von Neumann algebras associated with these regions and preferred vacuumlike vectors act in a suitable sense geometrical. Therefore, most applications of Tomita–Takesaki theory to quantum field theory in curved space–time so far have been restricted to a class of space–times possessing a structure which to certain extent mimics the geometrical features underlying the Bisognano–Wichmann situation, i.e., there are natural wedge regions and Killing flows leaving these wedge regions invariant. In this case, a variety of versions of a geometric action of modular objects associated with wedge regions and certain preferred states has been investigated in the works.[169–178] The pioneering work of this list is Ref. 169, where a situation analogous to the Bisognano–Wichmann setting is modeled on Schwarzschild–Kruskal space–time. An operator-algebraic version of it appears in Ref. 170. The works[171,172] deal with an investigation of this Bisognano–Wichmann-like situation on black-hole space–times for free scalar field models. In Refs. 173 and 174, Bisognano–Wichmann-like scenarios are investigated on de Sitter space–time, in Ref. 175 on black-hole space–times and in Ref. 88 on anti-de Sitter space–time.

An attractive line of thought is to try and characterize vacuum states on a generic space–time by a suitable form of geometric modular action with respect to von Neumann algebras associated with a class of distinguished regions (e.g., wedge regions, cf. also Ref. 173). On a generic space–time without isometries such a geometric action of modular objects cannot be expected to be given by point transformations on the underlying space–time manifold. A more general approach addressing this issue is developed in Ref. 87.

In Ref. 176 a somewhat different approach, compared to the works just cited, is taken towards the physical interpretation of modular objects in generally covariant quantum theories.

The type of the local von Neumann algebras of a quantum field theory is related to the spectra of their associated modular operators (Connes' invariant) and can, like on Minkowski space–time, be fixed on curved space–time via assumptions on the quantum field theory's short-distance scaling limits. This question is considered in Refs. 177 and 178.

REFERENCES

[1] P. A. M. Dirac, ''The Quantum Theory of the Electron,'' Proc. R. Soc. London, Ser. A **117**, 610–624 (1928).
[2] P. Jordan and E. Wigner, ''Über das Paulische Äquivalenzverbot,'' Z. Phys. **47**, 631–651 (1928).
[3] W. Heisenberg und W. Pauli, ''Zur Quantendynamik der Wellenfelder I,'' Z. Phys. **56**, 1–61 (1929).
[4] W. Heisenberg und W. Pauli, ''Zur Quantendynamik der Wellenfelder II,'' Z. Phys. **59**, 169–190 (1930).
[5] J. Bros and D. Buchholz, ''Towards a Relativistic KMS-Condition,'' Nucl. Phys. B **429**, 291–318 (1994).
[6] H. Lehmann, K. Symanzik, and W. Zimmermann, ''Zur Formulierung quantisierter Feldtheorien,'' Nuovo Cimento **1**, 425 (1955).
[7] A. S. Wightman, ''Quantum field theory in terms of vacuum expectation values,'' Phys. Rev. **101**, 860 (1956).
[8] R. Haag, *Local Quantum Physics* (Springer Verlag, Berlin, 1992).
[9] L. Schwartz, *Théorie des Distributions*, Vol. I (Hermann, Paris, 1957).
[10] L. Schwartz, *Théorie des Distributions*, Vol. II (Hermann, Paris, 1959).
[11] M. A. Naimark, *Normed Rings* (P. Noordhoff N. V., Groningen, 1959).
[12] J. von Neumann ''Mathematische Begründung der Quantenmechanik,'' Gött. Nachr. **Jg. 1927**, 1–12. Collected Works Vol. 1, No. 9.
[13] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
[14] I. E. Segal, ''Postulates for general quantum mechanics,'' Ann. Math. **48**, 930–948 (1947).
