Sofia Panagiota Corbà

Postdoktor i evolutionen av primordiala magnetfält

Ref nr: SU FV-4638-25-22
Datum för ansökan: 2026-01-18 14:24

Födelsedatum	1994-10-20
E-post	spcorba@umass.edu
Kön	Kvinna

Frågor

1. *Nuvarande sysselsättning (ange huvudsaklig sysselsättning)*
Anställd vid lärosäte utanför Sverige

2. *Högsta examen*
Doktors-/licentiatsexamen

3. *Från vilket land har du din högsta examen?*
Förenta staterna

4. *Har du din högsta examen från Stockholms universitet?*
Nej

5. *Ange datum när du tog din doktorsexamen*
2026-08-31

6. *NUVARANDE ANSTÄLLNING. Ange arbetsplats och jobbtitel samt när anställningen påbörjades..*
University of Massachusetts Amherst (USA)
Ph.D. Candidate in Theoretical Physics
2020-present (expected completion: Summer 2026)

7. *REFERENSER. Ange namn, telefon och e-post för 2–3 referenspersoner som kan komma att kontaktas.*
Lorenzo Sorbo
University of Massachusetts Amherst
email: sorbo@umass.edu

Michele Cicoli
Alma Mater Studiorum - Bologna University
email: michele.cicoli@unibo.it

Ben Heidenreich
University of Massachusetts Amherst
email: bheidenreich@umass.edu

8. *SPRÅKKUNSKAPER. Beskriv kort dina språkkunskaper.*
Italian: native
Greek: native
English: fluent

9. *FORSKNINGSPLAN/PROJEKTPLAN. Bifoga din plan som beskriver det tilltänkta projektet.*
Research_Proposal_Corba.pdf

10. *DOKTORSEXAMEN ELLER MOTSVARANDE. Ange doktorsexamen med ämne och lärosäte.*
Ph.D. in Theoretical Physics
University of Massachusetts Amherst, USA
Expected completion: Summer 2026

11. *EXAMENSBEVIS ELLER MOTSVARANDE. Bifoga examensbevis.*
master.pdf

12. *ÅBEROPADE PUBLIKATONER. Publikationer som åberopas till stöd för ansökan kan bifogas här.*
Corbà_2023_J._Cosmol._Astropart._Phys._2023_005.pdf

Corbà_2024_J._Cosmol._Astropart._Phys._2024_024.pdf

2504.13156v1.pdf

Eget uppladdat CV

Sofia P. Corbà
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Nordita, The Nordic Institute for Theoretical Physics

January 18, 2026

Dear members of the selection committee,

I am writing to express my interest in a postdoctoral position within the ERC Synergy Grant COSMOMAG with Prof. Axel Brandenburg. I am currently a Ph.D. candidate in theoretical physics at the University of Massachusetts Amherst (USA), working with Prof. Lorenzo Sorbo, and I expect to complete my degree in the summer of 2026. My research interests lie at the intersection of early universe cosmology, high energy theory, and gravitational wave physics.

My doctoral research has focused on the theoretical and phenomenological aspects of early universe cosmology. I have studied gravitational wave anisotropies in axion inflation models and their cross-correlations with CMB anisotropies, with the goal of characterizing the stochastic gravitational wave background predicted in this framework and identifying the theoretical signatures that distinguish it from its astrophysical counterpart. Within the same context of axion inflation, I am currently examining the backreaction effects of fermions on the inflaton dynamics. A significant part of my work has also focused on formal aspects of renormalization in quantum field theory on curved spacetime, where I proposed an alternative prescription for the traditionally used adiabatic renormalization and applied it to the power spectra of various inflationary models. I am currently testing this alternative prescription by calculating the conformal anomaly. More recently, I have begun exploring the physics of dynamical black holes during inflation in collaboration with Prof. Jennie Traschen. Prior to my Ph.D., in my master's thesis, *Backreaction from Magnetogenesis in String Inflation*, supervised by Prof. Michele Cicoli, I studied primordial magnetogenesis in string-inspired inflationary models obtained through moduli stabilization techniques in Type IIB flux compactifications.

Besides the technical and analytical skills I have developed, my research has taught me how to collaborate effectively with colleagues throughout the scientific process, how to take initiative, and how to bring a project to publication. In particular, through my most recent paper, which is a solo-authored manuscript, I have learned to conduct and manage independent research and to address complex scientific challenges on my own. Through participation in collaborations, seminars, journal clubs, and conferences, I have interacted with researchers across many areas, improving my ability to work within a research group and to present scientific results clearly.

Alongside my research at UMass, I served as a teaching assistant for both undergraduate and graduate courses, for which I received the Quinton Award for Excellence in Teaching in 2023. My teaching experience reflects strong communication skills and a commitment to community building. Managing these responsibilities alongside an active research program has taught me effective task administration, time management, and the coordination of multiple forms of work under tight deadlines. Beyond teaching, I am an active member of the Women in Physics organization and have served on the Physics Department's Climate Committee and the SEA Change initiative, working to promote equity, inclusion, and diversity within the academic community.

As a postdoctoral researcher, I would like to extend my current research on gravitational wave anisotropies to more general settings, focusing on variants of axion inflation that involve gauge field kinetic couplings, dynamical black holes, or primordial black holes. This study of gravitational wave signatures would contribute to the broader effort to characterize the stochastic gravitational wave background, which is essential to interpret and guide the intense current and future observational activity of ground-based (LIGO, Virgo, KAGRA, and the proposed Cosmic Explorer and Einstein Telescope) and space-based (LISA) detectors, as well as pulsar timing array observations expected in the coming years. At the same time, I intend to expand my work on formal aspects of quantum field theory to test and strengthen the alternative prescription I proposed for the renormalization of observables in curved spacetime. In this context, I plan to develop future projects on the renormalization of observables such as the electromagnetic stress-energy tensor in inflationary models, in order to produce reliable predictions for quantities like the seed magnetic fields that could have given rise to the observed large-scale cosmic magnetic fields. Finally, I look forward to expanding my work on black hole physics, an area I have only recently begun to explore and am excited to develop further, as well as advancing string-theoretic phenomenological approaches to cosmology.

I am especially excited to join the group of Prof. Axel Brandenburg, and to collaborate with Profs. Chiara Caprini, Andrii Neronov, and Franco Vazza on COSMOMAG, a collaborative project aimed at studying the origin of cosmological magnetic fields. This project naturally connects to my research experience and future plans, as I have previously worked on magnetogenesis in the context of my master's thesis, while my work on adiabatic renormalization could provide a valuable formalism for computing seed magnetic fields originating in the primordial universe. Moreover, the high quality and impact of the research conducted at Nordita, make it an ideal environment in which to grow scientifically and advance my academic goals. With my background in quantum field theory in curved spacetime and early universe physics, I am confident that I can contribute meaningfully to the group's research community. In addition, my experience with gravitational waves would allow me to contribute effectively to the more observational aspects of the project. Thank you for your consideration, and I look forward to the opportunity to discuss my application further.

Sincerely

Sofia P. Corbà

Sofia P. Corbà

Curriculum Vitae

UNIVERSITY OF MASSACHUSETTS AMHERST
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710 North Pleasant St, Amherst (MA) 01003, USA
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sites.google.com/view/spcswebpage/home-page

Nationality Greek, Italian

Academic Qualifications

2020-summer 2026 Ph.D. in Theoretical Physics, Department of Physics, University of Massachusetts Amherst, USA.
Supervisor: Prof. Lorenzo Sorbo.

2016-2019 M.Sc. in Theoretical Physics, Alma Mater Studiorum - University of Bologna, Italy.
Supervisor: Prof. Michele Cicoli. **Mark:** 110/110 cum laude.

2012-2016 B.Sc. in Physics, Alma Mater Studiorum - University of Bologna, Italy.
Supervisor: Prof. Michele Cicoli. **Mark:** 110/110 cum laude.

Research Experience

My research focuses on **theoretical cosmology** and its connections to **high-energy physics**, with a particular interest in **early-universe physics** and **inflation**. More recently, my interests have also expanded to the study of **black holes**.

1. **Research on Formal Aspects of Quantum Field Theory on Curved Spacetime:** Making predictions in quantum field theory requires a consistent method for **renormalizing observables**. In the presence of **gravity**, this process becomes more complicated than in flat spacetime due to **particle creation**. My work has focused on studying the **limitations** of the standard method of renormalization in curved spacetime, the **adiabatic renormalization**, and proposing an **alternative approach** to overcome them. I have applied this approach to analyze observables such as the **power spectrum** in various inflationary scenarios, and I am currently using it to calculate the **conformal anomaly**, whose well known result provides a **test** for the method.
2. **Research on Axion Inflation and Gravitational Waves:** **Axion inflation** is a class of models in which the inflaton is a pseudo-Nambu–Goldstone boson with a broken **shift symmetry**, allowing couplings to gauge and fermionic fields. Within this framework, I have studied the generation of **gravitational waves** and the **anisotropies** in the primordial stochastic background they produce. Studying these anisotropies can help **distinguish** a **cosmological** signal from an **astrophysical** one, providing information about the early universe. In the same context, I am currently analyzing aspects of **backreaction** of fermionic fields on the inflaton.
3. **Research on Black Holes during Inflation:** In slow-roll inflationary cosmologies, **dynamical black holes** can be studied within a metric that evolves **quasi-statically** through a sequence of **Schwarzschild-de Sitter** like metrics, with slowly varying mass and effective cosmological constant. In this framework, both the **black hole** and **cosmological horizon** areas grow as the inflaton rolls. I am currently working on extending this model to better understand dynamical black holes during inflation and their **cosmological signatures**.
4. **String Theory Inflation:** In my Master's thesis, *Backreaction from Magnetogenesis in String Inflation*, I studied **primordial magnetogenesis** in **string** inflationary models arising from Type IIB flux compactifications. The generated fields could serve as **seeds** for the large-scale magnetic fields observed today, while their backreaction on the inflaton may help **reconcile** the requirement of sufficient e-foldings of inflation with the **geometric constraints** imposed by the extra dimensions.

List of Main Publications

1. Sofia P. Corbà, Lorenzo Sorbo, **"On adiabatic subtraction in an inflating Universe"**, arXiv:2209.14362 [hep-th] (2022), JCAP 07 (2023), 005.
2. Sofia P. Corbà, Lorenzo Sorbo, **"Correlated scalar perturbations and gravitational waves from axion inflation"**, arXiv: 2403.03338 [astro-ph.CO] (2024), JCAP 10 (2024), 024.
3. Sofia P. Corbà, **"Gravitational wave anisotropies from axion inflation"**, arXiv:2504.13156 [astro-ph.CO] (2025). Accepted for publication in JCAP (in press).
4. Sofia P. Corbà, Lorenzo Sorbo, **"Calculation of the conformal anomaly using adiabatic renormalization"**. In preparation (2025).
5. Sofia P. Corbà, Lorenzo Sorbo, **"Study of backreaction from fermion production during axion inflation"**. In preparation (2025).

Conferences, Schools Attended and Talks

1. Visitor at Kavli Institute for Cosmological Physics (SPEAKER).
   University of Chicago.
   December 8, 2025, Chicago (IL), USA.
2. Parity Violation from Home 2025 (INVITED SPEAKER).
   November 18-21, 2025, Remote.
3. High Energy Theory Seminar (INVITED SPEAKER).
   University of Pennsylvania.
   November 17, 2025, Philadelphia (PA), USA.
4. MIT Cosmo Coffee series (SPEAKER).
   Massachusetts Institute of Technology.
   November 12, 2025, Cambridge (MA), USA.
5. COSMO-25 - 28th International Conference on Particle Physics & Cosmology (SPEAKER).
   Carnegie Mellon University.
   October 13-17, 2025, Pittsburgh (PA), USA.
6. Workshop - Primordial Black Holes: Theory Meets Experiment.
   Amherst Center for Fundamental Interactions - UMass Amherst.
   September 17-19, 2025, Amherst (MA), USA.
7. Summer School on Particle Physics 2025 (SPEAKER).
   ICTP International center for fundamental physics.
   June 16-27, 2025, Trieste (TS), Italy.
8. Invisibles School 2024 (POSTER PRESENTATION).
   Alma Mater Studiorum - University of Bologna.
   Jun 24-28, 2024, Bologna (BO), Italy.
9. DPF-Pheno 2024 (SPEAKER).
   University of Pittsburgh, Carnegie Mellon University.
   May 13-17, 2024, Pittsburgh (PA), USA.
10. Workshop - Surveying the Landscape.
    Amherst Center for Fundamental Interactions - UMass Amherst.
    April 15-17, 2024, Amherst (MA), USA.
11. ACFI Journal Club 2024 (SPEAKER).
    Amherst Center for Fundamental Interactions - UMass Amherst.
    April 10, 2024, Amherst (MA), USA.
12. DESY Theory Seminars - Journal Club 2022 (SPEAKER).
    Deutsches Elektronen-Synchrotron (DESY).
    November 10, 2022, Hamburg, Germany.
13. BS2019 - SEENET-MTP Balkan School on High Energy and Particle Physics:
    Theory and Phenomenology. Workshop on Advances in Fields, Particles and Cosmology.
    June 3-10, 2019, Ioannina, Greece.

References

1. Prof. Lorenzo Sorbo
   Department of Physics, University of Massachusetts Amherst, USA.
   email: sorbo@umass.edu
2. Prof. Jennie Traschen
   Department of Physics, University of Massachusetts Amherst, USA.
   email: traschen@umass.edu
3. Prof. Ben Heidenreich
   Department of Physics, University of Massachusetts Amherst, USA.
   email: bheidenreich@umass.edu
4. Prof. Michele Cicoli
   Department of Physics and Astronomy, University of Bologna, Italy.
   email: michele.cicoli@unibo.it

Teaching Experience

2020-present **As a Teaching Assistant:**

- P131 Intro Physics I, assistant of Team Based Learning (TBL) classrooms (Fall 2021, Spring 2022, Fall 2023, Spring 2024).
- Collaborated with Prof. Heath Hatch to design and co-teach P131 Intro Physics I (Fall 2024, Spring 2025).
- P131 Intro Physics I, co-instructor for Lab sessions (Spring 2024, Spring 2025).
- P151 General Physics I, assistant of TBL classrooms (Fall 2023, Fall 2025).
- P132 Intro Physics II, assistant of TBL classrooms (Fall 2022).
- P181 Introductory Mechanics, assistant of TBL classrooms and grading (Fall 2022).
- P564 Advanced Quantum Mechanics: Assistant of the course and grading (Spring 2023).
- P602 Statistical Mechanics: Assistant of the course and grading (Fall 2024).

As an Instructor:

- P151 General Physics I (Spring 2026).

Intensive Teaching Training:

Completion of the Intensive Teaching Training program to develop skills for becoming an independent instructor of large lecture courses, under the supervision of Prof. David Hamilton (Fall 2025).

Awards and Recognitions

Quinton Award for excellence in Teaching.
University of Massachusetts Amherst, USA (2023).

