Mode-Beating Model of ac Helicity Injection

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It is shown here that Bevir and Grey's ac helicity-injection current-generation scheme does *not* depend on Taylor's relaxation principle. Instead, ac helicity injection is identified as the beating of a compressional Alfvén mode with a resistive diffusion mode to produce a surface current. This has the interesting consequence that the mode frequencies can be much higher than Taylor relaxation would allow. Although Taylor's principle is not needed to *generate* the surface current, it is required for the current to *diffuse* into the plasma because it is also shown that the current will not diffuse in classically.

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The conventional way to drive the dc current required for toroidal plasma confinement and heating is transformer-type induction which has the shortcoming of being pulsed. There is consequently great interest in finding steady-state (i.e., nonpulsed) schemes for current drive. Bevir and Grey,[1] using the context of reversed-field toroidal pinches (RFP's), proposed one such scheme. Their scheme, based on Taylor's description[2] of RFP relaxation, involves very-low-frequency modulation ($\omega \ll \tau_T^{-1}$ where τ_T is the tearing time scale) of the toroidal and poloidal fields. Specifically, this modulation is such that $\mathbf{B}_{\rm pol} \sim \cos\omega t$, $\mathbf{B}_{\rm tor} \sim \sin\omega t$ and is designed to sustain the plasma's helicity $K = \int \mathbf{A}\cdot\mathbf{B}\, dV \sim$ (poloidal flux) $\times$ (toroidal flux). This scheme has been proposed by Shoenburg *et al.*[3] as a means of steady-state current drive for the Los Alamos ZT-40 RFP and by Janos[4] for the Princeton S-1 spheromak.

Jensen and Chu[5] further developed the ideas of Refs. 1 and 2 and derived the exact helicity-conservation equation

$$\partial K^*/\partial t + \nabla(\mathbf{Q}_{\rm ac} + Q_{\rm dc}) = -2\eta \mathbf{J}\cdot\mathbf{B}. \tag{1}$$

Here $K^* = \mathbf{A}\cdot\mathbf{B}$ is the helicity density, $\mathbf{Q}_{\rm dc} \equiv 2\Phi_{\rm elect}\mathbf{B}$ is the dc helicity flux ($\Phi_{\rm elect}$ is the voltage on an electrode), $\mathbf{Q}_{\rm ac} \equiv \mathbf{A}\times\partial\mathbf{A}/\partial t$ is the ac helicity flux, and the right-hand side of (1) corresponds to the resistive dissipation of helicity.

Jensen and Chu have shown that Bevir and Grey's scheme amounts to balancing the right-hand decay term with a time-averaged ac helicity injection:

$$\nabla\cdot\langle\tilde{\mathbf{A}}\times\partial\tilde{\mathbf{A}}/\partial t\rangle = -2\eta\mathbf{J}\cdot\mathbf{B}, \tag{2}$$

where $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_{\rm pol} + \tilde{\mathbf{A}}_{\rm tor}$ and $\tilde{\mathbf{A}}_{\rm pol} \sim \cos\omega t$ and $\tilde{\mathbf{A}}_{\rm tor} \sim \sin\omega t$.

The author[6] showed that Eq. (2) is equivalent to the beating between an oscillatory $\mathbf{E}\times\mathbf{B}$ velocity and an oscillatory magnetic field; i.e., Eq. (2) is equivalent to

$$\langle\tilde{\mathbf{U}}\times\tilde{\mathbf{B}}\rangle\cdot\mathbf{B} = \eta\mathbf{J}\cdot\mathbf{B}, \tag{3}$$

where $\tilde{\mathbf{U}} = \tilde{\mathbf{E}}\times\mathbf{B}/B^2$.