[15] R. Hagg, *Discussion des ''axioms'' et des propriètés asymptotiques d'une thèorie des champs locales avec particules composées*, in ''les Problèmes mathematiques de la théorie quantique des champs,'' Pub. CNRS. Vol. **85** (1959).
[16] R. Haag and B. Schroer, ''Postulates of quantum field theory,'' J. Math. Phys. **3**, 248–256 (1962).
[17] R. Haag and D. Kastler, ''An algebraic approach to quantum field theory,'' J. Math. Phys. **5**, 848–861 (1964).
[18] J. Glimm and A. Jaffe, *Collected Papers*, ''I Quantum Field Theory and Statistical Mechanics, II Constructive Quantum Field Theory'' (Birkhäuser, Boston, Basel, Stuttgart, 1985).
[19] K. Symanzik, ''Euclidean quantum field theory,'' in *Proceedings of the International School of Physikc ''Enrico Fermi,'' Varenna Course XLV,* edited by R. Jost (Academic, New York, 1969).
[20] K. Osterwalder and R. Schrader, ''Axioms for Euclidean Green's Functions,'' Commun. Math. Phys. **31**, 83–112 (1973).
[21] K. Osterwalder and R. Schrader, ''Axioms for Euclidean Green's Functions II,'' Commun. Math. Phys. **42**, 281–305 (1975).
[22] V. Glaser, ''On the Equivalence of Euclidean and Wightman Formulation of Field Theory,'' Commun. Math. Phys. **37**, 257–272 (1994).
[23] J. Yngvason, ''On the decomposition of Wightman functionals in the Euclidean frame-work,'' Rep. Math. Phys. **13**, 101–115 (1978).
[24] H.-J. Borchers and J. Yngvason, ''Necessary and Sufficient Conditions for Integral Representation of Schwinger Functionals at Schwinger Points,'' Commun. Math. Phys. **47**, 197–213 (1976).

[25] W. Zimmermann, ''Local Field Equation for A^4-Coupling in Renormalized Perturbation Theory,'' Commun. Math. Phys. **6**, 161–188 (1967).
[26] K. Wilson, ''Non-Lagrangian Models of Current Algebra,'' Phys. Rev. **179**, 1499–1512 (1969).
[27] H.-J. Borchers, *Translation Group and Particle Representations in Quantum Field Theory*, Lecture Notes in Physics, Vol. m40 (Springer, Heidelberg, 1996).
[28] M. Tomita, ''Quasi-standard von Neumann algebras,'' Preprint (1967).
[29] R. Haag, N. Hugenholtz, and M. Winnink, ''On the equilibrium state in quantum statistical mechanics,'' Commun. Math. Phys. **5**, 215–236 (1967).
[30] M. Winnink, An Application of C^*-Algebras to Quantum Statistical Mechanics of Systems in Equilibrium, thesis, Groningen (1968).
[31] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Mathematics, Vol. 128 (Springer-Verlag, Berlin, 1970).
[32] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I* (Springer-Verlag, New York, 1979).
[33] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras II* (Academic, New York, 1986).
[34] S. Stratila, *Modular Theory in Operator Algebras* (Abacus, Turnbridge Wells, Kent, England, 1981).
[35] H. Reeh and S. Schlieder, ''Eine Bemerkung zur Unitäräquivalenz von Lorentzinvarianten Feldern,'' Nuovo Cimento **22**, 1051 (1961).
[36] U. Haagerup, ''The standard form of von Neumann algebras,'' Math. Scand. **37**, 271–283 (1975).
[37] H. J. Brehmermann, R. Oehme, and J. G. Taylor, ''Proof of dispersion relation in quantized field theories,'' Phys. Rev. **109**, 2178–2190 (1958).
[38] H. Epstein, *Some Analytic Properties of Scattering Amplitudes in Quantum Field Theory*, in 1965 Brandeis Summer Institute (Gordon and Breach, New York, 1966).
[39] V. S. Vladimirov, ''The construction of envelopes of holomorphy for domains of special type,'' Dokl. Akad. Nauk SSSR **134**, 251 (1960).