Community Involvement

1. Participation in the Physics Department's Climate Committee meetings on equity, inclusion, and diversity.
2. Participation in Women in Physics group and events.
3. Participation in SEA Change group.

Research Proposal Sofia P. Corbà

I am writing to express my interest in a postdoctoral position within the ERC Synergy Grant **COSMOMAG**. I am currently a Ph.D. candidate in theoretical physics at the University of Massachusetts Amherst (USA), working with Prof. **Lorenzo Sorbo**, and I expect to complete my degree in the summer of 2026.

Overview

My research focuses on **theoretical cosmology** and its implications for fundamental interactions and physics beyond the Standard Model, with a particular interest in the **early universe**. The leading paradigm of primordial cosmology is **inflation**, which describes how, during the very first moments of its existence, the universe underwent a rapid, **quasi-exponential expansion**, naturally reaching an extremely **flat** and **homogeneous** configuration. During this phase, **quantum fluctuations** were stretched to **macroscopic** scales, becoming the seeds of the large-scale structure we observe today. This mechanism, which connects microscopic quantum processes to macroscopic cosmological observables, makes inflation a natural framework for studying **quantum field theory in curved spacetime** and its connection with general relativity. This is the context in which I have developed my research.

I began studying the physics of the early universe in my master's thesis, *Backreaction from Magnetogenesis in String Inflation*, supervised by Prof. **Michele Cicoli**, where I analyzed string theory-inspired models of inflation and primordial magnetogenesis. During my Ph.D., I have worked closely with Prof. **Lorenzo Sorbo** on both **phenomenological** and **formal** aspects of early universe cosmology, including **gravitational wave anisotropies** and **backreaction** effects in axion inflation models, as well as the **renormalization** of observables in **curved spacetime**. More recently, my interests have also expanded to the study of **black holes**. In particular, I am currently collaborating with Prof. **Jennie Traschen** on a project investigating **dynamical** black holes in slow-roll inflation, where the metric evolves **quasi-statically** through a sequence of **Schwarzschild-de Sitter** metrics.

Research on Axion Inflation and Gravitational Waves

Given the **ultraviolet (UV) sensitivity** of generic scalar potentials, building a robust model of inflation requires controlling the quantum corrections arising from UV modes. One class of models addressing this problem is **axion inflation**, where the inflaton is a **pseudo-Nambu–Goldstone** boson with a broken **shift symmetry**. Because of this symmetry, the axion inflaton naturally couples to **gauge fields**, which are amplified during inflation and in turn **source scalar** and **tensor** fluctuations. Consequently, this model produces two types of fluctuations: the **standard vacuum** fluctuations generated by the accelerated expansion of the background and the fluctuations **sourced** by the amplified gauge fields [1, 2]. The same shift symmetry also allows couplings to **fermionic** fields.

Past and Ongoing Research

In my paper [3], published in JCAP in October 2024, I investigated the correlation between **scalar** and **tensor** fluctuations produced in axion inflation. The correlator receives **two contributions**: one from gravitational waves correlated with **vacuum** scalar fluctuations, and another from gravitational waves correlated with **sourced** scalar fluctuations. My analysis showed that the **former effect dominates**, with the normalized correlator being of the order of $10^{-4} - 10^{-2}$. Its observability, subject to instrumental noise and the intrinsic variance of the isotropic component, depends on the **amplitude of the anisotropies** in the gravitational wave spectrum. In the subsequent paper [4], accepted for publication in JCAP in November 2025, I calculated this amplitude, finding that it can reach values as large as $\mathcal{O}(10^{-1})$, thus showing that axion inflation **can indeed produce observable anisotropies**.

In this same context, I am currently working on a project investigating **backreaction** effects in axion inflation models that include couplings to **fermionic** fields [5]. In particular, I am exploring whether there exist **parameter values** for which particle production can occur on parametrically **sub-horizon** scales. This would allow the analysis to be carried out analytically within a **perturbative regime**, analogously to what happens in [6].

Future Research

Going forward, I plan to expand the study of gravitational wave anisotropies from axion inflation to more general settings. Evidence indicates that adding a **kinetic** coupling together with the commonly studied axial coupling could, even if weakly, affect the **gravitational wave energy density** and **power spectrum** [7]. A key question is how such couplings could influence the anisotropies and whether **characteristic signatures** might be **detected** by upcoming observations.

Research on Formal Aspects of Quantum Field Theory on Curved Spacetime

Making predictions in quantum field theory requires a consistent method for **renormalizing** observables. In the presence of gravity, this becomes more complicated than in flat spacetime due to **particle creation**. A significant part of my work has focused on developing a consistent method of **renormalization** of observables in **curved spacetime**.

Past and Ongoing Research

One renormalization technique traditionally used in cosmology is **adiabatic** renormalization [8, 9]. This leads to finite observables by subtracting from the bare results, mode by mode, the corresponding quantities obtained using the positive-frequency Wentzel-Kramers-Brillouin (WKB) approximation. The WKB expansion is obtained as a recursive series, and, according to the standard prescription, the truncation order should correspond to the **degree of ultraviolet divergence of the operator** being renormalized. While this method works very well in the UV regime, it often generates **infrared (IR) artifacts** at finite momenta [10], due to the fact that the order of truncation is fixed.

In [11], published in JCAP in July 2023, I proposed a **revision** of the standard approach, recognizing that the adiabatic expansion is generally **asymptotic**, and as such has an **optimal** truncation that gives the best approximation to the exact solution [12]. This optimal truncation **depends on the system's parameters**, including the momenta, thus resolving the unphysical behavior at intermediate scales. In my paper, I focused on the renormalization of the **power spectrum** of scalar fluctuations during and after inflation across various scenarios. Currently, I am working on a new project that applies optimal truncation to the calculation of the **conformal anomaly**, the expectation value of which is well known and can serve as a **test** for this approach.

Future Research

In models where the inflaton **couples to gauge fields**, such as axion inflation, particularly when kinetic couplings are included, or in some string theory-inspired scenarios, **primordial magnetic fields** can be produced. To obtain reliable predictions and constrain these models through observations, it is necessary to properly renormalize the **magnetic energy density**. As a future project, I plan to apply the alternative prescription I introduced in [11] to renormalize the electromagnetic **stress-energy tensor**. Once the optimal truncation, as a function of momentum, is determined, the renormalized tensor is obtained from the bare one by subtracting, mode by mode (i.e., under the integral sign), the corresponding terms obtained using the asymptotic expansion. This provides a **consistent** result for the generated magnetic fields, which could then serve as **seeds** for the large-scale magnetic fields observed in the universe today, potentially explaining their **origin**.

Research on Black Holes during Inflation

In slow-roll inflationary cosmologies, **dynamical black holes** can be studied within a metric that evolves **quasi-statically** through a sequence of **Schwarzschild-de Sitter like metrics**, with slowly varying mass and effective cosmological constant [13]. In this framework, both the **black hole** and **cosmological** horizon areas grow as the inflaton rolls.

Ongoing Research

Recently, I have started a project with Prof. **Jennie Traschen** in which we **translate** the inflaton's field evolution, together with the evolution of the black hole mass and effective cosmological constant found in the Schwarzschild-de Sitter metric, into the language of **standard inflationary cosmology** in the **far field**. In this context, we are studying various **cosmological observables** to have a clearer understanding of how dynamical black holes modify standard inflationary dynamics and what observational signatures could **confirm** or **rule out** their existence.

Future Research

In this direction, a future project I plan to focus on is the study of **dynamical black holes**, described within the slowly varying Schwarzschild-de Sitter metric, in the case where inflation is driven by an **axion inflaton**. A separate question concerns **primordial** black holes formed **after** axion inflation, when the fluctuations generated during inflation re-enter the horizon and **collapse**. In particular, it would be interesting to examine how the **charge** of these primordial black holes is affected by the presence of the **gauge fields** sourced by the inflaton.

Research on String Theory Inflation and Magnetogenesis

In all the previous topics, inflation was studied within the **effective** framework of quantum field theory in curved spacetime. Another way to obtain inflationary models is through the UV-complete theory of **string theory**. Since string theory is formulated in ten dimensions, deriving predictions for the four-dimensional physics, relevant to cosmology, requires **compactifying** the six extra dimensions on a suitable manifold, known as a Calabi-Yau. An important consequence of this compactification and the subsequent **dimensional reduction** is the emergence of a large number of massless scalar fields, the **moduli**, with **flat potentials** that make them natural candidates to drive inflation.

Past Research

In my Master's thesis, I studied two **string theory inflationary models** arising from Type IIB flux compactifications: Kähler Inflation and Fibre Inflation. In these models, the inflaton **couples to gauge fields**, which in turn **backreact** by slowing the inflationary evolution. In my work, I examined two important effects: first, the **generation of magnetic fields** that could explain the large-scale magnetic fields observed today, and second, the **slowdown of the inflaton**, which may help alleviate the **tension** between the phenomenological requirement of sufficient e-foldings of inflation and the **geometric constraints** imposed by the size of extra dimensions [14].

Future Research Goals

The study of the gravitational wave anisotropies I carried out in [3, 4] contributes to the broader effort to **characterize** the cosmological stochastic gravitational wave background and to **distinguish** it from its **astrophysical** counterpart. This line of research has gained increasing importance because of the intense current and future observational activity of **ground-based detectors** (LIGO, Virgo, KAGRA, and the proposed Cosmic Explorer and Einstein Telescope) and **space-based missions** (LISA), particularly following the 2023 **NANOGrav detection** [15]. Building on this, I aim to develop **quantitative predictions** for observables such as the spectral shape, polarization, and cross-correlations of the stochastic gravitational wave background across different early universe models, with the goal of guiding the interpretation of **upcoming observational data**. At the same time, I would be interested in the opportunity to explore areas such as black holes, with which I am only beginning to familiarize myself, as well as late-time cosmology and string theory, as these fields can contribute to a broader understanding of the universe.

I am especially excited to join the group of Prof. **Axel Brandenburg**, and to collaborate with Profs. **Chiara Caprini**, **Andrii Neronov**, and **Franco Vazza** on COSMOMAG, a collaborative project aimed at studying the **origin of cosmological magnetic fields**. This project naturally connects to my research experience and future plans, as I have previously worked on **magnetogenesis** in the context of my master's thesis, while my work on adiabatic renormalization could provide a valuable formalism for computing **seed magnetic fields** originating in the primordial universe. With my background in quantum field theory in curved spacetime and early universe physics, I am confident that I can contribute meaningfully to the group's research community. In addition, my experience with gravitational waves would allow me to contribute effectively to the more **observational aspects** of the project.

References

[1] L. Sorbo, *Parity violation in the cosmic microwave background from a pseudoscalar inflaton*, *Journal of Cosmology and Astroparticle Physics* **2011** (June, 2011) 003–003.

[2] E. Pajer and M. Peloso, *A review of Axion Inflation in the era of Planck*, *Class. Quant. Grav.* **30** (2013) 214002, [1305.3557].

[3] S. P. Corbà and L. Sorbo, *Correlated scalar perturbations and gravitational waves from axion inflation*, *JCAP* **10** (2024) 024, [2403.03338].

[4] S. P. Corbà, *Gravitational wave anisotropies from axion inflation*, 2504.13156.

[5] P. Adshead, L. Pearce, M. Peloso, M. A. Roberts and L. Sorbo, *Phenomenology of fermion production during axion inflation*, *JCAP* **06** (2018) 020, [1803.04501].

[6] P. Creminelli, S. Kumar, B. Salehian and L. Santoni, *Dissipative inflation via scalar production*, *JCAP* **08** (2023) 076, [2305.07695].

[7] M. Teuscher, R. Durrer, K. Martineau and A. Barrau, *Gravitational Waves sourced by Gauge Fields during Inflation*, 2510.00869.

[8] L. Parker, *Quantized fields and particle creation in expanding universes. i*, *Phys. Rev.* **183** (Jul, 1969) 1057–1068.

[9] L. Parker, *Quantized fields and particle creation in expanding universes. ii*, *Phys. Rev. D* **3** (Jan, 1971) 346–356.

[10] L. Parker, *Amplitude of Perturbations from Inflation*, hep-th/0702216.

[11] S. P. Corbà and L. Sorbo, *On adiabatic subtraction in an inflating Universe*, *JCAP* **07** (2023) 005, [2209.14362].

[12] R. Dabrowski and G. V. Dunne, *Superadiabatic particle number in Schwinger and de Sitter particle production*, *Phys. Rev. D* **90** (2014) 025021, [1405.0302].

[13] R. Gregory, D. Kastor and J. Traschen, *Evolving Black Holes in Inflation*, *Class. Quant. Grav.* **35** (2018) 155008, [1804.03462].

[14] M. Cicoli, D. Ciupke, C. Mayrhofer and P. Shukla, *A geometrical upper bound on the inflaton range*, *Journal of High Energy Physics* **2018** (May, 2018) .

[15] NANOGrav collaboration, G. Agazie et al., *The NANOGrav 15 yr Data Set: Evidence for a Gravitational-wave Background*, *Astrophys. J. Lett.* **951** (2023) L8, [2306.16213].

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

TRANSCRIPT OF RECORDS - DEGREE CERTIFICATE

ARCHIVE NUMBER: 86591 *reg. 124558*

MATRICULATION NUMBER: 0000816075

NAME OF THE STUDENT: Family Name: CORBA' First Name: SOFIA PANAGIOTA GENDER: F

DATE, PLACE AND COUNTRY OF BIRTH: Date (dd/mm/yyyy): 20/10/1994 Place TRIPOLIS Country GRECIA

CLASS (MAIN FIELD OF STUDY FOR THE QUALIFICATION): Class n. LM-17 Physics
DEGREE PROGRAMME: Physics (Second cycle degree programme)
OFFICIAL LENGTH OF THE PROGRAMME: 2 academic years
ADMINISTRATIVE OFFICE: Bologna
LANGUAGE OF INSTRUCTION : Italian

MATRICULATION DATE (dd/mm/yyyy): 03/11/2016

FIRST ACADEMIC YEAR OF ENROLLMENT: 2016/2017

QUALIFICATION AND TITLE AWARDED:
Qualification: Laurea magistrale in Physics
Grade: 110 e lode(1) - Average grade: 29,92/30 (109,71/110)
Date (dd/mm/yyyy): 20/03/2019
Title: Dottore magistrale

FINAL EXAMINATION
Type: DISSERTATION AND RELATIVE DISCUSSION
Title: Backreaction from magnetogenesis in string inflation
Supervising Professor: CICOLI MICHELE

LEARNING ACTIVITIES SUCCESSFULLY COMPLETED IN THE LAST ATTENDED PROGRAMME

Learning activities	Grade	ECTS Scale	Date (dd/mm/yy)	SSD	CFU/ECTS
Dynamics of Stellar Systems	30	B(2)	13/12/2017	FIS/05	6
Field Theory 1	29	C(2)	22/02/2017	FIS/02	6
Field Theory 2	30	B(2)	27/07/2017	FIS/02	6
Group Theory	30	B(2)	12/09/2017	FIS/02	6
Nuclear Reactions	30 e lode	A(2)	03/07/2018	FIS/04	6
Particles and Fields	30	B(2)	18/06/2018	FIS/02	6
Preparation for the Final Examination	ID		11/01/2019		3
PREPARAZIONE PROVA FINALE	ID		22/02/2019		30
Relativity 1	30 e lode	A(2)	10/02/2017	FIS/02	6
Relativity 2	30	B(2)	13/10/2017	FIS/02	6

Learning activities	Grade	ECTS Scale	Date (dd/mm/yy)	SSD	CFU/ECTS
Statistical Mechanics 1	30 e lode	A(2)	02/02/2018	FIS/02	6
Statistical Mechanics 2	30 e lode	A(2)	28/03/2018	FIS/02	6
Theoretical Physics 1	30 e lode	A(2)	10/03/2017	FIS/02	6
Theoretical Physics 2	30 e lode	A(2)	12/06/2017	FIS/02	6
Final examination	Successfully Completed		20/03/2019		15

USEFUL CREDITS (RECOGNISED AND/OR OBTAINED IN THE LAST DEGREE PROGRAMME): 120

Notes

(1)
Final Examination taken at the School of 10 - Science

The Board evaluates the candidate through his/her study curriculum and the final examination; the Board expresses its evaluation as a mark out of one hundred and ten. The examination is passed with a minimum score of 66/110. In the event of the maximum score being awarded (110/110), the Board may unanimously decide to award the "cum laude" honour.