In this Letter, I would like to identify $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{B}}$ in terms of known plasma modes, discuss how this process relates to the Taylor principle, and finally present some preliminary experimental measurements of one of the plasma modes. The governing equations are the linearized momentum equation

$$\rho\frac{\partial\tilde{\mathbf{U}}}{\partial t} = -\nabla\left(\tilde{P} + \frac{\mathbf{B}\cdot\tilde{\mathbf{B}}}{\mu_0}\right) + \frac{\mathbf{B}\cdot\nabla\tilde{\mathbf{B}}}{\mu_0} + \frac{\tilde{\mathbf{B}}\cdot\nabla\mathbf{B}}{\mu_0}, \tag{4}$$

and the linearized magnetic equation

$$\partial\tilde{\mathbf{B}}/\partial t = \nabla\times(\tilde{\mathbf{U}}\times\mathbf{B}) + (\eta/\mu_0)\nabla^2\tilde{\mathbf{B}}. \tag{5}$$

Because of the toroidal and poloidal symmetry of the driving method (cf. Fig. 2 of Ref. 3), $\partial/\partial\theta = \partial/\partial\phi = 0$, where θ and ϕ are, respectively, the poloidal and toroidal angles. Thus, both equilibrium and perturbed quantities depend only on the minor radius r. To simplify the analysis, the toroidal geometry will be replaced by a slab geometry, i.e., $r\to x$, $\theta\to y$, $\phi\to z$. Figure 1(a) shows a sketch of these slab coordinates; all quantities depend only on x, a is the minor radius, and the directions of the fields and currents are indicated.

Since $\nabla\cdot\tilde{\mathbf{B}} = 0$, it follows that $\tilde{B}_x = 0$ and for low-β plasmas (i.e., $\tilde{P} \ll \mathbf{B}\cdot\tilde{\mathbf{B}}/\mu_0$) Eq. (4) becomes

$$\rho\frac{\partial\tilde{\mathbf{U}}}{\partial t} = -\hat{\mathbf{x}}\frac{\partial}{\partial x}\left(\frac{\mathbf{B}\cdot\tilde{\mathbf{B}}}{\mu_0}\right) \tag{6}$$

because $\tilde{\mathbf{B}}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\tilde{\mathbf{B}} = 0$. If an $\exp(-i\omega t)$ time dependence is assumed and Eq. (6) is substituted into Eq. (5) then one obtains

$$-i\omega\tilde{\mathbf{B}} = -\mathbf{B}\frac{\partial^2}{\partial x^2}\left(\frac{\mathbf{B}\cdot\tilde{\mathbf{B}}}{i\omega\rho\mu_0}\right) + \frac{\eta}{\mu_0}\frac{\partial^2}{\partial x^2}\tilde{\mathbf{B}}. \tag{7}$$

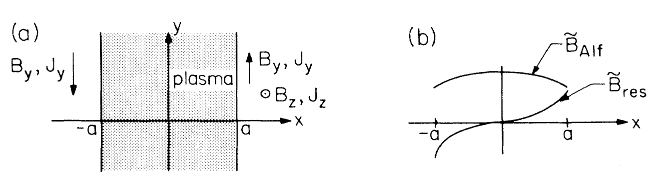

FIG. 1. (a) Slab coordinates; symmetry of $\mathbf{B}_z \sim \mathbf{B}_{\rm tor}$ field and $\mathbf{B}_y \sim \mathbf{B}_{\rm pol}$ field. (b) Symmetries of Alfvén and resistive modes for $\mathbf{B}_{\rm tor} \gg \mathbf{B}_{\rm pol}$.

The first (i.e., Alfvén) term on the right-hand side of Eq. (7) is $\sim(\omega\tau_A^2)^{-1}$, where $\tau_A = a/v_A$ is the Alfvén time ($v_A^2 \equiv B^2/\rho\mu_0$ is the Alfvén velocity); the second (i.e., resistive) term is $\sim\tau_R^{-1}$, where $\tau_R^{-1} \equiv \eta/\mu_0 a^2$ is the resistive diffusion time. For all plasmas of interest $\tau_R >> \tau_A$ and for the lower frequencies involved $\omega\tau_A << 1$, so that the Alfvén term should dominate the resistive term [i.e., $(\omega\tau_A^2)^{-1} >> \tau_R^{-1}$].

Consider the component of Eq. (7) parallel to **B**, i.e.,

$$-i\omega\mathbf{B}\cdot\tilde{\mathbf{B}} = -B^2\frac{\partial^2}{\partial x^2}\left(\frac{\mathbf{B}\cdot\tilde{\mathbf{B}}}{i\omega\rho\mu_0}\right) + \frac{\eta}{\mu_0}\mathbf{B}\cdot\frac{\partial^2}{\partial x^2}\tilde{\mathbf{B}}. \tag{8}$$

Since $(\omega\tau_A^2)^{-1} >> \tau_R^{-1}$ the resistive term may be dropped. Assuming an $\exp(ikx)$ spatial dependence, one finds

$$\mathbf{B}\cdot\tilde{\mathbf{B}}(\omega^2 - k^2 v_A^2) = 0, \tag{9}$$

i.e., $\mathbf{B}\cdot\tilde{\mathbf{B}} \neq 0$ corresponds to the compressional (fast) Alfvén wave.