[40] H. J. Borchers, ''Über die Vollständigkeit lorentzinvarianter Felder in einer zeitartigen Röhre,'' Nuovo Cimento **19**, 787–796 (1961).
[41] J. Bros, H. Epstein, V. Glaser, and R. Stora, *Quelques Aspects Globaux des Problemes d'Edge-of-the-Wedge*, in ''Hyperfunctions and Theoretical Physics (Nice Rencontre 1973),'' Lecture Notes in Mathematics, Vol. 449 (Springer-Verlag, Berlin, 1975).
[42] R. Jost and H. Lehmann, ''Integraldarstellungen kausaler Kommutatoren,'' Nuovo Cimento **5**, 1598–1610 (1957).
[43] F. J. Dyson, ''Integral representation of causal commutators,'' Phys. Rev. **110**, 1460–1464 (1958).
[44] J. Bros, A. Messiah, and R. Stora, ''A problem of analytic completion related to the Jost–Lehmann–Dyson formula,'' J. Math. Phys. **2**, 639–651 (1961).
[45] S. Hawking, ''Particle Creation by Black Holes,'' Commun. Math. Phys. **43**, 199–220 (1975).
[46] W. G. Unruh, ''Notes on black-hole evaporation,'' Phys. Rev. D **14**, 870–892 (1976).
[47] D. Buchholz and R. Verch, ''Scaling Algebras and Renormalization Group in Algebraic Quantum Field Theory,'' Rev. Math. Phys. **7**, 1195–1240 (1995).
[48] M. H. Stone, ''On unbounded operators in Hilbert space,'' Ind. Math. Soc. **15**, 155–192 (1951).
[49] A. E. Nussbaum, ''Reduction Theory for Unbounded Closed Operators in Hilbert Space,'' Duke Math. J. **31**, 33–44 (1964).
[50] C. D'Antoni, S. Doplicher, K. Fredenhagen, and R. Longo, ''Convergence of Local Charges and Continuity Properties of W^*-inclusions,'' Commun. Math. Phys. **110**, 325–348 (1987).
[51] H.-J. Borchers, ''On the use of modular groups in quantum field theory,'' Ann. Inst. Henri Poincaré **64**, 331–382 (1995).
[52] G. K. Pedersen, *C^*-Algebras and their Automorphism Groups* (Academic, London, 1979).
[53] M. Florig, ''On Borchers' Theorem,'' Lett. Math. Phys. **46**, 289–293 (1998).
[54] H.-J. Borchers, ''On the embedding of von Neumann subalgebras,'' Commun. Math. Phys. **205**, 69–79 (1999).
[55] H.-J. Borchers, ''The CPT-Theorem in Two-dimensional Theories of Local Observables,'' Commun. Math. Phys. **143**, 315–332 (1992).
[56] H.-W. Wiesbrock, ''Half-Sided Modular Inclusions of von Neumann Algebras,'' Commun. Math. Phys. **157**, 83 (1993).
[57] H.-W. Wiesbrock, ''Half-Sided Modular Inclusions of von Neumann Algebras,'' Commun. Math. Phys. **184**, 683–685 (1997).
[58] H. J. Borchers, ''On Modular Inclusion and Spectrum Condition,'' Lett. Math. Phys. **27**, 311–324 (1993).
[59] H.-W. Wiesbrock, ''A comment on a recent work of Borchers,'' Lett. Math. Phys. **25**, 157–159 (1992).
[60] H.-J. Borchers, ''Half-sided Translations and the Type of von Neumann Algebras,'' Lett. Math. Phys. **44**, 283–290 (1998).
[61] R. V. Kadison, ''Remarks on the Type of von Neumann Algebras of Local Observables in Quantum Field Theory,'' J. Math. Phys. **4**, 1511 (1963).