ECTS Scale	Grade	% of students who have obtained such grade
A	110 e lode	51
C	108 - 110	23
D	101 - 107	19
E	66 - 100	7

(2)
Exam taken at the School of 10 - Science
ECTS grading scale - Institutional grading system of the School of 10 - Science (second cycle degree programmes)

ECTS Scale	Grade	% of students who have obtained such grade
A	30 e lode	15
B	30	32

ECTS Scale	Grade	% of students who have obtained such grade
C	29	9
C	28	16
D	27	11
D	26	6
D	25	4
E	24	3
E	23	1
E	22	1
E	21	0
E	20	1
E	19	0
E	18	1

Passing grade for each exams or learning activity can range from 18 to 30. The highest possible grade is "30 e lode" (30L), i.e. 30 with honours.
For some exams and activities there is no grade, but only an "approved" (ID).
The percentages of students obtaining a given grade are rounded up to the nearest whole number. The highest percentage is calculated by the difference between 100 and the sum of the percentages of the students obtaining the other grades.
1 CFU = Credit Unit = 1 ECTS = 25 working hours (teaching, independent study, examinations, tutorials)
N.A. = Not applicable in a different Faculty in the Univesity system before 1999 reform or in a different University.
SSD = Scientific field/Discipline
RC = Recognised
RP = Replaced
SO = Sobstitute

The Italian University System

(DM 509/99 and DM 270/2004)

Since 1999, Italian university studies have been reformed so as to meet the objectives of the "Bologna process". The university system is now organised in 3 cycles: the Laurea, the 1st cycle academic degree, grants access to the 2nd cycle, and the *Laurea specialistica/magistrale*, the main degree of the 2nd cycle, gives access to 3rd cycle courses awarding the *Dottorato di ricerca*. In addition to the three sequential degrees mentioned above, the system offers other programmes with their respective degrees.

First cycle. First cycle studies consist exclusively in *Corsi di Laurea*, aimed at guaranteeing students an adequate command of general scientific methods and contents as well as specific professional skills. The general access requirement is the school leaving qualification awarded on completion of 13 years of global schooling and after the relevant State examinations; also comparable foreign qualifications may be accepted. Admission to individual degree courses may be subject to specific course requirements. *Laurea* courses last 3 years. The *Laurea* (1st degree) is awarded to students who have earned 180 credits; the completion of a training period and the defence of a thesis may also be required. The *Laurea* grants access to competitions for the civil service, to regulated and non-regulated professions, and to 2nd cycle courses.

Second cycle. Second cycle studies include the following typologies:
A) *Corsi di Laurea specialistica/Corsi di Laurea magistrale*; they are aimed at providing students with an advanced level of education for the exercise of a highly qualified activity in specific areas. Access is usually by a Laurea or a comparable foreign degree; admission is subject to specific course requirements determined by individual universities; workload: 120 credits; length: 2 years. The awarding of the degree, *Laurea specialistica/magistrale* (2nd cycle degree of the “Bologna process”) is conditional on the defence of a thesis. The change of the name from *Laurea specialistica* into *Laurea magistrale* was decided in 2004.
A limited number of 2nd cycle programmes (dentistry, human medicine, pharmacy, veterinary medicine, architecture, law), are defined *Corsi di Laurea specialistica/magistrale a ciclo unico* (one-block LS/LM courses); access is by the school leaving diploma or a comparable foreign qualification; admission is subject to selective entrance exams; each degree course is organised in just one-block of 5 years and 300 credits (only human medicine requires 6 years and 360 credits). All *Lauree specialistiche/magistrali* grant access to competitions for the civil service, to regulated and non-regulated professions, research doctorate programmes and all the other degree courses of the 3rd cycle.
B) *Corsi di Master universitario di primo livello*. They consist in advanced scientific courses or higher continuing education studies open to the holders of a *Laurea* or a comparable foreign degree; admission may be subject to additional conditions. Length: minimum 1 year; workload: 60 credits at least. The *Master universitario* di primo livello does not give access to the 3rd cycle.

Third cycle. Third cycle studies include the following typologies:
A) *Corsi di Dottorato di Ricerca* aim at training students for very advanced scientific research; they adopt innovative teaching methodologies, updated technologies, training periods abroad and supervised activities in specialized research centres. Admission requires a *Laurea specialistica/ magistrale* (or a comparable foreign degree) and to pass a specific competition; studies last a minimum of 3 years; the doctoral student must work out an original dissertation to be defended in the final examination.
B) *Corsi di specializzazione* are devised to provide students with knowledge and abilities as requested in the practice of highly qualified professions; they mainly concern medical, clinical and surgical specialities. Admission requires a *Laurea specialistica/magistrale* (or a comparable foreign degree) and the passing of a competitive examination; course length varies in relation to subject fields. The final degree, *Diploma di specializzazione*, gives the right to the title as *Specialista.*
C) *Corsi di Master universitario di secondo livello* consist in advanced scientific courses or higher continuing education studies, open to the holders of an LS or a comparable foreign degree. Length: minimum 1 year; workload: 60 credits at least.

Credits: degree courses are usually structured in credits. A university credit generally corresponds to 25 hours of global work per student, time for personal study included. The average workload of a full time student is conventionally fixed at 60 credits per year.

Classes of degree courses: all degree courses sharing educational objectives and teaching-learning activities are organised in groups called *classi*. The content of individual degree courses is autonomously determined by universities; however, when establishing a degree course, individual institutions have to adopt some general requirements fixed at national level. Degrees belonging to the same class have the same legal validity.

Academic titles: the *Laurea* confers the title "*Dottore*", the *Laurea specialistica/magistrale* that of *Dottore magistrale*, the *Dottorato di ricerc*a that of "*Dottore di ricerca*".

Joint degrees: Italian universities may establish degree courses in cooperation with foreign partner universities; on completion of integrated curricula joint or double/multiple degrees are awarded.

Esenzione: Art. 11 dell'Allegato B DPR 642-1972

Dott. Michele Menna. Issued on: 03/04/2019

Journal of Cosmology
and Astroparticle
Physics

PAPER

On adiabatic subtraction in an inflating Universe

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

On adiabatic subtraction in an inflating Universe

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Received October 12, 2022
Revised March 16, 2023
Accepted June 15, 2023
Published July 4, 2023

Abstract. Adiabatic subtraction is a popular method of renormalization of observables in quantum field theories on a curved spacetime. When applied to the computation of the power spectra of light ($m \ll H$) fields on de Sitter space with flat Friedmann-Lemaître-Robertson-Walker slices, the *standard prescriptions* of adiabatic subtraction, traceable back to [1, 2], lead to results that are significantly different from the standard expectations not only in the ultraviolet ($k \gg aH$) but also at intermediate ($m \ll k/a \lesssim H$) wavelengths. In this paper we review those results and we contrast them with the power spectra obtained using an alternative prescription for adiabatic subtraction applied to quantum field theoretical systems by Dabrowski and Dunne [3, 4]. This prescription eliminates the intermediate-wavelength effects of renormalization that are found when using the standard one.

Keywords: cosmological perturbation theory, quantum field theory on curved space, inflation, quantum cosmology

ArXiv ePrint: 2209.14362

JCAP07(2023)005

 https://doi.org/10.1088/1475-7516/2023/07/005

Contents

1 Introduction

One of the key predictions of primordial inflation is that quantum fluctuations of light fields are amplified into the seeds of the large scale structure of our Universe. As is the case on Minkowskian backgrounds, also for quantum fields on a curved spacetime, one needs to address the presence of ultraviolet divergences in the expressions of the physical observables. On Minkowskian backgrounds, divergences of observables that are quadratic in the fields (such as the total energy of a quantum field) can be subtracted "by hand", an operation, associated with the normal ordering of creation and annihilation operators, that is justified by the fact that the divergent quantities are unobservable constants. Things get more subtle in time-dependent and/or curved backgrounds.

In this article we study the renormalization of the power spectrum of a massless or light scalar field in de Sitter and quasi-de Sitter space. We will focus on the method of *adiabatic subtraction*, the original version of which was proposed by Parker [1, 2] to renormalize the energy-momentum tensor of scalar fields in expanding universes. This method leads to finite observables by subtracting from the bare results, mode by mode, the same quantities obtained by replacing the mode functions with their positive-frequency Wentzel-Kramers-Brillouin (WKB) approximation. This is a natural generalization of the subtraction of divergent constants performed to obtain finite results on trivial backgrounds. The WKB approximation is obtained as a recursive series, and, according to the standard prescription, the order at which it should be evaluated is related to the degree of ultraviolet divergence of the operator that we try to renormalize. More specifically, one is instructed [1, 2, 5–7] to truncate the WKB series to the minimum order that allows to obtain a UV-finite result.

By considering a light scalar on (quasi) de Sitter, we can compare our study of adiabatic subtraction with those existing in the literature. In particular, the method of adiabatic subtraction has been applied by Parker [8] to the standard calculation of the power spectrum $\mathcal{P}_k^\phi$ of a light ($m \ll H$) scalar field during inflation with Hubble parameter H. In that paper it was found that the renormalized power spectrum converges to the standard result $\frac{H^2}{4\pi^2}$ for $k/a \ll m \ll H$, while for $m \ll k/a \lesssim H$ one can evaluate it to $\sim \frac{H^2}{4\pi^2} \times \frac{3\,m^2\,a^2}{4\,k^2}$. Related analyses can be found in [9–12]. While the large scale results match the standard expectation, at intermediate scales, where causality arguments would require quasi-constant

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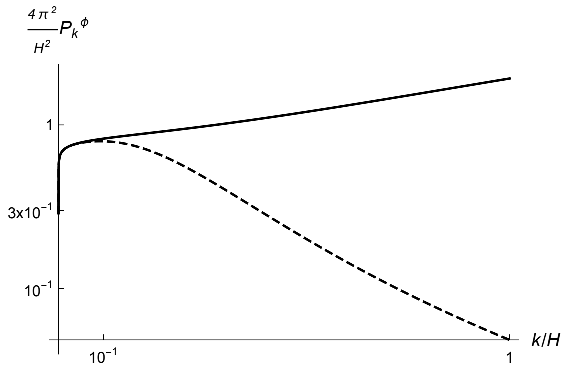

Figure 1. The power spectrum of a scalar field of mass $m^2 = .1\, H^2$ on de Sitter space. The solid line represents the non-renormalized result. The dashed line gives the renormalized result according to eq. (18) of [8].

power spectra, the renormalized spectra show a rapid running. This behavior can be seen for instance in figure 1, where we show the non-renormalized and the renormalized power spectra (obtained from eq. (18) of [8]) of a scalar with mass $m^2 = .1\, H^2$. It is worth stressing that the study of scalar metric perturbations during inflation, which are restricted by diffeomorphism invariance, would require a more specific formulation than the one we are using here, and is beyond our scope.

Several authors have discussed the result of [8]. Shortly after [8], it was pointed out in [13] that the two-point function, being finite when computed at distinct points, should not need renormalization.[1] The authors of [13], however, also noted that adiabatic regularization of the power spectrum, a fundamental tool to renormalize the stress-energy tensor, leads to "unpleasant features" in the regularized power spectra that persist if one considers the next order in the WKB series. In [14] it has been noted that a time-dependent value of the Hubble parameter makes the adiabatically subtracted component smaller at later times. This view was restated in [15], where it was stressed that the effect of renormalization should not affect the cosmological scales $k \ll a\, H$. In [16], the same authors argued that these problems are alleviated if one assumes a radiation-dominated period prior to inflation, that effectively provides an infrared cutoff to the modes of the scalar perturbations. The paper [17] was written in response to these objections. In [18] it was shown that adiabatic subtraction can be recast in the form of redefinition of parameters of the Lagrangian.[2] Very recently, finally, the authors of [19] have argued that the unusual behavior for superhorizon modes with physical wavelengths shorter than m^{-1} is eliminated by implementing adiabatic subtraction only for modes with wavelengths shorter than an infrared cutoff.

[1]Reasons why the two-point function should be renormalized at all have been discussed, e.g., in [6], page 84. We stress here that finite values for quantities of physical interest such as the stress-energy tensor of a scalar field $\langle T_{\mu\nu}(x)\rangle$ can be obtained by computing objects such as $\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\langle\phi(x)\phi(y)\rangle\right)_{y=x}$, which are divergent and can be renormalized using adiabatic subtraction. Then, for consistency, one should also apply adiabatic subtraction to $\langle\phi(x)\phi(y)\rangle$.

[2]In Minkowski-space renormalization, the introduction of counterterms is associated with the existence of observables, such as the energy, sensitive to the divergent quantity under consideration. In the case of the power spectrum, one can think of a field χ interacting with ϕ through a coupling proportional to $\phi^2\chi^2$. In this case the expectation value $\langle\phi^2(x)\rangle = \lim_{y\to x}\langle\phi(x)\phi(y)\rangle$ contributes to the effective mass of χ, and would be observable by measuring χ's dispersion relation. The counterterm, in this example, appears in the bare mass of the field χ.