Now consider the component of Eq. (7) orthogonal to **B** (i.e., the component obtained by taking the dot product of Eq. (7) with $\hat{\mathbf{x}}\times\hat{\mathbf{B}}$):

$$-i\omega\tilde{\mathbf{B}}_\perp = \frac{\eta}{\mu_0}\frac{\partial^2}{\partial x^2}\tilde{\mathbf{B}}_\perp. \tag{10}$$

Although the Alfvén term in Eq. (7) is, in general, much larger than the resistive term, for the orthogonal direction the Alfvén term vanishes leaving the resistive term to dominate (as in a tearing layer).

The dispersion of the resistive diffusion "mode" is

$$i\omega = k^2\eta/\mu_0 \tag{11}$$

or $\mathrm{Re}k = \mathrm{Im}k = \delta_s^{-1}$, where $\delta_s = (2\eta/\omega\mu_0)^{1/2}$ is the resistive skin depth.

The structure of these two orthogonal modes can be understood by considering (for example) tokamak geometry, where $\mathbf{B}_{\text{tor}} >> \mathbf{B}_{\text{pol}}$. In this case **B** is mainly toroidal and it is seen that the Alfvén mode is dominantly toroidal with a subdominant poloidal component (for an RFP the toroidal and poloidal directions interchange, but the concept of a compressional mode parallel to **B** and a diffusion mode orthogonal to **B** still holds). From Fig. 1(a) it is seen that the (toroidal) Alvén field must be symmetric in x while the (poloidal) resistive mode must be antisymmetric, i.e.,

$$\tilde{\mathbf{B}}_{\text{Alf}} = B_{\text{Alf}}\hat{\mathbf{B}}\cos(k_A x)\sin\omega t, \tag{12}$$

where $k_A = \omega/v_A$ and $\hat{\mathbf{B}} = \mathbf{B}/B$, and

$$\tilde{\mathbf{B}}_{\text{res}} = B_{\text{res}}\hat{\mathbf{x}}\times\hat{\mathbf{B}}\{\exp[(x-a)/\delta_s]\cos[(x-a)/\delta_s + \omega t] - \exp[-(x+a)/\delta_s]\cos[(x+a)/\delta_s - \omega t]\}. \tag{13}$$

Figure 1(b) shows the relative symmetries of these Alfvén and resistive modes (their subdominant directions will have the opposite symmetries).

Now consider how Eq. (3) describes the mode mixing. Equation (6) shows that the Alfvén mode has an associated plasma velocity; but that the resistive mode does not. Integration of Eq. (6) using Eq. (12) gives

$$\tilde{\mathbf{U}}_{\text{Alf}} = -v_A\hat{\mathbf{x}}(B_{\text{Alf}}/B)\sin k_A x\cos\omega t. \tag{14}$$

Substitution of Eqs. (12)–(14) in Eq. (3) shows that $\langle\tilde{\mathbf{U}}_{\text{Alf}}\times\tilde{\mathbf{B}}_{\text{Alf}}\rangle = 0$, but $\langle\tilde{\mathbf{U}}_{\text{Alf}}\times\tilde{\mathbf{B}}_{\text{res}}\rangle \neq 0$. Furthermore, both $\tilde{\mathbf{U}}_{\text{Alf}}$ and $\tilde{\mathbf{B}}_{\text{res}}$ are antisymmetric in x so that their prouct is symmetric. Because the resistive mode is finite only for $|x| \approx a$ the product becomes

$$\langle\tilde{\mathbf{U}}\times\tilde{\mathbf{B}}\rangle = \langle\tilde{\mathbf{U}}_{\text{Alf}}\times\tilde{\mathbf{B}}_{\text{res}}\rangle_{|x|\approx a|} = \hat{\mathbf{B}}v_A\sin(k_A a)B_{\text{Alf}}B_{\text{res}}/2B = \begin{cases}\hat{\mathbf{B}}\omega a B_{\text{Alf}}B_{\text{res}}/2B, & |x|\approx a,\\ 0, & |x| << a.\end{cases} \tag{15}$$

where $\sin(k_A a) \approx k_A a$ has been used since $\omega << \tau_A^{-1}$. Combining Eqs. (3) and (15) gives the current drive as

$$\eta\mathbf{J}\cdot\mathbf{B} = \begin{cases}\omega a B_{\text{res}}B_{\text{Alf}}/2, & |x|\approx a,\\ 0, & |x| << a.\end{cases} \tag{16}$$