[62] M. Guenin and B. Misra, ''On the von Neumann Algebras Generated by Field Operators,'' Nuovo Cimento **30**, 1272–1290 (1963).
[63] J. Bisognano and E. H. Wichmann, ''On the duality condition for a Hermitean scalar field,'' J. Math. Phys. **16**, 985–1007 (1975).
[64] J. Bisognano and E. H. Wichmann, ''On the duality condition for quantum fields,'' J. Math. Phys. **17**, 303–321 (1976).
[65] S. Trebels, Über die geometrische Wirkung modularer Automorphismen, thesis, Göttingen (1997).
[66] R. Jost, ''Eine Bemerkung zum CTP Theorem,'' Helv. Phys. Acta **30**, 409–416 (1957).
[67] D. Buchholz, *On the Structure of Local Quantum Fields with Non-trivial Interactions*, in ''Proceedings of the Interna-

tional Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics, Leipzig, 1977'' (Teubner-Texte zur Mathematik, 1978), p. 146–153.
[68] P. D. Hislop and R. Longo, ''Modular structure of the local algebra associated with a free massless scalar field theory,'' Commun. Math. Phys. **84**, 71–85 (1982).
[69] H.-J. Borchers and J. Yngvason, ''Modular groups of quantum fields in thermal states,'' J. Math. Phys. **40**, 601–624 (1999).
[70] J. Yngvason, ''A Note on Essential Duality,'' Lett. Math. Phys. **31**, 127–141 (1994).
[71] D. Guido, ''Modular covariance, PCT, spin and statistics,'' Ann. Inst. Henri Poincaré **64**, 383–398 (1995).
[72] B. Kuckert, Spin & Statistics, Localization Regions, and Modular Symmetries in Quantum Field Theory, thesis, Hamburg (1998).
[73] W. Pauli, *Exclusion Principle, Lorentz Group and Reflection of Space-Time and Charge*, Niels Bohr and the Development of Physics, edited by W. Pauli, I. Rosenfeld, and V. Weisskopf (Pergamon, London, 1955).
[74] J. Schwinger ''The Theory of Quantized Fields I,'' Phys. Rev. **82**, 914–927 (1951).
[75] G. Lüders, ''On the equivalence of invariance under time reversal and under particle–antiparticle conjugation for relativistic field theories,'' Danske Vidensk. Selskab, Mat.-fysiske Meddelelser **28**, no 5, 1–17 (1954).
[76] G. Lüders and B. Zumino, ''On the connection between spin and statistics,'' Phys. Rev. **110**, 1450–1453 (1958).
[77] D. Hall and A. S. Wightman, ''A theorem on invariant analytic functions with applications to relativistic quantum field theory,'' Danske Vidensk. Selskab, Mat.-fysiske Meddelelser **31**, no 5, 1–41 (1957).
[78] R. Streater, ''Local Fields with the Wrong Connection Between Spin and Statistics,'' Commun. Math. Phys. **5**, 88–96 (1967).
[79] A. I. Oksak and I. T. Todorov, ''Invalidity of the TCP-Theorem for Infinite-Component Fields,'' Commun. Math. Phys. **11**, 125 (1968).
[80] H.-J. Borchers, ''When does Lorentz Invariance imply Wedge-Duality,'' Lett. Math. Phys. **35**, 39–60 (1995).
[81] R. Brunetti, D. Guido, and R. Longo, ''Group cohomology, modular theory and space-time symmetries,'' Rev. Math. Phys. **7**, 57–71 (1994).
[82] D. Guido and R. Longo, ''An Algebraic Spin and Statistic Theorem,'' Commun. Math. Phys. **172**, 517–533 (1995).
[83] H.-J. Borchers, ''Half-sided Modular Inclusion and the Construction of the Poincaré Group,'' Commun. Math. Phys. **179**, 703–723 (1996).
[84] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon, New York, 1963).
[85] R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, RI, 1965).
[86] G. W. Mackey, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, New York, 1968).