The result of [8] highlights a couple of undesirable features of the standard formulation of adiabatic subtraction:

1. adiabatic subtraction leads to artifacts (significant corrections to the classical result far from the UV regime) for modes that are not adiabatically evolving. One such example is given by the modes with $m \lesssim k/a \lesssim H$, see figure 1. One can argue that this is not so upsetting, because if the proper frequencies are not adiabatically evolving, then the concept of particle itself becomes ill-defined. Nevertheless, it would be preferable to *(i)* have a trustworthy definition of integral quantities, such as, e.g., $\langle \phi(\mathbf{x},\, t)^2 \rangle$, that is given as the integral on all scales of the power spectrum; and *(ii)* that the value of the power spectrum takes physically sensible values at all wavelengths;

2. in the standard, textbook [5, 6] prescription for adiabatic subtraction, the order of WKB approximation depends on the degree of ultraviolet divergence of the operator under consideration. For instance, the calculation of $\langle \phi(\mathbf{x},\, t)^2 \rangle$ will require subtraction up to the second order in the WKB expansion, while to compute $\langle (\nabla\phi(\mathbf{x},\, t))^2 \rangle$ one needs to go to fourth order. A prescription where the order of truncation of the WKB expansion is independent of the operator under consideration might be preferable.

A related subtlety is that adiabaticity depends on the choice of time. For a massless scalar in cosmic time t the proper frequency is given by $\omega_k^2 = k^2\, e^{-2Ht} - \frac{9}{4}H^2$, that behaves adiabatically, $|d\omega_k/dt| \ll |\omega_k|^2$ at late times $t \to \infty$. On the other hand, in conformal time τ, $d\tau = e^{-Ht}\, dt$, the proper frequency of canonically normalized modes $\omega_k^2 = k^2 - \frac{2}{\tau^2}$ does *not* behave adiabatically at late times, as $|d\omega_k/d\tau| \simeq |\omega_k|^2/\sqrt{2}$ for $\tau \to 0^-$. The notion of adiabaticity thus seems to depend on the choice of time. While this ambiguity can be dealt with by considering terms such as $\frac{\ddot{a}}{a}$ appearing in the proper frequency as higher order terms in the adiabatic expansion, we will show below that this issue is resolved by assuming a definition of time that maintains $\omega_k(t)^2 > 0$ at all moments. Such a definition of time has been used in the past, see for instance [7, 11].

In this paper we will explore the implications, for the calculation of power spectra, of an alternative prescription for the order of truncation of the WKB approximation. This prescription, based on the findings in [20–22], has been explicitly applied to quantum field theoretical systems in [3, 4]. As we will discuss in sections 2 and 3 below, the adiabatic expansion is generally an asymptotic expansion and as such has an optimal truncation, i.e., there is an order of the expansion that gives an exponentially good approximation to the exact solution. This order has nothing to do with the degree of divergence of the operator one has to renormalize but depends on the parameters of the system. This order is also generally dependent on the momentum k. At least in principle, this allows us to avoid the generation of unphysical behavior at intermediate momenta such as that observed when applying the textbook prescription [5, 6]. Remarkably, the asymptotic behavior of the mode functions which underlies the adiabatic solution is exactly the manifestation of the notion of particle production, that can be evaluated analytically by focusing on the Stokes phenomenon [23]. In other words, if the WKB expansion is not asymptotic and can be resummed exactly, then no particle production will occur.

In section 4, we apply the general results presented in the previous sections to specific cases of inflationary spectra. We find, in agreement with the result found in [8], that the renormalized spectrum for a massless, minimally coupled scalar in exact de Sitter space with flat spatial slices is identically vanishing. For a light, massive field in exact de Sitter space,

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on the other hand, the application of the prescription [3, 4] gives, unlike the results of [8], a power spectrum that is approximately constant for all scales $k \lesssim aH$. Finally, we apply the general results discussed above to the case of a massless scalar in an FLRW Universe that performs a smooth transition from a quasi-de Sitter to a radiation-dominated stage.

Section 5 contains our conclusions and a discussion of these results.

2 Bogolyubov coefficients, adiabatic subtraction

Quantum field theory is plagued with divergences. The simplest divergence that is encountered is that of the expectation value, for a free theory on a Minkowskian background, of quadratic operators such as the energy density. Such a divergence is cured by subtracting its (formally infinite) "vacuum contribution" by hand, or equivalently by replacing the operators under consideration with their normal-ordered version.[3]

In the case in which the field is quantized on a time-dependent background, however, the vacuum of the theory will also be generally evolving, making the concept of "vacuum contribution" ambiguous. In this case, finite expectation values for quadratic operators are usually obtained by applying *adiabatic subtraction*, i.e., by subtracting, mode by mode, from the expectation value of the operator under consideration, the expectation value of the same quantity evaluated in the adiabatic approximation [1].

We will now review how this prescription can also be phrased in terms of a time-dependent normal ordering — a picture that can be traced back to [24, 25].

Consider a general system on a time-dependent background described by a scalar field $\hat{\phi}(\mathbf{x},\, t)$ that we quantize as

$$\hat{\phi}(\mathbf{x},\, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{kx}} \left[\phi(k,\, t)\, \hat{a}_{\mathbf{k}} + \phi(k,\, t)^*\, \hat{a}^\dagger_{-\mathbf{k}}\right] \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{kx}}\, \hat{\phi}(\mathbf{k},\, t)\,, \tag{2.1}$$

where $\phi(k,\, t)$ satisfies the equation

$$\ddot{\phi}(k,\, t) + \omega_k(t)^2\, \phi(k,\, t) = 0\,. \tag{2.2}$$

As we will discuss below, in order for the adiabatic subtraction to be well defined, we require $\omega_k(t)^2$ to be positive. We also assume, for the sake of presentation in this section, that $\omega_k(t) \geq 0$ satisfies the adiabaticity conditions $|\dot{\omega}_k|/\omega_k^2 \to 0$ and $|\ddot{\omega}_k|/\omega_k^3 \to 0$ both as $t \to -\infty$ and as $t \to +\infty$, while at intermediate times $\omega_k(t)$ will generally evolve non adiabatically. Under these conditions, the general solution to eq. (2.2) for $t \to \pm\infty$ is a linear combination of $\phi^{\rm ad}_{(0)}(k,\, t)$ and $\phi^{\rm ad}_{(0)}(k,\, t)^*$, with

$$\phi^{\rm ad}_{(0)}(k,\, t) = \frac{1}{\sqrt{2\, W_{(0)}(k,\, t)}} e^{-i\int_{t_0}^t W_{(0)}(k,\, t')\, dt'}\,, \qquad W_{(0)}(k,\, t) \equiv \omega_k(t) \geq 0\,, \tag{2.3}$$

where the value of t_0 (as long as it is real) is irrelevant.

We choose the initial condition to be positive frequency only

$$\phi(k,\, t \to -\infty) = \phi^{\rm ad}_{(0)}(k,\, t)\,, \tag{2.4}$$

which implies that the operator $\hat{a}_{\mathbf{k}}$ ($\hat{a}^\dagger_{\mathbf{k}}$) annihilates (creates) quanta of ϕ at $t \to -\infty$.

[3]It is worth stressing that not all the divergences in a theory appear in the expectation values of quadratic operators. Nevertheless, the present paper will deal only with these ones, which are relevant for observables such as power spectra or occupation numbers.

Then we solve eq. (2.2). Since the adiabaticity condition is assumed to be satisfied at late times, for $t \to +\infty$ the solution must take the form

$$\phi(k,\, t \to +\infty) = \alpha(k)\, \phi^{\rm ad}_{(0)}(k,\, t) + \beta(k)\, \phi^{\rm ad}_{(0)}(k,\, t)^*\,, \tag{2.5}$$

where $\alpha(k)$ and $\beta(k)$ are constants: the *Bogolyubov coefficients*. Thus, the field $\hat{\phi}(\mathbf{x},\, t)$ at late times reads

$$\begin{aligned}\hat{\phi}(\mathbf{x},\, t \to +\infty) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}} \Big\{ \Big[\alpha(k)\, \phi^{\rm ad}_{(0)}(k,\, t) + \beta(k)\, \phi^{\rm ad}_{(0)}(k,\, t)^*\Big]\, \hat{a}_\mathbf{k} \\ + \Big[\alpha(k)^*\, \phi^{\rm ad}_{(0)}(k,\, t)^* + \beta(k)^*\, \phi^{\rm ad}_{(0)}(k,\, t)\Big]\, \hat{a}^\dagger_{-\mathbf{k}} \Big\}\,. \end{aligned} \tag{2.6}$$

We can therefore define the new operators

$$\begin{aligned} \hat{b}_\mathbf{k} &\equiv \alpha(k)\, \hat{a}_\mathbf{k} + \beta(k)^*\, \hat{a}^\dagger_{-\mathbf{k}}\,, \\ \hat{b}^\dagger_\mathbf{k} &\equiv \alpha(k)^*\, \hat{a}^\dagger_\mathbf{k} + \beta(k)\, \hat{a}_{-\mathbf{k}}\,, \end{aligned} \tag{2.7}$$

in terms of which the field $\hat{\phi}(\mathbf{x},\, t)$ takes the form

$$\hat{\phi}(\mathbf{x},\, t \to +\infty) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}} \Big[\phi^{\rm ad}_{(0)}(k,\, t)\, \hat{b}_\mathbf{k} + \phi^{\rm ad}_{(0)}(k,\, t)^*\, \hat{b}^\dagger_{-\mathbf{k}}\Big]\,. \tag{2.8}$$

This equation shows that $\hat{b}_\mathbf{k}$ is seen as an annihilation operator by an observer born at $t \to +\infty$. In the literature [24, 25] the choice of these creation/annihilation operators is justified by the fact that these are the operators that diagonalize the Hamiltonian at late times.

Let us now evaluate, for instance, the Hamiltonian operator $\hat{H}(t)$, which is time dependent since the system is on a time-dependent background. It is easy to see that

$$\begin{aligned} \hat{H}(t \to -\infty) &= \int d^3\mathbf{k} \frac{\omega_k(t \to -\infty)}{2} \left(\hat{a}_\mathbf{k}\, \hat{a}^\dagger_\mathbf{k} + \hat{a}^\dagger_\mathbf{k}\, \hat{a}_\mathbf{k}\right)\,, \\ \hat{H}(t \to +\infty) &= \int d^3\mathbf{k} \frac{\omega_k(t \to +\infty)}{2} \left(\hat{b}_\mathbf{k}\, \hat{b}^\dagger_\mathbf{k} + \hat{b}^\dagger_\mathbf{k}\, \hat{b}_\mathbf{k}\right)\,, \end{aligned} \tag{2.9}$$

whose vacuum expectation value $\langle 0|\hat{H}(t \to \pm\infty)|0\rangle$ is divergent. This divergence can be eliminated by computing the expectation value of the *normal ordered* Hamiltonian operator instead, $\langle 0| : \hat{H} : |0\rangle$. *We require that an early observer normal orders the $\hat{a}_\mathbf{k}$ and $\hat{a}^\dagger_\mathbf{k}$ operators, whereas a late observer normal orders the $\hat{b}_\mathbf{k}$ and $\hat{b}^\dagger_\mathbf{k}$ operators.* On the other hand, since we are working in the Heisenberg picture, the state $|0\rangle$ is independent of time, so that it is annihilated by the $\hat{a}_\mathbf{k}$ operators, but not by the $\hat{b}_\mathbf{k}$ operators.

The consequence of this prescription is that

$$\langle 0| : \hat{H}(t \to -\infty) : |0\rangle \to 0\,, \qquad \langle 0| : \hat{H}(t \to +\infty) : |0\rangle \to \int d^3\mathbf{k}\, \omega_k(t \to +\infty)\, |\beta(k)|^2\,, \tag{2.10}$$

so that, as it is well known, $|\beta(k)|^2$ can be interpreted as the occupation number of created particles at late times.

We can generalize the above procedure to any quantity that is quadratic in the fields. In particular, this can be applied to the power spectrum $\mathcal{P}^\phi_k$, that we define through

$$\langle 0|\hat{\phi}(\mathbf{x},\, t)\hat{\phi}(\mathbf{y},\, t)|0\rangle = \int \frac{d^3\mathbf{k}}{4\pi\, k^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\, \mathcal{P}^\phi_k(t)\,, \tag{2.11}$$

which gives $\mathcal{P}_k^\phi(t) = \frac{k^3}{2\pi^2}\,|\phi(k,\,t)|^2$. A straightforward calculation then shows, using eq. (2.5), that the normal ordered power spectrum can be written, both for $t\to-\infty$ and for $t\to+\infty$, as

$$\mathcal{P}_k^\phi(t\to\pm\infty) = \frac{k^3}{2\pi^2}\left[|\phi(k,\,t\to\pm\infty)|^2 - \frac{\Im\left\{\phi(k,\,t\to\pm\infty)\,\dot\phi(k,\,t\to\pm\infty)^*\right\}}{W_{(0)}(k,\,t\to\pm\infty)}\right], \tag{2.12}$$

which can be shown to be equivalent to

$$\begin{aligned}\mathcal{P}_k^\phi(t\to\pm\infty) &= \frac{k^3}{2\pi^2}\,\left[|\phi(k,\,t\to\pm\infty)|^2 - |\phi_{(0)}^{\rm ad}(k,\,t\to\pm\infty)|^2\right]\\ &= \frac{k^3}{2\pi^2}\,\left[|\phi(k,\,t\to\pm\infty)|^2 - \frac{1}{2\,\omega_k(t\to\pm\infty)}\right].\end{aligned} \tag{2.13}$$

The first line of this equation shows that time-dependent normal ordering amounts to subtracting from the original expression the same one with the mode functions evaluated in the adiabatic approximation. This procedure thus justifies in a natural way *adiabatic subtraction* as a method of subtracting the divergent part of the expectation values of operators in time-dependent settings.

However, in the way in which it is described above, adiabatic subtraction is not always sufficient to subtract *all* the UV divergences. A related issue is that one should require a prescription for the definition of occupation numbers also at finite, even if large, times. To deal with these questions, one modifies the above derivation to include *higher orders* in the adiabatic expansion in the definition of $\phi^{\rm ad}(k,\,t)$, and to generalize eq. (2.5) to all values of the time.

More specifically, we define the $(2n)-$th order adiabatic approximation $\phi_{(2n)}^{\rm ad}(k,\,t)$ to the exact solution of eq. (2.2) as

$$\phi_{(2n)}^{\rm ad}(k,\,t) = \frac{1}{\sqrt{2\,W_{(2n)}(k,\,t)}}e^{-i\int_{t_0}^t W_{(2n)}(k,t')\,dt'}\,, \tag{2.14}$$

where $W_{(2n)}(k,\,t)$ is found as follows.