A physical interpretation of this process is as follows: For the Alvén mode, the plasma position quivers radially with the velocity $\tilde{\mathbf{U}}_{\text{Alf}}$ and the field lines associated with this mode are glued to the plasma—i.e., do not cut across the plasma. In contrast, for the resistive mode, there is no associated plasma motion and so the field lines associated with the resistive mode are not glued to the plasma. Hence, in the plasma frame, the resistive field lines (pointed in the $\hat{\mathbf{x}}\times\hat{\mathbf{B}}$ direction) cut radially across the plasma with the quiver velocity $\tilde{\mathbf{U}}_{\text{Alf}}$. The sign of the resistive field reverses when $\tilde{\mathbf{U}}_{\text{Alf}}$ reverses, and so the rate of change of flux always has the same sign. Thus, there is a steady pumping of poloidal flux into the skin layer at $|x| \approx a$, which (from Faraday's law) produces a steady electric field in the $\hat{\mathbf{B}}$ direction.

Note that this model did not invoke tearing or the Taylor principle. To emphasize this point, the current could also be produced in a torus of nonplasma material provided the torus is (i) resistive and (ii) compressible in the minor radius. For example, an automobile tire inner tube made of moderately conducting rubber could be used. Here the radial velocity $\tilde{\mathbf{U}}_{\text{Alf}}$ could be produced by modulation of the inner tube pressure while the diffusion mode would be produced by induction (same configuration as a tokamak Ohmic heating transformer).

Large field modulation amplitudes will have undesirably large amounts of reactive energy and also might destroy the plasma equilibrium; hence, a measure of "efficiency" for this scheme would be the current driven for a given percentage of modulation. Equation (16) shows that this efficiency scales with ω. References 1–3 and Ref. 6 assumed that the upper bound on ω was τ_T^{-1} because they assumed that Taylor relaxation (i.e., tearing) was necessary for current generation. In contrast, this model did not invoke tearing for current generation, and so τ_T^{-1} is *not* the upper bound for ω [here, $\omega < (\tau_R/\tau_A)\tau_A^{-1}$ was assumed, a much less restrictive condition].

Tearing, however, is required to diffuse the flux from the skin layer into the bulk plasma, because classical diffusion will not work. To see this, consider Eq. (5) recast[7] in quasilinear form,

$$\partial \mathbf{B}/\partial t + \nabla \times \eta\mu_0^{-1}\nabla \times \mathbf{B} = \nabla \times \langle \tilde{\mathbf{U}}_{\mathrm{Alf}} \times \tilde{\mathbf{B}}_{\mathrm{res}} \rangle . \quad (17)$$

The right-hand side of Eq. (17) is the nonlinear driving term, while the left-hand side describes the classical diffusion of the field associated with the driven current. It is trivial to see that, unlike conventional skin currents, the nonlinear skin current is a *steady-state* (i.e., $\partial \mathbf{B}/\partial t = 0$) solution of Eq. (17) because the nonlinear skin current is $\eta \mathbf{J} = \eta\mu_0^{-1}\nabla\times\mathbf{B} = \langle \tilde{\mathbf{U}}_{\mathrm{Alf}} \times \tilde{\mathbf{B}}_{\mathrm{res}}\rangle$. Thus, the nonlinear skin current will stay localized so far as Eq. (17) (i.e., classical diffusion) is concerned. However, a stationary localized skin current is in violation of Taylor's principle[2,5] which states that the plasma will relax via tearing to the force-free state $\mathbf{J} = \sigma \mathbf{B}$, where σ is a *constant throughout the plasma*. Hence, one would expect Taylor relaxation (i.e., tearing) to cause the nonlinear skin current to penetrate the plasma. These tearing modes will, in general, have $\partial/\partial\theta \neq 0$, $\partial/\partial\phi \neq 0$ and so the anomalous penetration will involve a decay of the highly symmetric modes described by Eqs. (12) and (13) into less symmetric modes.