[87] D. Buchholz and S. J. Summers, ''An algebraic characterization of vacuum states in Minkowski space,'' Commun. Math. Phys. **155**, 449–458 (1993).
[88] D. Buchholz, O. Dreyer, M. Florig, and S. Summers, ''Geometric Modular Action and Spacetime Symmetry Groups,'' Rev. Math. Phys. (to be published).
[89] D. Buchholz, M. Florig, and S. J. Summers, ''An algebraic characterization of vacuum states in Minkowski space,'' II, preprint hep-th/9905178.
[90] K. Fredenhagen and J. Hertel, ''Local algebras of observables and point like localized fields,'' Commun. Math. Phys. **80**, 555–561 (1981).
[91] H.-J. Borchers, ''On Poincaré transformations and the modular group of the algebra associated with a wedge,'' Lett. Math. Phys. **46**, 295–301 (1998).
[92] B. Schroer and H.-W. Wiesbrock, Modular Theory and Geometry, Preprint (1998).
[93] H.-W. Wiesbrock, ''Symmetries and Modular Intersections of von Neumann algebras,'' Lett. Math. Phys. **39**, 203–212 (1997).
[94] H.-W. Wiesbrock, ''Modular Intersections of von Neumann Algebras in Quantum Field Theory,'' Commun. Math. Phys. **193**, 269–285 (1998).
[95] A. Connes, ''Caractérisation des algèbres de von Neumann comme espaces vectoriels ordonnés,'' Ann. Inst. Fourier **24**, 121–155 (1974).
[96] R. Powers, ''Representations of uniformly hyperfinite algebras and their associated von Neumann rings,'' Ann. Math. **86**, 138–171 (1967).
[97] H. Araki and E. J. Woods, ''A classification of factors,'' Pub. R.I.M.S., Kyoto Univ. **4**, 51–130 (1968).
[98] A. Connes, ''Un classification de factors de type III,'' Ann. Sci. Ecole Norm. Sup. **6**, 133–252 (1973).
[99] A. Connes, ''Classification of injective factors,'' Ann. Math. **104**, 73–115 (1976).
[100] U. Haagerup, ''Connes' bicentralizer problem and uniqueness of injective factors of type III_1,'' Acta Math. **158**, 95–148 (1987).
[101] H. Araki, ''On the algebra of all local observables,'' Prog. Theor. Phys. **32**, 844–854 (1964).
[102] H. J. Borchers, ''On the structure of the algebra of field operators,'' Nuovo Cimento **24**, 214–236 (1962).
[103] S. Doplicher, R. V. Kadison, D. Kastler, and D. W. Robinson, ''Asymptotically Abelian systems,'' Commun. Math. Phys. **6**, 101–120 (1967).
[104] S. Doplicher, D. Kastler, and D. W. Robinson, ''Covariance algebras in field theory and statistical mechanics,'' Commun. Math. Phys. **3**, 1–28 (1966).
[105] D. Ruelle, ''States of Physical Systems,'' Commun. Math. Phys. **3**, 133–150 (1966).
[106] O. E. Lanford and D. Ruelle, ''Integral representations of invariant states on C^*-algebras,'' J. Math. Phys. **8**, 1460–1463 (1967).
[107] E. Størmer, ''Large Groups of Automorphisms of C^*-Algebras,'' Commun. Math. Phys. **5**, 1–22 (1967).

[108] A. S. Wightman, ''La théorie quantique locale et la théorie quantique des champs,'' Ann. Inst. Henri Poincare, Sect. A **I**, 403–420 (1964).
[109] W. Driessler, ''Comments on Lightlike Translations and Applications in Relativistic Quantum Field Theory,'' Commun. Math. Phys. **44**, 133–141 (1975).
[110] H. J. Borchers, ''Energy and momentum as observables in quantum field theory,'' Commun. Math. Phys. **2**, 49–54 (1966).