We start by inserting into eq. (2.2) the Ansatz

$$\phi(k,\,t) = \frac{1}{\sqrt{2\,W(k,\,t)}}e^{-i\int_{t_0}^t W(k,t')\,dt'}\,, \tag{2.15}$$

which implies that $W(k,\,t)$ satisfies the equation

$$W(k,\,t)^2 = \omega_k(t)^2 + \sqrt{W(k,\,t)}\frac{d^2}{dt^2}\left(\frac{1}{\sqrt{W(k,\,t)}}\right). \tag{2.16}$$

The solution to this equation can be found iteratively by expanding it as a series in time derivatives. Then $W_{(2n)}(k,\,t)$ is found as the truncation at the $(2n)$-th order of this derivative (i.e., adiabatic) expansion:

$$\begin{aligned}&W_{(0)}(k,\,t) = \omega_k(t)\,,\\ &W_{(2)}(k,\,t) = \omega_k(t)\left[1+\frac{\sqrt{\omega_k(t)}}{2\,\omega_k(t)^2}\frac{d^2}{dt^2}\left(\frac{1}{\sqrt{\omega_k(t)}}\right)\right],\\ &W_{(4)}(k,\,t) = \omega_k(t)\left[1+\frac{\sqrt{\omega_k(t)}}{2\,\omega_k(t)^2}\frac{d^2}{dt^2}\left(\frac{1}{\sqrt{\omega_k(t)}}\right)-\frac{\sqrt{\omega_k(t)}}{4\,\omega_k(t)^2}\frac{d^2}{dt^2}\left(\frac{1}{2\omega_k(t)^2}\frac{d^2}{dt^2}\left(\frac{1}{\sqrt{\omega_k(t)}}\right)\right)\right],\\ &W_{(6)}(k,\,t) = \dots\,.\end{aligned} \tag{2.17}$$

In particular, eq. (2.13) turns out to be valid because, under the assumption of adiabaticity at early and late times, one has $W_{(2n)}(k,\, t \to \pm\infty) \to W_{(0)}(k,\, t \to \pm\infty) = \omega_k(t \to \pm\infty)$.

Then, we define the Bogolyubov coefficients $\alpha_{(2n)}(k,\, t)$ and $\beta_{(2n)}(k,\, t)$, *for all values of the time* t, through

$$\phi(k,\, t) = \alpha_{(2n)}(k,\, t)\, \phi^{\rm ad}_{(2n)}(k,\, t) + \beta_{(2n)}(k,\, t)\, \phi^{\rm ad}_{(2n)}(k,\, t)^* \,. \tag{2.18}$$

It is important to stress at this point that the definition (2.18) of Bogolyubov coefficients requires $\phi^{\rm ad}_{(2n)}(k,\, t)$ and $\phi^{\rm ad}_{(2n)}(k,\, t)^*$ to be linearly independent quantities, which implies that $W_{(2n)}(k,\, t)$ should be real. Since $W_{(2n)}(k,\, t)$ is an approximation of $\omega_k(t)$, *Bogolyubov coefficients can be consistently defined only at times for which* $\omega_k(t)$ *is real.*

In principle one can also study situations where $\omega_k(t)$ transitions from real to imaginary and then again to real values [26], but if we want the occupation number to be defined at all times, we will require $\omega_k(t)$ to be real at all times. We will see this in particular in section 4, where we will adopt a definition of time for which $\omega_k(t)$ is real also for super-horizon modes.

Finally, by requiring $\omega_k(t)^2 \geq 0$ one does not incur in the problem, found in [16], of having to deal with an imaginary component in the spectrum in eq. (2.13).[4]

Eq. (2.18) does not determine $\alpha_{(2n)}(k,\, t)$ and $\beta_{(2n)}(k,\, t)$ uniquely, so that we need a second prescription. Among various options that give equivalent results at $t \to \pm\infty$, we choose that obtained by taking the time derivative of eq. (2.18) while keeping $\alpha_{(2n)}(k,\, t)$ and $\beta_{(2n)}(k,\, t)$ constant:

$$\begin{aligned}\dot\phi(k,\, t) &= \frac{\alpha_{(2n)}(k,\, t)}{\sqrt{2\, W_{(2n)}(k,\, t)}} \left(-i W_{(2n)}(k,\, t) - \frac{\dot W_{(2n)}(k,\, t)}{2\, W_{(2n)}(k,\, t)}\right) e^{-i\int_{t_0}^t W_{(2n)}(k,\, t')\, dt'} \\ &\quad + \frac{\beta_{(2n)}(k,\, t)}{\sqrt{2\, W_{(2n)}(k,\, t)}} \left(i W_{(2n)}(k,\, t) - \frac{\dot W_{(2n)}(k,\, t)}{2\, W_{(2n)}(k,\, t)}\right) e^{i\int_{t_0}^t W_{(2n)}(k,\, t')\, dt'} \,. \end{aligned} \tag{2.19}$$

Eqs. (2.18) and (2.19) can be inverted to give

$$\begin{aligned}\alpha_{(2n)}(k,\, t) &= \frac{1}{2}\sqrt{2\, W_{(2n)}(k,\, t)}\, e^{i\int_{t_0}^t W_{(2n)}(k,\, t')\, dt'} \left[\phi(k,\, t) + i \frac{\dot\phi(k,\, t)}{W_{(2n)}(k,\, t)}\right] , \\ \beta_{(2n)}(k,\, t) &= \frac{1}{2}\sqrt{2\, W_{(2n)}(k,\, t)}\, e^{-i\int_{t_0}^t W_{(2n)}(k,\, t')\, dt'} \left[\phi(k,\, t) - i \frac{\dot\phi(k,\, t)}{W_{(2n)}(k,\, t)}\right] , \end{aligned} \tag{2.20}$$

and we can thus define the time dependent creation/annihilation operators

$$\begin{aligned}\hat b^{(2n)}_{\mathbf{k}}(t) &\equiv \alpha_{(2n)}(k,\, t)\, \hat a_{\mathbf{k}} + \beta_{(2n)}(k,\, t)^*\, \hat a^\dagger_{-\mathbf{k}} \,, \\ \hat b^{(2n)}_{\mathbf{k}}(t)^\dagger &\equiv \alpha_{(2n)}(k,\, t)^*\, \hat a^\dagger_{\mathbf{k}} + \beta_{(2n)}(k,\, t)\, \hat a_{-\mathbf{k}} \,. \end{aligned} \tag{2.21}$$

that are seen as annihilation/creation operators for an observer born at time t.

[4] Trying to be more sophisticated, in the case in which $\omega_k(t)$ is imaginary, one might look for two linearly independent solutions of eq. (2.2) with $\omega_k(t)^2 < 0$, one complex conjugate of the other, to allow for a decomposition like that of eq. (2.1). One would then find that it is not possible for any of such decompositions to diagonalize the Hamiltonian.

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In particular, after normal ordering of the $\hat{b}_{\mathbf{k}}^{(2n)}(t)$ and $\hat{b}_{\mathbf{k}}^{(2n)}(t)^\dagger$ operators, the power spectrum of the field $\hat{\phi}(\mathbf{x},\, t)$ reads

$$\mathcal{P}_k^{\phi}(t) = \frac{k^3}{2\pi^2}\left[|\phi(k,\, t)|^2 - \frac{1}{2\, W_{(2n)}(k,\, t)}\right], \tag{2.22}$$

where all the dependence on the order $(2n)$ of the adiabatic expansion lies in the $W_{(2n)}(k,\, t)$ at the denominator in the last term, since $\phi(k,\, t)$ is the exact solution to the equation of motion.

3 Truncation vs resummation of the adiabatic expansion

As discussed in the previous section, in order to apply adiabatic subtraction we need to truncate the adiabatic expansion at the right adiabatic order. The next questions is then: which order? According to [5–7], the prescription is to keep in $W_{(2n)}(k,\, t)$ all the terms, *but no more*, that are necessary to cancel all the UV divergences in the operator under consideration.[5] While one might argue that this prescription has the advantage of being the least intrusive way of generating UV-finite results, there are, as we have discussed in the Introduction, a couple of reasons for concern. First, it is not clear why the order of truncation should depend on the observable. For instance, for the same field, this prescription instructs to use $W_{(2)}(k,\, t)$ when computing the power spectrum, and $W_{(4)}(k,\, t)$ when computing the energy density. Second, while this prescription creates a good behavior for the observables at $k \to \infty$, it often generates artifacts (i.e. features in the observables where the subtracted component overwhelms the bare one) at finite momenta, such as those observed in [8].

In this article we discuss an alternative approach, see [3, 4], where the order of truncation is not related to the degree of divergence of the operator we try to renormalize. In fact, the WKB series is in general an asymptotic expansion and as such it has an optimal truncation which gives the best possible approximation to the exact solution. Truncating the adiabatic expansion to the optimal order, besides being a less arbitrary choice, also removes infrared artifacts, as the order of the truncation generally depends on the wavenumber k and therefore the subtracted function will not be the same in the UV and the IR regime. More importantly, as we will see below, optimal truncation of the WKB series, at least in the regime of large truncation order, leads to a *universal* functional dependence of the Bogolyubov coefficients.

The asymptotic nature of the WKB expansion is associated with the fact that the adiabatic solutions are defined only locally, as we will now discuss.

The WKB approximation requires the adiabatic conditions $|\dot{\omega}_k| \ll \omega_k^2$ and $|\ddot{\omega}_k| \ll \omega_k^3$ to be satisfied. The points in the complex-t plane where $\omega_k = 0$ are called *poles* or *turning points*, and each of them is surrounded by a region where the adiabatic approximation is violated. For the sake of the presentation, let us assume that the adiabatic conditions are satisfied as $t \to \pm\infty$, i.e., that the concept of particle is well defined at early and at late times. Let us also assume that there is a path Γ in the complex-t plane, see figure 2, that allows us to go from $t \to -\infty$ to $t \to +\infty$ while staying arbitrarily far from the turning points (which is guaranteed

[5]Moreover, the textbook prescription is to further expand $\frac{1}{2\, W_{(2n)}(k,\, t)}$ in eq. (2.22) in a derivative expansion:

$$\frac{1}{2\, W_{(2n)}} \simeq \frac{1}{2\, W_{(0)}} - \frac{W_{(2)} - W_{(0)}}{2\, W_{(0)}^2} + \ldots\,. \tag{3.1}$$

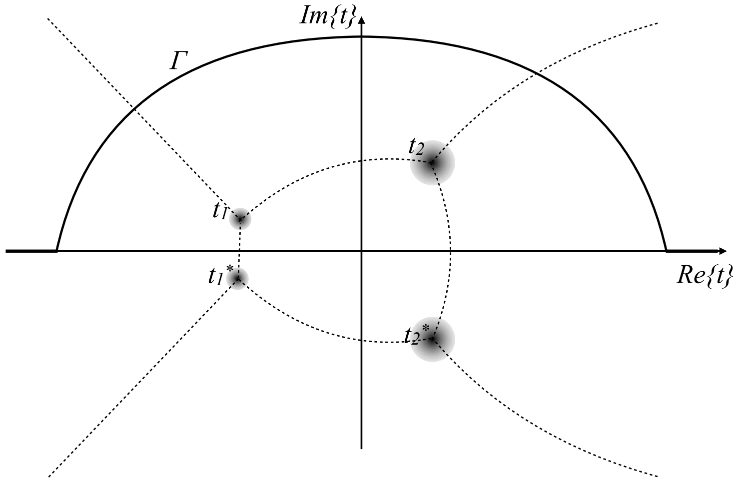

Figure 2. A schematic representation of the complex-t plane. In this example the theory has four turning points, coming in complex conjugate pairs, t_1, t_1^*, t_2, t_2^*. Each turning point is surrounded by a shaded area where the adiabaticity condition is violated. The path Γ allows us to go from $t \to -\infty$ to $t \to +\infty$ while traveling arbitrarily far from the turning points. However, along its path, it must cross Stokes lines (marked as dotted lines), where a negative frequency component is added to the WKB solution.

as long as the number of turning points is finite). This would imply that, starting at $t \to -\infty$ with a positive frequency solution $\propto e^{-i\int^t \omega_k\, dt'}$, and using the WKB approximation along Γ, we get solutions that are only positive frequency also as $t \to +\infty$, i.e., $\beta(k,\, t \to +\infty) = 0$. Clearly there is something wrong here, since in general, if the adiabatic condition is violated at finite times, particles should be created and $\beta(k,\, t \to +\infty)$ should not vanish!

What has gone wrong? The issue is that the WKB approximation is defined only *locally*, and the mode functions at initial time cannot be analytically continued through the whole complex plane. The borders of regions of validity of such local approximations are called *Stokes lines*, and the generation of a negative frequency component as one crosses a Stokes line is known as the *Stokes phenomenon* [27].

The origin of the Stokes phenomenon is that in any given Stokes region the WKB perturbative expansion fails to capture the full expression of the solution, i.e., a part of the solution does not appear *at any order* in the WKB approximation,[6] and as a consequence is not generated by the simple analytical continuation to $t \to +\infty$ of the positive frequency solution. In other words, the Stokes phenomenon signals the fact that the WKB expansion cannot be resummed everywhere to get the exact mode function, i.e., that the WKB expansion is asymptotic. We thus obtain the chain of equivalences

$$\text{asymptotic WKB expansion } \Leftrightarrow \text{Stokes lines} \Leftrightarrow \text{frequency mixing} \Leftrightarrow \text{particle creation.}$$

On the other hand, if the WKB expansion *can* be resummed, then the exact mode functions can be written, in the entire complex-t plane, as $\frac{1}{\sqrt{2W(k,t)}} e^{-i\int^t W(k,t')\,dt'}$ for some

[6]This is similar to the case of a function such as $f(x) = g(x) + e^{-1/x^2}$ where $g(x)$ can be expanded as a Taylor series around $x = 0$. If we try to evaluate $f(x)$ at finite values of x by using its Taylor expansion around $x = 0$, we will never recover the component e^{-1/x^2}, irrespective to the order at which we perform the Taylor expansion.

function $W(k,\,t)$ (with boundary condition $W(k,\,t\to-\infty)\to\omega_k(t\to-\infty)$) and there is no particle creation, $\beta(k,\,t)=0$. In this case it is apparent how the prescription [3, 4] is radically different from that of [5–7]: while [3, 4] leads to $\beta(k,\,t)=0$ if the WKB series can be resummed exactly, [5–7] would truncate the WKB series to a finite order, yielding $\beta(k,\,t\to+\infty)\neq0$.

As we have seen above, the Stokes lines represent the boundaries of the regions with well defined local WKB expansions. On those lines we have the greatest disparity between the exponentials appearing in the positive and negative frequency solutions. Thus, the Stokes lines are determined by the condition

$$F_k(t)\equiv-2i\int_{t_c}^{t}\omega_k(t')\,dt'=\text{purely real}\,,\tag{3.2}$$

where we denoted as t_c a turning point ($\omega_k(t_c)=0$) in the complex-t plane with positive imaginary part. $F_k(t)$ is called the *singulant* variable.