The requirement for less-symmetric modes can also be seen by considering what happens if ω is set to be below τ_T^{-1}, and the mode symmetry is assumed to be as given in Eqs. (13) and (14). The diffusion-mode skin depth would broaden, but this broadening could not extend the nonlinear skin current all the way to the magnetic axis, because both $\tilde{\mathbf{U}}_{\mathrm{Alf}}$ and $\tilde{\mathbf{B}}_{\mathrm{res}}$ are antisymmetric and so vanish on the magnetic axis (i.e., $x = 0$ in slab coordinates).

A test for the diffusion mode has been performed on the high-repetition-rate Encore tokamak at California Institute of Technology (vacuum-chamber major and minor radii of 0.38 and 0.12 m, respectively, Ar gas, $n \sim 10^{11}$–10^{12} cm^3, $T_e \sim 5$–10 eV, $I_{\mathrm{tor}} \sim 1$–5 kA, $\mathbf{B}_{\mathrm{tor}} \sim 0.1$–$1.5$ kG, 15 shots/sec, plasma duration ~ 2–4 msec). Because the Ohmic heating system is driven by an amplifier[8] it is straightforward to modulate $\mathbf{B}_{\mathrm{pol}}$. Figure 2 shows the experimental setup. A magnetic probe,[9] oriented to measure $\mathbf{B}_{\mathrm{pol}}$, traverses the plasma minor radius. The probe $\partial B/\partial t$ signal is integrated, filtered, and then digitized. The digitizer output is displayed as a function of time and position, as shown in Fig. 3(a). Here, darker regions correspond to larger-amplitude signals, the vertical axis corresponds to the signal for 400 μsec after a trigger reference, and the horizontal axis corresponds to the probe minor-radius position. The tilt of the r-t characteristics in Fig. 3(a) is caused by the cosine phase dependence in Eq. (13). By fitting the data of Fig. 3(a) at each radial position with the function $\mathbf{B}_{\mathrm{pol}} = \lambda \cos\omega t + \mu \sin\omega t$, the local phase $[\tan^{-1}(\mu/\lambda)]$ and amplitude $[(\lambda^2+\mu^2)^{1/2}]$ were determined [cf. Fig. 3(b)]. The radial derivative of the phase gave Rek while the amplitude e folding gave Imk. Measurements were made from 5 kHz (limited by requirement that plasma duration $\gg \omega^{-1}$) to 130 kHz (limited by amplifier cutoff).

Equation (11) predicts that Rek and Imk should

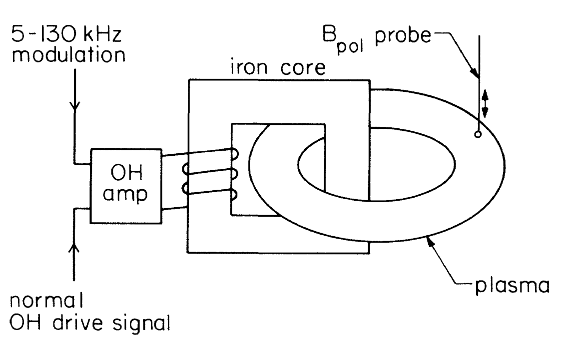

FIG. 2. Experimental setup.

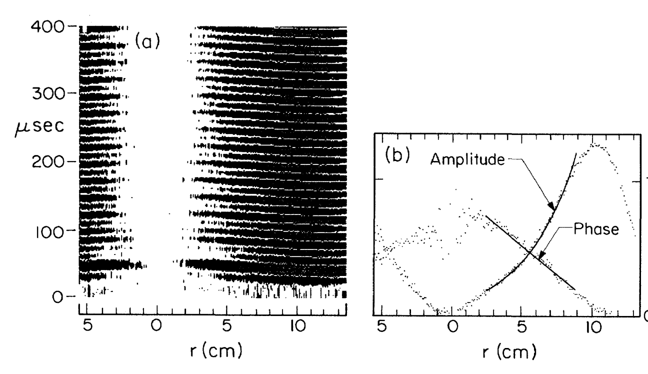

FIG. 3. (a) Typical data; $\omega/2\pi = 40$ kHz, $B_{\mathrm{tor}} = 690$ G, 3.5×10^{-4} Torr argon, $V_{\mathrm{loop}} = 8$ V, $I_{\mathrm{tor}} = 1$ kA. (b) Amplitude and phase for (a); solid lines are least-mean-squares fits used to evaluate Rek, Imk.