[111] H. J. Borchers, ''A Remark on a Theorem of B. Misra,'' Commun. Math. Phys. **4**, 315–323 (1967).
[112] R. Longo, ''Notes on Algebraic Invariants for Non-commutative Dynamical Systems,'' Commun. Math. Phys. **69**, 195–207 (1979).
[113] S. Doplicher and R. Longo, ''Standard and split inclusions of von Neumann algebras,'' Invent. Math. **73**, 493 (1984).
[114] K. Fredenhagen, ''On the Modular Structure of Local Algebras of Observables,'' Commun. Math. Phys. **97**, 79–89 (1985).
[115] R. Haag, H. Narnhofer, and U. Stein, ''On quantum field theory in gravitational background,'' Commun. Math. Phys. **94**, 219–238 (1984).
[116] K. Fredenhagen, ''On the Modular Structure of Local Algebras of Observables,'' Commun. Math. Phys. **97**, 79–89 (1985).
[117] D. Buchholz and E. H. Wichmann, ''Causal independence and energy level density of states in local quantum field theory,'' Commun. Math. Phys. **106**, 321 (1986).
[118] D. Buchholz, C. D'Antoni, and K. Fredenhagen, ''The universal structure of local algebras,'' Commun. Math. Phys. **84**, 123–135 (1987).
[119] H. J. Borchers, ''On the Implementation of Automorphism Groups,'' Commun. Math. Phys. **14**, 305–314 (1969).
[120] H.-J. Borchers, ''C^*-Algebras and Automorphism Groups,'' Commun. Math. Phys. **88**, 95–103 (1983).
[121] H. J. Borchers, ''Symmetry Groups of *C^*-algebras and Associated von Neumann Algebras,'' in Dynamics of Complex and Irregular Systems*, edited by Ph. Blanchard, L. Streit, M. Sirugue-Collin, and D. Testard (World Scientific, Singapore, 1993), pp. 12–21.
[122] B. Iochum, *Cones Autopolaires et Algèbres de Jordan*, Lecture Notes in Math., Vol. 1049 (Springer-Verlag, Heidelberg, 1983).
[123] W. Driessler, ''On the type of local algebras in quantum field theory,'' Commun. Math. Phys. **53**, 295–297 (1977).
[124] H.-J. Borchers and M. Wollenberg, ''On the relation between types of local algebras in different global representations,'' Commun. Math. Phys. **137**, 161–173 (1991).
[125] M. Takesaki, ''Conditional Expectations in von Neumann Algebras,'' J. Funct. Anal. **9**, 306–321 (1972).
[126] H. Kosaki, ''Extension of Jones' Theory on Index to Arbitrary Factors,'' J. Funct. Anal. **66**, 123–140 (1986).
[127] L. Kadison and D. Kastler, *Cohomological aspects and relative separability of finite Jones index factors*, Nachr. Akad. d. Wissensch. Göttingen (1992), pp. 95–105.
[128] H.-J. Borchers, Conditional Expectation and Half-sided Translations, Preprint (1999).
[129] L. Ge and R. Kadison, ''On tensor products of von Neumann algebras,'' Invent. Math. **123**, 453–466 (1966).
[130] J. Tomijama, ''On the projections of norm one in W^* algebras,'' Proc. Jpn. Acad. **33**, 608–612 (1957).
[131] H. J. Borchers, ''Local Rings and the Connection of Spin with Statistics,'' Commun. Math. Phys. **1**, 281–307 (1965).
[132] S. Doplicher, R. Haag, and J. E. Roberts, ''Fields, observables and gauge transformations I,'' Commun. Math. Phys. **13**, 1 (1969).
[133] S. Doplicher, R. Haag, and J. E. Roberts, ''Fields, observables and gauge transformations II,'' Commun. Math. Phys. **15**, 173 (1969).
[134] D. Buchholz and K. Fredenhagen, ''Locality an the structure of particle states,'' Commun. Math. Phys. **84**, 1 (1982).