Stokes [27] provided a *connection formula* which allows us to find the mode function on one side of the Stokes line once we have it on the other side: in order to have the concordance between different asymptotic representations defined in different regions of the complex-t plane, the multiplier of the sub-dominant exponential must have a jump which is equal to i (the imaginary unit) times the multiplier of the dominant one. If we assume the mode function at early times to be

$$\phi(k,\,t\text{ on one side of Stokes line})=\frac{1}{\sqrt{2\omega_k(t)}}e^{-i\int_{t_0}^{t}\omega_k(t')dt'}\,,\tag{3.3}$$

then, after crossing the Stokes line, it will have the form

$$\phi(k,t\text{ on other side of Stokes line})=\frac{1}{\sqrt{2\omega_k(t)}}\left[e^{-i\int_{t_0}^{t}\omega_k(t')dt'}-i\,e^{-2i\int_{t_0}^{t_c}\omega_k(t')dt'}e^{i\int_{t_0}^{t}\omega_k(t')dt'}\right].\tag{3.4}$$

A study of the microscopic structure of the Stokes phenomenon has been performed by Dingle [20] and Berry [21, 22]. These authors have found a formula for the order n^{optimal} at which the WKB expansion should be truncated to yield the best approximation to $\beta(k,\,t)$. More importantly, they have resolved the thickness of the Stokes line, arguing that the evolution of the multiplier of the subdominant mode function (i.e., the Bogolyubov coefficient $\beta(k,\,t)$) in a neighborhood of the Stokes line has a universal and smooth form, at least as long as $n^{\text{optimal}}\gg1$. Dunne and Dabrowsky [3, 4] have verified the consistency of those formulae for the cases of Schwinger effect and of creation of massive scalars in a closed de Sitter Universe.

The fact that the optimal truncation of the WKB series gives the best approximation to the universal form of $\beta(k,\,t)$ is the main reason behind the proposal for adiabatic subtraction, alternative to that of [5, 6], that we study in this paper.

Such a universal form is obtained in terms of a natural Stokes-line crossing variable $\sigma_k(t)$ given by

$$\sigma_k(t)\equiv\frac{\Im\{F_k(t)\}}{\sqrt{2\,\Re\{F_k(t)\}}}\,,\tag{3.5}$$

where the singulant function $F_k(t)$ is defined in eq. (3.2). The Bogolyubov coefficient $\beta(k,\,t)$, as we cross the Stokes' line, is given by

$$\beta(k,\,t)\simeq\frac{i}{2}\left[1+\text{erf}\left(\sigma_k(t)\right)\right]e^{-F_k^{(0)}}\,,\tag{3.6}$$

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that indeed ranges from 0 to $i\, e^{-F_k^{(0)}}$, where

$$F_k^{(0)} = -i \int_{t_c}^{t_c^*} \omega_k(t')\, dt' \tag{3.7}$$

is the integral of $\omega_k(t)$ taken along the Stokes' line between two complex conjugate turning points, and is therefore real (and, with appropriate choice of branches, positive).

Also, the order of the optimal truncation of the WKB series is given by

$$n^{\text{optimal}} \simeq \text{Int}\left[F_k^{(0)}/2\right], \tag{3.8}$$

which shows that this whole discussion is strictly speaking valid only as long as $F_k^{(0)} \gg 1$. In fact, the thickness of the Stokes' line, as given by the size of the region across which $\beta(k,\, t)$ accumulates most of its variation, is given by the range of t for which $\sigma_k(t)$ ranges from $O(-1)$ to $O(+1)$, which is shown by eq. (3.5) to scale as $1/\sqrt{F_k^{(0)}}$. Equivalently, one can estimate the size of the regions about t_c for which the adiabaticity condition is not satisfied. To do so, we assume that t_c is a simple zero and set without loss of generality $\Re\{t_c\} = 0$. Then, linearizing, $\omega_k(t) \simeq \dot\omega_k(t_c) \times (t - t_c)$, we obtain that $|\dot\omega_k(t)| \gtrsim \omega_k(t)^2$ for $|t - t_c| \lesssim 1/\sqrt{|\dot\omega_k(t_c)|}$, so that if we want t_c and t_c^* to be distant enough that their regions of non-adiabaticity do not overlap, we must require $|t_c - t_c^*| = O(|t_c|) \gg 1/\sqrt{|\dot\omega_k(t_c)|}$. Estimating the singulant for the linearized expression of $\omega_k(t)$ we obtain $F_k(t) \approx \int_{t_c}^{t} \dot\omega_k(t_c) \times (t' - t_c)\; dt' \approx \dot\omega_k(t_c)\,(t - t_c)^2 \Rightarrow F_k^{(0)} \approx \dot\omega_k(t_c)\, t_c^2$, so that the condition that the poles are distinct, $O(|t_c|) \gg 1/\sqrt{|\dot\omega_k(t_c)|}$ is equivalent to $F_k^{(0)} \gg 1$.

If the condition $F_k^{(0)} \gg 1$ is not satisfied, then the use of the expressions discussed above is not justified. In the absence of any specific formula for the case $F_k^{(0)} = O(1)$, however, we will still truncate the WKB series at the value of n for which the first local minimum of $|W_{(2n)} - W_{(2n-2)}|$ is reached.

4 Adiabatic subtraction for a scalar field in an inflating Universe

After having seen general prescriptions for adiabatic subtraction, our goal is to apply these results to the case of a spatially flat (quasi) de Sitter Universe.

The action of a massive test scalar field ϕ on a spatially flat FLRW background is given by

$$\mathcal{S}_\phi = \int \frac{d^3k}{(2\pi)^3}\, dt\, a^3 \left(\frac{1}{2}|\dot\phi(k,\, t)|^2 - \frac{k^2}{2\, a^2}\, |\phi(k,\, t)|^2 - \frac{m^2}{2}\, |\phi(k,\, t)|^2\right). \tag{4.1}$$

In order to work with real frequencies at all times, we define a new time variable (this definition of time has been used in the past, see for instance [7, 11])

$$d\theta = \frac{dt}{a(t)^3}\,, \tag{4.2}$$

such that the action for $\phi(k,\, t)$ reads

$$\mathcal{S}_\phi = \int \frac{d^3k}{(2\pi)^3}\, d\theta \left(\frac{1}{2}\left|\frac{d\phi(k,\, \theta)}{d\theta}\right|^2 - \frac{1}{2}\left(k^2\, a^4 + m^2\, a^6\right) |\phi(k,\, \theta)|^2\right). \tag{4.3}$$

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The field $\phi(k,\,\theta)$ is thus already canonically normalized, and the frequency of the mode with momentum k is given by $\omega_k = \sqrt{k^2\,a^4 + m^2\,a^6}$, that is positive definite.

Let us now consider different regimes, starting from the simplest case of a massless field on an exact de Sitter space in flat slicing.

4.1 A massless scalar on exact de Sitter space in flat slicing

In this case $\omega_k = k\,a^2$ with $a(t) = e^{Ht}$, or, using

$$d\theta = \frac{dt}{a(t)^3} \Rightarrow \theta = -\frac{e^{-3Ht}}{3\,H}\,, \qquad\qquad -\infty < \theta < 0\,,$$
$$a(\theta) = (-3H\theta)^{-1/3}\,, \qquad\qquad \omega_k(\theta) = \frac{k}{(-3H\theta)^{2/3}}\,, \tag{4.4}$$

where we have set $\theta = -1/3H$ at the end of inflation, $t = 0$, $a = 1$.

The equation of motion for mode functions,

$$\frac{d^2\phi(k,\,\theta)}{d\theta^2} + \frac{k^2}{(-3H\theta)^{4/3}}\,\phi(k,\,\theta) = 0\,, \tag{4.5}$$

with positive frequency only solutions as $\theta \to -\infty$, reads

$$\begin{aligned}\phi(k,\,\theta) &= \frac{H}{\sqrt{2k^3}}\left(1 - i\,\frac{k}{H}\,(-3H\theta)^{1/3}\right)e^{i\,\frac{k}{H}\,(-3H\theta)^{1/3}}\\ &= \frac{H}{\sqrt{2k^3}}\left(1 - i\,\frac{k}{a\,H}\right)e^{i\,\frac{k}{a\,H}}\,,\end{aligned} \tag{4.6}$$

which is the well-known expression for the mode functions of a massless scalar on a de Sitter space with flat slices.

Remarkably, this solution can be written in the form

$$\phi(k,\,\theta) = \frac{e^{-i\int^{\theta} W(k,\,\theta')\,d\theta'}}{\sqrt{2W(k,\,\theta)}}\,, \tag{4.7}$$

if we choose

$$W(k,\,\theta) = \frac{k^3\,a(\theta)^2}{k^2 + H^2\,a(\theta)^2}\,. \tag{4.8}$$

Since, in the case of a massless scalar on de Sitter space, the mode functions can be written in the form (4.7), the WKB series can be resummed exactly, which implies that the Bogolyubov coefficient $\beta(k,\,t)$ vanishes. One could reach an analogous conclusion by observing that by moving along the real negative θ-axis, we encounter no Stokes lines in the complex-θ plane, as $\omega(\theta)$ has no zeros at finite θ.

We thus conclude that the prescription we are examining in this paper implies *no particle production* for a massless scalar in exact de Sitter space in flat slicing.

Let us compare our result with that of the prescription of [5, 6]. In this case, one is instructed to truncate the WKB series to second order,

$$W_{(2)}(k,\,\theta) = k\,a(\theta)^2\left(1 - \frac{H^2\,a(\theta)^2}{k^2}\right)\,, \tag{4.9}$$

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so that applying eq. (2.22) with $n = 1$ and using eq. (4.6) we obtain

$$\mathcal{P}_k^\phi(\theta) = \frac{H^2}{4\pi^2}\left[\left(1+\frac{k^2}{a(\theta)^2\,H^2}\right)-\left(\frac{k^2}{a(\theta)^2\,H^2}\,\frac{1}{1-\frac{a(\theta)^2\,H^2}{k^2}}\right)\right], \tag{4.10}$$

where the term in the first $(\dots)$ is proportional to $|\phi_k(\theta)|^2$ whereas the second term is proportional to $1/W_{(2n)}(k,\,\theta)$. If, in the spirit of the adiabatic approximation, we expand the second term to $O(k^0)$, we obtain that the prescription [5–7] leads to $\mathcal{P}_k^\phi(\theta) = \mathcal{O}(k^{-2})$, which agrees with the result obtained by setting $m = 0$ in [8] and, more importantly, which is the same result we obtained above using the prescription of [3, 4].

The vanishing of the power spectrum of a massless, minimally coupled scalar on de Sitter space with flat slices deserves a couple of comments. First off, it is in agreement with results from earlier literature. For instance, the formulae in [28], when applied to this case, would also give a vanishing spectrum. Second, its disagreement with the usual expectation $\langle\phi(\mathbf{x},\,t)\rangle \propto H^3\,t$ can be traced back to the absence of an infrared cutoff in de Sitter space *in flat slicing*. The usual result $\langle\phi(\mathbf{x},\,t)\rangle \propto H^3\,t$ can be indeed recovered in closed de Sitter space, that has a natural infrared cutoff given by the wavelength of the modes that left the horizon when the scale factor of the Universe was at its minimum [28, 29].

Finally, let us note that, even if they end up giving the same result, the (vanishing) power spectra obtained by using the two prescriptions are qualitatively different. In the case of the standard prescription, the final result is obtained from a truncation followed by a Taylor expansion of the series in aH/k, which is therefore reliable only in the limit $k \gg aH$. In the second case the vanishing renormalized spectrum is the result of a resummation of the entire series. As we will see in the next example, the two prescriptions will give, in general, different results.

4.2 A massive scalar on exact de Sitter space in flat slicing

In the case $m \neq 0$, with a de Sitter background, one can still find an exact solution to the mode equation,

$$\phi(k,\,\theta) = \sqrt{\frac{\pi}{4\,H\,a(\theta)^3}}\,e^{i\nu\pi/2+i\pi/4}\,H_\nu^{(1)}\left(\frac{k}{a(\theta)\,H}\right), \qquad \nu \equiv \sqrt{\frac{9}{4}-\frac{m^2}{H^2}}, \quad m < \frac{3}{2}H, \tag{4.11}$$

which, however, cannot be written in the form (4.7) for any function $W(k,\,\theta)$, as we will now discuss.

First, unlike the massless case, for $m \neq 0$ the frequency $\omega_k = \sqrt{k^2\,a(\theta)^4 + m^2\,a(\theta)^6}$ does have zeros at finite values of θ given by

$$\theta_\pm = \pm\frac{i}{3H}\frac{m^3}{k^3}, \tag{4.12}$$

from which Stokes lines emanate. One of those lines crosses the real θ axis at a point $\theta_{\text{Stokes, Real}}$ determined by solving the equation

$$i\int_{\theta_+}^{\theta_{\text{Stokes, Real}}}\sqrt{k^2\,a(\theta)^4 + m^2\,a(\theta)^6}\,d\theta = \text{real}, \tag{4.13}$$

i.e., changing variable back to physical time t and integrating,

$$i\frac{m}{H}\left[\log\left(e^y+\sqrt{e^{2y}+1}\right)-\sqrt{1+e^{-2y}}\right]_{y=Ht_{\text{Stokes, Real}}-\log(k/m)}-\frac{\pi\,m}{2\,H}=\text{Real}\,, \tag{4.14}$$

so that $t_{\text{Stokes, Real}}$ can be found by solving numerically the equation $\log\left(e^y+\sqrt{e^{2y}+1}\right)-\sqrt{1+e^{-2y}}=0$, yielding

$$Ht_{\text{Stokes, Real}}\simeq .411+\log(k/m)\,. \tag{4.15}$$

As a consequence, particle creation happens approximately when the scale factor e^{Ht} crosses k/m, a consequence of the de Sitter symmetry $t\to t+\Delta t$, $k\to k\,e^{-H\,\Delta t}$.

The existence of a Stokes line shows that the WKB series is asymptotic. The singulant reads

$$F_k^{(0)}=i\int_{\theta_-}^{\theta_+}\sqrt{k^2\,a(\theta)^4+m^2\,a(\theta)^6}\,d\theta=i\int_{[\log(k/m)+i\pi/2]/H}^{[\log(k/m)-i\pi/2]/H}\sqrt{k^2\,e^{-2Ht}+m^2}\,dt=\pi\frac{m}{H}\,, \tag{4.16}$$

which means that the WKB approximation will be a good one for $m\gg H$. In this case, eq. (3.6) gives $|\beta(k,\,t)|^2\propto e^{-2\pi m/H}$ in agreement with derivations of the rate of creation of heavy particles based on Schwarzschild-de Sitter metric, such as that in [30].

A second, more pedestrian way to see that the series is asymptotic is to solve the eq. (2.16) by brute force, at least for the first few terms. By using an algebraic manipulation program we obtain

$$\begin{aligned}W(k,t)\sim ka^2\Bigg[&\left(1+\frac{1}{2}\frac{m^2a^2}{k^2}+O(m^4)\right)+\frac{a^2H^2}{k^2}\left(-1-\frac{1}{4}\frac{m^2a^2}{k^2}+O(m^4)\right)\\&+\frac{a^4H^4}{k^4}\left(1-\frac{5}{2}\frac{m^2a^2}{k^2}+O(m^4)\right)+\frac{a^6H^6}{k^6}\left(-1+\frac{217}{8}\frac{m^2a^2}{k^2}+O(m^4)\right)\\&+\frac{a^8H^8}{k^8}\left(1-\frac{3249}{8}\frac{m^2a^2}{k^2}+O(m^4)\right)+\frac{a^{10}H^{10}}{k^{10}}\left(-1+\frac{39523}{4}\frac{m^2a^2}{k^2}+O(m^4)\right)+\ldots\Bigg]\,,\end{aligned} \tag{4.17}$$

which shows that, while the $O(m^0)$ terms can be resummed to give the massless result (4.8), the coefficients of the $O(m^2)$ terms are rapidly increasing, signaling the asymptotic nature of the WKB series in the massive case.