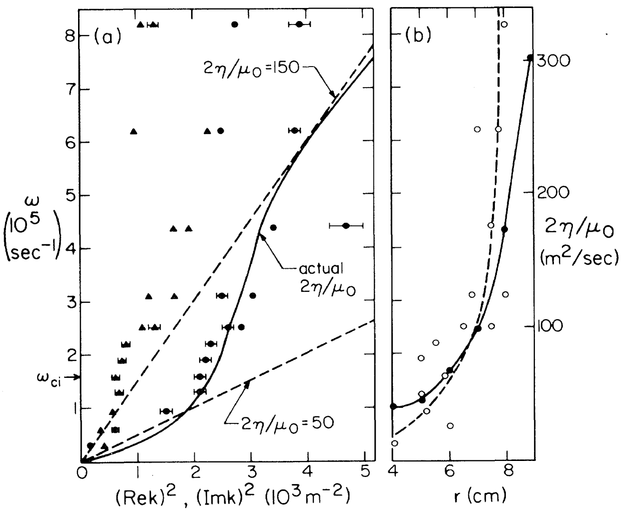

FIG. 4. ω vs $(\mathrm{Re}\,k)^2$ (circles) and $(\mathrm{Im}\,k)^2$ (triangles); solid line is theoretical prediction using locally measured $2\eta/\mu_0$, $B_{\mathrm{tor}}=690$ G, 6×10^{-5} Torr argon, $V_{\mathrm{loop}}=8$ V, $I_{\mathrm{tor}}=2$ kA. (b) Measurement of $2\eta/\mu_0$ (right-hand scale) for each ω (open circles and hand-fitted dashed line, both left-hand scale).

satisfy the dispersions $\omega/(\mathrm{Re}\,k)^2=\omega/(\mathrm{Im}\,k)^2=2\eta/\mu_0$. Figure 4(a) plots the measured $(\mathrm{Im}\,k)^2$ and $(\mathrm{Re}\,k)^2$ vs ω. It was found that there was a radial gradient in $2\eta/\mu_0$ (corresponding to the plasma temperature gradient) and also that the signal location moved radially outward as ω increased. To compare the Fig. 4(a) data with theory, $2\eta/\mu_0$ was measured—cf. Fig. 4(b)—at the actual signal location for each ω. In Fig. 4(b) the dashed line is a hand-drawn fit to the locus of signal positions (open circles, left-hand scale) and the solid line is the locally measured $2\eta/\mu_0$ obtained from the relation $2\eta/\mu_0=2(V_{\mathrm{loop}}/2\pi R)/[r^{-1}\partial(rB_{\mathrm{pol}})/\partial r]$, where R is the major radius. The actual value of $2\eta/\mu_0$ for each frequency is then used to give the solid line in Fig. 4(a).

There is excellent agreement between theory and data for $(\mathrm{Re}\,k)^2$ but $(\mathrm{Im}\,k)^2$ is small by a factor of 2–3. This represents either anomalous skin penetration, measurement error caused by random plasma radial motion smearing out a sharper skin layer, or possibly a $\nabla\eta$ effect.

Significantly, the resistive diffusion mode is unaffected by ion cyclotron resonance [cf. ω_{ci} in Fig. 4(a)] and continues on to frequencies several times ω_{ci}, i.e., beyond the validity of magnetohydrodynamics theory. Since the diffusion mode has $\tilde{\mathbf{E}}\parallel\mathbf{B}$, it is not surprising that it persists to such high frequencies. Of equal significance, the perpendicularly propagating compressional Alfvén wave is also unaffected at $\omega=\omega_{ci}$, and so ac helicity injection should also work above ω_{ci}, which will greatly increase the efficiency (as defined above). The Rotamak[10] may be an example of high-frequency ac helicity injection in a slightly different geometry, since the Rotamak involves two quadrature phased orthogonal magnetic fields oscillating at $\omega>\omega_{ci}$.

An experiment to modulate the toroidal field as well is under construction.

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