[135] S. Doplicher and J. E. Roberts, ''Endomorphisms of C^*-algebras, cross products and duality of compact groups,'' Ann. Math. **98**, 157 (1989).
[136] A. Guichardet, Tensor Products of C^*-Algebras, Mat. Inst. Univ. Aarhus, Lecture notes no **12**, (1969).
[137] D. R. Davidson, Classification of Subsystems of Local Algebras, Dissertation, U.C. Berkeley (1988).
[138] H. J. Borchers, ''Field Operators as C^∞ Functions in Spacelike Directions,'' Nuovo Cimento **33**, 1600–1613 (1964).
[139] H.-J. Borchers, ''On the Lattice of Subalgebras Associated with the Principle of Half-sided Modular Inclusion,'' Lett. Math. Phys. **40**, 371–390 (1996).
[140] H.-J. Borchers and D. Buchholz, ''Global Properties of Vacuum States in de Sitter Space,'' Ann. Inst. Henri Poincaré **70**, 23–40 (1999).
[141] D. Buchholz and P. Junglas, ''On the existence of equilibrium states in local quantum field theory,'' Commun. Math. Phys. **121**, 255–270 (1989).
[142] D. Buchholz, S. Doplicher, and R. Longo, ''On Noether's Theorem in Quantum Field Theory,'' Ann. Phys. **170**, 1–17 (1986).
[143] A. Connes ''Sur le théorème de Radon Nikodym pour les poids normaux fidèles semifinis,'' Bull. Sci. Math. **97**, 253–258 (1973).
[144] A. Connes and M. Takesaki, ''Flow of weights on a factor of type III,'' Tôhoku Math. J. **29**, 473–575 (1977).
[145] K. Fredenhagen, K.-H. Rehren, and B. Schroer, ''Superselection sectors with permutation group statistics and exchange algebras, I,'' Commun. Math. Phys. **125**, 201–226 (1989).
[146] J. Fröhlich and F. Gabbiani, ''Braid statistics in local quantum theory,'' Rev. Math. Phys. **2**, 251–353 (1991).
[147] A. Wassermann, *Operator Algebras and Conformal Field Theory*, Proceedings ICM Zürich, 1994 (Birkhäuser, 1995), pp. 966–979.
[148] A. Wassermann, ''Operator algebras and conformal field theory,'' Invent. Math. **133**, 467–538 (1998).

[149] T. Loke, Operator algebras and conformal field theory for the discrete series representations of Diff S^1, Ph.D. thesis, Cambridge (UK), 1994.
[150] R. Brunetti, D. Guido, and R. Longo, ''Modular structure and duality in conformal quantum field theory,'' Commun. Math. Phys. **156**, 201–220 (1993).
[151] D. Guido and R. Longo, ''The conformal spin and statistics theorem,'' Commun. Math. Phys. **181**, 11–36 (1996).
[152] D. Buchholz and H. Schulz-Mirbach, ''Haag duality in conformal quantum field theory,'' Rev. Math. Phys. **2**, 105–125 (1990).
[153] M. Müger, ''Quantum double actions on operator algebras and orbifold quantum field theories,'' Commun. Math. Phys. **191**, 137–181 (1998).
[154] D. Guido, R. Longo, and H.-W. Wiesbrock, ''Extensions of conformal nets and superselection structures,'' Commun. Math. Phys. **192**, 217–244 (1998).
[155] M. Müger, ''Superselection structure of massive quantum field theories in $1+1$ dimensions,'' Rev. Math. Phys. **10**, 1147–1170 (1998).
[156] D. Schlingemann, ''Construction of kink sectors for two-dimensional quantum field theory models: an algebraic approach,'' Rev. Math. Phys. **10**, 851–891 (1998).
[157] M. Müger, ''On soliton automorphisms in massive and conformal theories,'' Rev. Math. Phys. (to be published).