The case $m\ll H$ is the one of greatest phenomenological interest. In this case, the formulae of [21, 22] are not strictly speaking valid, as we discussed at the end of section 3 above, and we will simply truncate the WKB series where $|W_{(2n+2)}-W_{(2n)}|$ displays a minimum. In particular, we have

$$\begin{aligned}W_{(0)}(k,\,t)&=a(t)^3\sqrt{q(t)^2+m^2}\,,\\W_{(2)}(k,\,t)-W_{(0)}(k,\,t)&=-a(t)^3\,H^2\,\frac{9\,m^4+22\,m^2\,q(t)^2+8\,q(t)^4}{8\,\left(q(t)^2+m^2\right)^{5/2}}\,,\qquad q(t)\equiv\frac{k}{a(t)}\,,\end{aligned} \tag{4.18}$$

so that both for $q(t)\ll m\ll H$ and for $m\ll q(t)\lesssim H$ we have $|W_{(0)}(k,\,t)|\ll|W_{(2)}(k,\,t)-W_{(0)}(k,\,t)|$. As a consequence, for $m\ll H$ and $k\lesssim a\,H$ we will only keep the zeroth order of the WKB series. This gives the power spectrum

$$\mathcal{P}_k^\phi=\frac{k^3}{2\pi^2}\left[\frac{\pi}{4Ha^3}\left|H_\nu^{(1)}\left(\frac{k}{aH}\right)\right|^2-\frac{1}{2\sqrt{k^2a^4+m^2a^6}}\right]\,,\qquad m\ll H\,,\quad k\lesssim aH\,. \tag{4.19}$$

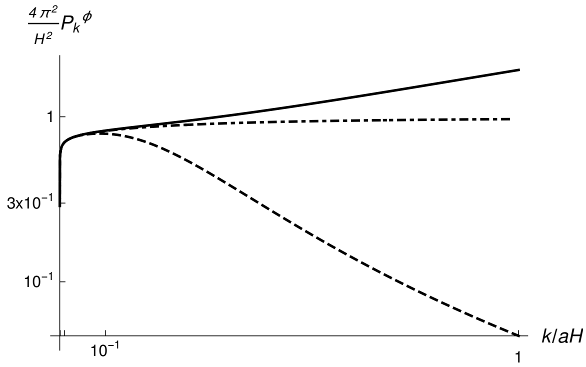

Figure 3. The power spectrum of a scalar field of mass $m^2 = .1H^2$ on de Sitter space. The solid line represents the non-renormalized result. The dashed line shows the renormalized result obtained by using the standard prescription of adiabatic subtraction, i.e. by removing terms up to the second adiabatic order. The dot dashed line represents the renormalized result obtained with the method of the optimal truncation, i.e. removing, for superhorizon modes, only the zeroth order of the WKB series.

In figure 3 we show the non-renormalized power spectrum and the two versions of the normalized one obtained by using the two different prescritions. As we can see the use of the prescription [3, 4] eliminates the first of the "undesirable features" listed in the Introduction, namely the fact that the standard prescription for adiabatic subtraction leads to a significant running of the renormalized power spectrum for $m \ll k/a \lesssim H$. With the method of optimal truncation on the other hand we get an almost constant spectrum.

4.3 A massless scalar in quasi-de Sitter, followed by radiation domination

We have shown in section 4.1 above that, by applying the prescription of [5, 6], the power spectrum of a massless scalar in exact 3+1-dimensional de Sitter space is identically vanishing. It is however easy to see that super-horizon modes for which $k \ll a\,H$ are not evolving adiabatically in this system, since

$$\frac{1}{\omega^2}\left|\frac{d\omega}{d\theta}\right| = 2\,\frac{aH}{k} \gg 1\,, \qquad \text{for } k \ll a\,H\,. \tag{4.20}$$

It is therefore more interesting to consider the situation where at late times all modes re-enter the horizon and are adiabatically evolving. Such a situation can be realized by considering a background metric where a quasi-de Sitter phase is followed by a radiation dominated Universe.

To study such a system, we consider a massless scalar field on the top of an FLRW Universe whose scale factor evolution is given by

$$a(t) = 2\,e^{Ht}\frac{(1+H^2t^2)^{1/4}}{e^{Ht}+2\,(1+H^2t^2)^{1/4}}\,, \tag{4.21}$$

which gives a de Sitter metric $a(t) \sim e^{Ht}$ for $t \ll -1/H$ and radiation dominated cosmology $a(t) \sim 2\sqrt{Ht}$ for $t \gg 1/H$. This choice of the form of $a(t)$ also leads to $\dot{H} < 0$ at all times, as required by energy conditions.

For a scale factor given by eq. (4.21), the proper frequency $\omega_k(\theta) = k\, a^2(\theta)$ has complex zeros at $\theta = \theta_c$ and $\theta = \theta_c^*$, corresponding to $Ht = \pm i$, and we get

$$F_k^{(0)} = -i \int_{\theta_c}^{\theta_c^*} \omega_k(\theta)\, d\theta = -i\, k \int_{-i}^{i} a(t)^2 \, \frac{dt}{a(t)^3} \simeq 2.88\, \frac{k}{H}\,. \tag{4.22}$$

Numerical evaluation shows that the Stokes line from θ_c to θ_c^* crosses the real axis at a value of θ corresponding to $t \simeq .34\, H^{-1}$, which can be identified as the time at which production of quanta of ϕ occurs.

We have computed numerically the spectrum of ϕ as a function of time. The bare spectrum is shown, for $t = 0$ and $t = 100\, H^{-1}$ in figure 4. Note that, as expected, the power spectrum converges to $(H/2\pi)^2$ in the limit of long wavelengths, while it goes as k^2/a^2 at short wavelengths.

In figure 5 we show, instead, the power spectrum for the field ϕ obtained by subtracting a regularized optimally truncated version of $W(t)$.

The regularization is built as follows. We define the functions $\tilde{W}_{(2i)}(t)$ through

$$W_{(2n)}(t) = k \left[\tilde{W}_{(0)}(t) + \frac{\tilde{W}_{(2)}(t)}{k^2} + \frac{\tilde{W}_{(4)}(t)}{k^4} + \ldots + \frac{\tilde{W}_{(2n)}(t)}{k^{2n}}\right], \tag{4.23}$$

where $\tilde{W}_{(0)}(t) = a(t)^2$. Next, we define weighted averages of $|\tilde{W}_{(2i)}(t)|$ as

$$\hat{W}_{(2i)}(t) = \int_{-\infty}^{\infty} |\tilde{W}_{(2i)}(t + t_1)|\, e^{-H^2 t_1^2}\, dt_1\,, \tag{4.24}$$

and the regularized Heaviside Θ functions as

$$\Theta_Q(t;\, 2m,\, 2n) = \frac{1}{2}\left[1 + \tanh\left(2Q \frac{\hat{W}_{(2m)}(t) - \hat{W}_{(2n)}(t)}{\hat{W}_{(2m)}(t) + \hat{W}_{(2n)}(t)}\right)\right], \tag{4.25}$$

that converge to the Heaviside step function for $Q \to \infty$. In our numerical evaluation we set $Q = 5$.

Finally, the regularized optimal truncation of $W(t)$ is obtained as

$$\begin{aligned} W_{\rm reg}^{\rm optimal}(t) = {}& k\, \tilde{W}_{(0)}(t)\, \Theta_5(t;2,0) + k\left(\tilde{W}_{(0)}(t) + \frac{\tilde{W}_{(2)}(t)}{k^2}\right)\Theta_5(t;0,2)\,\Theta_5(t;4,2) \\ & + k\left(\tilde{W}_{(0)}(t) + \frac{\tilde{W}_{(2)}(t)}{k^2} + \frac{\tilde{W}_{(4)}(t)}{k^4}\right)\Theta_5(t;0,2)\,\Theta_5(t;2,4)\,\Theta_5(t;6,4) + \ldots \end{aligned} \tag{4.26}$$

so that $W_{\rm reg}^{\rm optimal}(t) = k\,\tilde{W}_{(0)}(t)$ for $\hat{W}_{(2)}(t) \gg \hat{W}_{(0)}(t)$, and $W_{\rm reg}^{\rm optimal}(t) = k\left(\tilde{W}_{(0)}(t) + \frac{\tilde{W}_{(2)}(t)}{k^2}\right)$ for $\hat{W}_{(0)}(t) \gg \hat{W}_{(2)}(t)$ and $\hat{W}_{(2)}(t) \ll \hat{W}_{(4)}(t)$ (i.e., $\hat{W}_{(2)}(t)$ is a local minimum of the $\hat{W}_{(2i)}(t)$), etc...

The weighted average $\hat{W}_{(2i)}(t)$ is introduced to eliminate spurious effects originating from the fact that the functions $\tilde{W}_{(2i)}(t)$ are generally oscillating, and therefore cross zero and appear to be small even if they have a large amplitude, for the relevant values of time t. The regularized Θ function is used to lead to a smooth spectrum.

As figure 5 shows, the use of this regularized optimal truncation eliminates the singularity in the power spectrum around $k \simeq .5\, H$ that emerges when one uses the second

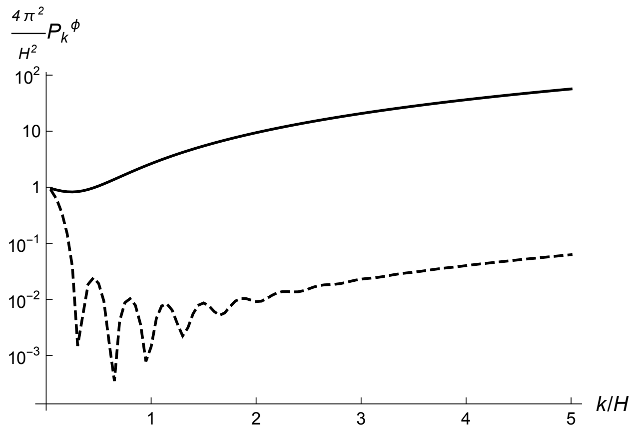

Figure 4. The unsubtracted power spectrum (in units of $(H/2\pi)^2$) of a massless scalar in an expanding Universe with the expression of the scale factor given by eq. (4.21). The spectrum is evaluated at $t = 0$ (solid line) and at $t = 100\,H^{-1}$ (dashed). Modes with $k \gtrsim .9\,H$ satisfy $k/a > \dot{a}/a$ for the entire evolution, so that they never cross the horizon. For those modes the power spectrum goes as k^2/a^2.

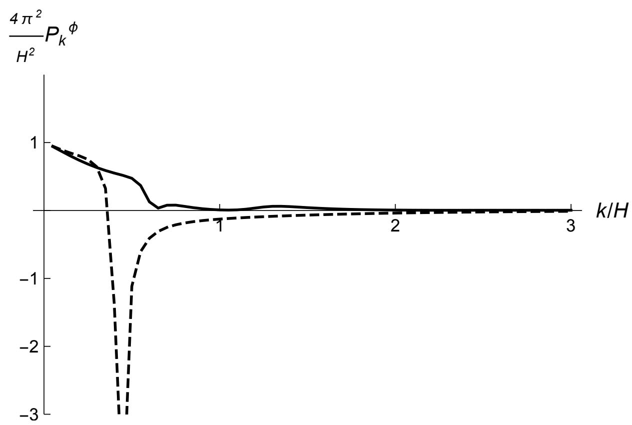

Figure 5. The subtracted power spectra (in units of $(H/2\pi)^2$) of a massless scalar in an expanding Universe with the expression of the scale factor given by eq. (4.21). The spectra are evaluated at $t = 0$. Solid: the spectrum obtained by subtracting a regularized version (see the main text leading to eq. (4.26)) of the optimally truncated expression for $W(t)$. Dashed: the spectrum obtained by subtracting $W_{(2)}(t)$.

order adiabatic subtraction, which originates from the fact that for that value of k one has $W_{(2)}(k,\, t = 0) \simeq 0$.

On the other hand, the expression for $W_{\rm reg}^{\rm optimal}(t)$ rapidly converges to $W_{(2)}(k,\, t)$ as t grows. In figure 6 we show the subtracted spectra for $t = 100\,H^{-1}$. However, already at $t = 5\,H^{-1}$ the two subtracted spectra are indistinguishable.

5 Conclusions and discussion

In this work we have revisited the method of adiabatic subtraction for the renormalization of the power spectrum of a massless and light scalar field in (quasi) de Sitter space, in flat slicing. First, we have reviewed the equivalence between adiabatic subtraction and normal ordering

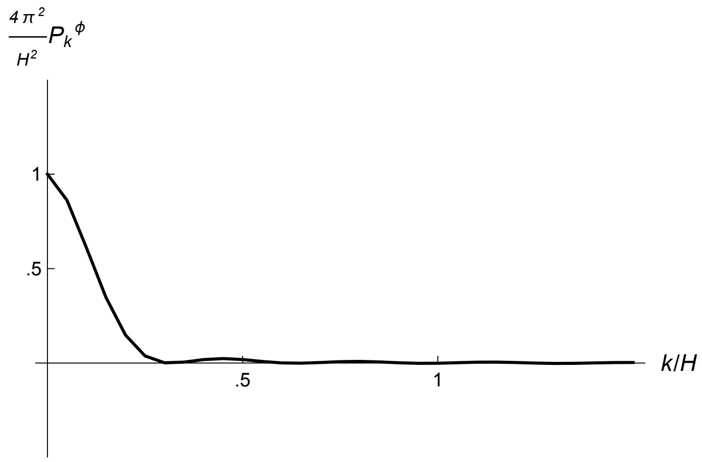

Figure 6. The subtracted power spectra (in units of $(H/2\pi)^2$) of a massless scalar in an expanding Universe with the expression of the scale factor given by eq. (4.21). The specta are evaluated at $t = 100\,H^{-1}$. The spectrum obtained by subtracting $W_{\rm reg}^{\rm optimal}(t)$ and that obtained by subtracting $W_{(2)}(t)$ are, for this large value of t, numerically equivalent.

of the time-dependent creation/annihilation operators that instantaneously diagonalize the Hamiltonian. This requires the proper frequency of the mode functions to be real at all times, and we have found a definition of time that guarantees this condition to be satisfied.

The main question we have tackled is to what order one should truncate the WKB expansion of the adiabatic modes that have to be subtracted to provide the renormalized result. The standard prescription [1, 2, 5–7] to truncate the WKB series to the lowest order that allows to cancel all divergences, while having the advantage of leading to (relatively) simple calculations, can generate artifacts at intermediate momenta [8]. An alternative option [3, 4] is based on the fact that the WKB approximation is generally an asymptotic one, which naturally results into truncating the WKB series to the value that gives the closest approximation to the "actual expression" of the Bogolyubov coefficient β, that has been argued to take a universal form [20–22].