[158] K.-H. Rehren, ''Spin-statistics and CPT for solitons,'' Lett. Math. Phys. **46**, 95–110 (1998).
[159] S. Doplicher and J. E. Roberts, ''Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics,'' Commun. Math. Phys. **131**, 51–107 (1990).
[160] D. Buchholz, G. Mack, and I. T. Todorov, ''The current algebra on the circle as a germ of local field theories,'' Nucl. Phys. B (Proc. Suppl.) **5B**, 20–56 (1988).
[161] K. Fredenhagen, K.-H. Rehren, and B. Schroer, ''Superselection sectors with permutation group statistics and exchange algebras, II,'' Rev. Math. Phys. **Special Issue** , 113–157 (1992).
[162] K. Fredenhagen and M. Jörß, ''Conformal Haag–Kastler nets, pointlike localized fields and the existence of operator product expansions,'' Commun. Math. Phys. **176**, 541–554 (1996).
[163] F. Nill, unpublished (1994).
[164] K.-H. Rehren, *Weak C* Hopf symmetry*, in Group XXI, Proceedings Goslar, 1996, edited by H.-D. Doebner *et al.* (Heron, Sofia, 1997), pp. 62–69.
[165] G. Mack and V. Schomerus, ''Conformal field algebras with quantum symmetry from the theory of superselection sectors,'' Commun. Math. Phys. **134**, 139–196 (1990).
[166] K.-H. Rehren, ''Field algebras for anyons and plektons,'' Commun. Math. Phys. **132**, 461–483 (1990).
[167] R. Longo and K.-H. Rehren, ''Nets of subfactors,'' Rev. Math. Phys. **7**, 567–597 (1995).
[168] J. Böckenhauer and D. Evans, ''Modular invariants, graphs and α-induction for nets of subfactors, I–III,'' Commun. Math. Phys. **197**, 361–386 (1998), hep-th/9805023, and preprint in preparation.
[169] G. L. Sewell, ''Quantum fields on manifolds: PCT and gravitationally induced thermal states,'' Ann. Phys **141**, 201 (1982).
[170] S. J. Summers and R. Verch, ''Modular inclusion, the Hawking temperature, and quantum field theory in curved spacetime,'' Lett. Math. Phys. **37**, 145 (1996).
[171] B. S. Kay, ''The double-wedge algebra for quantum fields on Schwarzschild and Minkowski spacetime,'' Commun. Math. Phys. **100**, 57 (1985).
[172] B. S. Kay and R. M. Wald, ''Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon,'' Phys. Rep. **207**, 49 (1991).
[173] H.-J. Borchers and D. Buchholz, ''Global properties of vacuum states in de Sitter space,'' Ann. Inst. Henri Poincaré **70**, 23 (1999).
[174] J. Bros, H. Epstein, and U. Moschella, ''Analyticity properties and thermal effects for general quantum field theory on de Sitter space,'' Commun. Math. Phys. **196**, 535 (1998).
[175] D. Guido, R. Longo, J. E. Roberts, and R. Verch, Charged sectors, spin and statistics in quantum field theory on curved spacetimes,'' preprint, math-ph/9906019.
[176] A. Connes and C. Rovelli, ''Von Neumann algebra automorphisms and the time-thermodynamics relation in generally covariant quantum theories,'' Class. Quantum Grav. **11**, 2899 (1994).
[177] R. Verch, ''Scaling algebras, the renormalization group and the principle of local stability in algebraic quantum field theory,'' in *Operator Algebras and Quantum Field Theory*, edited by S. Doplicher, R. Longo, J. E. Roberts, and L. Zsido (International, Boston, 1997).
[178] M. Wollenberg, ''Scaling limits and type of local algebras over curved spacetime,'' in *Operator Algebras and Topology*, edited by W. B. Arveson *et al.*, Pitman Research Notes in Mathematics, Series 270 (Longman, Harlow, 1992).