We applied these prescriptions to the massless and massive minimally coupled scalar field on exact de Sitter space in flat slicing, and we have compared the resulting power spectra. In the massless case, the fact the WKB series can be resummed implies that we do not have particle production, and the renormalized power spectra turn out to vanish using both prescriptions. This result, in line with those in [6, 28] is due to the fact that de Sitter space in flat slicing, unlike de Sitter space in closed slicing, does not have a built-in infrared cutoff, which is responsible for the growth $\langle\phi(\mathbf{x},\, t)^2\rangle \propto H^3\, t$.

In the massive case the results are quite different: optimal truncation requires, for super-horizon modes $k \lesssim a\,H$, to remove only the zeroth order WKB contribution, leading to the standard quasi scale-invariant spectrum, while the usual adiabatic renormalization removes terms up to the second adiabatic order and leads to a significant running for scales $m \ll k/a \lesssim H$. Subsequently, we applied our method to a more complicated system, a massless scalar field which undergoes a phase of slow roll inflation followed by a radiation dominated era. In this case, even though the field is massless, the power spectrum is non vanishing and takes the standard expression $\sim (H/2\pi)^2$ at large scales, while at shorter scales the prescription based on the optimal truncation of the WKB series eliminates some of the artifacts that are generated by the standard second order truncation.

The prescription to truncate the WKB expansion at its optimal order is justified by the consideration that the process of particle production can be identified with the Stokes phenomenon. The connection formula that extends the WKB approximation to the whole complex plane is referring to asymptotic series truncated at their least terms. More recently, it has been found that truncating the WKB series at the optimal order, *as long as such an optimal order is much larger than* $O(1)$, leads to a sum that approximates well the universal behavior across the Stokes line found by Dingle and Berry [20–22]. This theory is effective when the regions of non-adiabaticity in the complex plane are small enough. In this work, we used the optimal truncation of the WKB series even when it happens at a low order, i.e., when the thickness of the Stokes lines becomes comparable with their length or when two regions of non-adiabaticity overlap. It would be interesting to see whether this choice is justified by an argument analogous to that of [20–22].

Acknowledgments

We thank Paolo Creminelli, John Donoghue, David Kastor, Mehrdad Mirbabayi, Marco Peloso, Borna Salehian, Luca Santoni, Gianmassimo Tasinato and Jennie Traschen for useful discussions. A special thanks goes to Gerald Dunne for his patient explanations. This work is partially supported by the US-NSF grants PHY-1820675 and PHY-2112800.

References

[1] L. Parker, *Quantized fields and particle creation in expanding universes. 1.*, *Phys. Rev.* **183** (1969) 1057 [INSPIRE].

[2] L. Parker, *Quantized fields and particle creation in expanding universes.* 2, *Phys. Rev. D* **3** (1971) 346 [*Erratum ibid.* **3** (1971) 2546] [INSPIRE].

[3] R. Dabrowski and G.V. Dunne, *Superadiabatic particle number in Schwinger and de Sitter particle production*, *Phys. Rev. D* **90** (2014) 025021 [arXiv:1405.0302] [INSPIRE].

[4] R. Dabrowski and G.V. Dunne, *Time dependence of adiabatic particle number*, *Phys. Rev. D* **94** (2016) 065005 [arXiv:1606.00902] [INSPIRE].

[5] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge, U.K. (1984) [DOI:10.1017/CBO9780511622632] [INSPIRE].

[6] L.E. Parker and D. Toms, *Quantum field theory in curved spacetime: quantized field and gravity*, Cambridge University Press, Cambridge, U.K. (2009) [DOI:10.1017/CBO9780511813924] [INSPIRE].

[7] L. Parker and S.A. Fulling, *Adiabatic regularization of the energy momentum tensor of a quantized field in homogeneous spaces*, *Phys. Rev. D* **9** (1974) 341 [INSPIRE].

[8] L. Parker, *Amplitude of perturbations from inflation*, hep-th/0702216 [INSPIRE].

[9] I. Agullo, J. Navarro-Salas, G.J. Olmo and L. Parker, *Revising the predictions of inflation for the cosmic microwave background anisotropies*, *Phys. Rev. Lett.* **103** (2009) 061301 [arXiv:0901.0439] [INSPIRE].

[10] I. Agullo, J. Navarro-Salas, G.J. Olmo and L. Parker, *Reexamination of the power spectrum in de Sitter inflation*, *Phys. Rev. Lett.* **101** (2008) 171301 [arXiv:0806.0034] [INSPIRE].

[11] M.M. Glenz and L. Parker, *Study of the spectrum of inflaton perturbations*, *Phys. Rev. D* **80** (2009) 063534 [arXiv:0905.2624] [INSPIRE].

[12] I. Agullo, J. Navarro-Salas, G.J. Olmo and L. Parker, *Revising the observable consequences of slow-roll inflation*, *Phys. Rev. D* **81** (2010) 043514 [arXiv:0911.0961] [INSPIRE].

[13] F. Finelli, G. Marozzi, G.P. Vacca and G. Venturi, *The impact of ultraviolet regularization on the spectrum of curvature perturbations during inflation*, *Phys. Rev. D* **76** (2007) 103528 [arXiv:0707.1416] [INSPIRE].

[14] Y. Urakawa and A.A. Starobinsky, *Adiabatic regularization of primordial perturbations generated during inflation*, in the proceedings of the 19$^{\rm th}$ *workshop on general relativity and gravitation in Japan*, (2009) [INSPIRE].

[15] R. Durrer, G. Marozzi and M. Rinaldi, *On adiabatic renormalization of inflationary perturbations*, *Phys. Rev. D* **80** (2009) 065024 [arXiv:0906.4772] [INSPIRE].

[16] G. Marozzi, M. Rinaldi and R. Durrer, *On infrared and ultraviolet divergences of cosmological perturbations*, *Phys. Rev. D* **83** (2011) 105017 [arXiv:1102.2206] [INSPIRE].

[17] I. Agullo, J. Navarro-Salas, G.J. Olmo and L. Parker, *Remarks on the renormalization of primordial cosmological perturbations*, *Phys. Rev. D* **84** (2011) 107304 [arXiv:1108.0949] [INSPIRE].

[18] T. Markkanen, *Renormalization of the inflationary perturbations revisited*, *JCAP* **05** (2018) 001 [arXiv:1712.02372] [INSPIRE].

[19] C. Animali, P. Conzinu and G. Marozzi, *On adiabatic renormalization with a physically motivated infrared cut-off*, *JCAP* **05** (2022) 026 [arXiv:2201.05602] [INSPIRE].

[20] R.B. Dingle, *Asymptotic expansions: their derivation and interpretation*, Academic Press, London, U.K. (1973).

[21] M.V. Barry, *Uniform asymptotic smoothing of Stokes's discontinuities*, *Proc. Roy. Soc. Lond. A* **422** (1989) 7 [INSPIRE].

[22] M.V. Berry, *Waves near Stokes lines*, *Proc. Roy. Soc. Lond. A* **427** (1990) 265.

[23] C.K. Dumlu and G.V. Dunne, *The Stokes phenomenon and Schwinger vacuum pair production in time-dependent laser pulses*, *Phys. Rev. Lett.* **104** (2010) 250402 [arXiv:1004.2509] [INSPIRE].

[24] A.A. Grib and S.G. Mamaev, *On field theory in the friedman space*, *Yad. Fiz.* **10** (1969) 1276 [INSPIRE].

[25] A.A. Grib, S.G. Mamaev and V.M. Mostepanenko, *Particle creation from vacuum in homogeneous isotropic models of the universe*, *Gen. Rel. Grav.* **7** (1976) 535 [INSPIRE].

[26] J.F. Dufaux et al., *Preheating with trilinear interactions: tachyonic resonance*, *JCAP* **07** (2006) 006 [hep-ph/0602144] [INSPIRE].

[27] G.G. Stokes, *On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series: a letter to the editor*, *Acta Math.* **26** (1902) 393.

[28] S. Habib, C. Molina-Paris and E. Mottola, *Energy momentum tensor of particles created in an expanding universe*, *Phys. Rev. D* **61** (2000) 024010 [gr-qc/9906120] [INSPIRE].

[29] B. Allen and A. Folacci, *The massless minimally coupled scalar field in de Sitter space*, *Phys. Rev. D* **35** (1987) 3771 [INSPIRE].

[30] G.W. Gibbons and S.W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, *Phys. Rev. D* **15** (1977) 2738 [INSPIRE].

Journal of Cosmology
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PAPER

Correlated scalar perturbations and gravitational waves from axion inflation

To cite this article: Sofia P. Corbà and Lorenzo Sorbo JCAP10(2024)024

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Received: March 21, 2024
Revised: August 7, 2024
Accepted: September 20, 2024
Published: October 8, 2024

Correlated scalar perturbations and gravitational waves from axion inflation

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Abstract: The scalar and tensor fluctuations generated during inflation can be correlated, if arising from the same underlying mechanism. In this paper we investigate such correlation in the model of axion inflation, where the rolling inflaton produces quanta of a U(1) gauge field which, in turn, source scalar and tensor fluctuations. We compute the primordial correlator of the curvature perturbation, ζ, with the gravitational energy density, Ω_{GW}, at frequencies probed by gravitational wave detectors. This two-point function receives two contributions: one arising from the correlation of gravitational waves with the scalar perturbations generated by the standard mechanism of amplification of vacuum fluctuations, and the other coming from the correlation of gravitational waves with the scalar perturbations sourced by the gauge field. Our analysis shows that the former effect is generally dominant. For typical values of the parameters, the correlator, normalized by the amplitude of ζ and by the fractional energy in gravitational waves at interferometer frequencies, turns out to be of the order of $10^{-4} \div 10^{-2}$.

Keywords: axions, Inflation and CMBR theory, physics of the early universe, primordial gravitational waves (theory)

ArXiv ePrint: 2403.03338

https://doi.org/10.1088/1475-7516/2024/10/024

JCAP10(2024)024

Contents

1 Introduction

The theory of inflation constitutes the dominant paradigm of primordial cosmology. Besides solving the most important problems of the standard Hot Big Bang model, it is able to provide an explanation, in excellent agreement with observations, for the origin of the temperature anisotropies present in the Cosmic Microwave Background (CMB) radiation and of the density fluctuations that characterize the large scale structure of the Universe. Among the many different inflationary scenarios, axion inflation is one of those giving a satisfying solution to the problem of UV sensitivity of the inflaton potential. In this model, proposed for the first time in 1990 as natural inflation [1], the inflaton is a pseudo-Nambu-Goldstone Boson that enjoys a (softly broken) shift symmetry, i.e., a symmetry under the transformation $\phi \to \phi + \text{const}$, which protects its potential against large radiative corrections.

The axionic inflaton is naturally coupled to gauge fields through the operator $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}/f$, where f is the axion decay constant. In the presence of such coupling, the rolling zero mode of the inflaton acts as a source for the modes of the gauge field. As a result, quanta of the gauge field are amplified into classical modes, which in turn source, through a process of inverse decay, both scalar and tensor fluctuations. Since, due to the pseudoscalar nature of the inflaton, only one of the two helicities of the gauge field experiences a tachyonic instability, the spectra of the tensor modes of different helicities have different amplitudes. This scenario has multiple phenomenological predictions, including nongaussianities [2], deviations from scale invariance [3], formation of a population of primordial black holes [4], generation of primordial chiral gravitational waves at CMB [5] or interferometer [6] frequencies, baryogenesis [7], as well as the possible generation of cosmologically relevant magnetic fields [8, 9] — see [10] for a review.

By comparing these phenomenological predictions with observations we can constrain the relevant parameters characterizing the models of axion inflation. More specifically, there are two significant observational lengthscales. At large scales, probed by CMB measurements, the primary constraint arises from the non-observation of primordial nongaussianities for the scalar fluctuations. In axion inflation the sourced scalar fluctuations are highly nongaussian.

Consequently, the model can be viable only if the sourced component of scalar modes is subdominant compared to that generated by the standard amplification of vacuum fluctuations. This is equivalent to stating that the amplitude of the gauge field, which sources the scalar and tensor fluctuations, must be relatively small. Therefore, the sourced component of tensor fluctuations is also small at this stage.

At smaller scales, corresponding to modes that left the horizon closer to the end of inflation, the situation becomes more interesting. For simple inflationary potentials, the inflaton's velocity increases as inflation progresses and therefore the population of gauge quanta, whose amplitude depends exponentially on the inflaton's velocity, becomes more sizable towards the end of inflation. As a consequence, sourced gravitational waves of shorter wavelengths, which are remarkably those probed by gravitational wave experiments, can have a much larger amplitude and might even be directly detectable [6] by a variety of observatories. Also in this regime we need the scalar fluctuations to remain bounded to avoid an overproduction of primordial black holes [10, 11].

A natural follow-up to the recent observational evidence [12–14] of a stochastic gravitational wave background (SGWB) is the search for anisotropies, in analogy to the scalar anisotropies observed in the CMB (see, e.g., [15] for a recent analysis of LIGO/Virgo/KAGRA and [16] for LISA's reach in this respect). Study of these anisotropies can allow us to distinguish between the astrophysical and cosmological origin of the SGWB. Furthermore, cosmological tensor anisotropies may be correlated with the scalar anisotropies of the CMB if they arise from the same underlying mechanisms [17]. Exploring such correlations can give important information about the cosmological background of gravitational waves, thus providing insights about the physics of the Early Universe. Reference [18] performed a study of the statistics of these anisotropies while [19] studied the consequences of a non-trivial primordial scalar-tensor-tensor nongaussianity on the energy density of gravitational waves.

In this work we compute the correlation between the curvature perturbation $\zeta(\mathbf{x})$ and the energy density $\Omega_{GW}(\mathbf{x}) = \dot{h}_{ij}(\mathbf{x})\,\dot{h}_{ij}(\mathbf{x})/(12\,H_0^2)$ of the tensor modes within the framework of axion inflation. The computation is conducted at frequencies tested by gravitational detectors, and the correlator is normalized by both the square root of the scalar power spectrum and the average value of $\Omega_{GW}(\mathbf{x})$. The two point function receives two contributions, reflecting the fact that scalar fluctuations are generated both from the vacuum, through the standard amplification process, and by modes of the gauge field, through the inverse decay process. More specifically, we will study the two following situations:

- the rolling inflaton has fluctuations that are generated by the standard mechanism of amplification of vacuum fluctuations in an expanding Universe. The rolling inflaton then sources quanta of the gauge field, which in turn source gravitational waves. The fluctuations in the inflaton are thus imprinted in the fluctuations in the gravitational waves. We study this correlator in section 3.1;

- the rolling inflaton sources quanta of the gauge field, which in turn source *both* scalar fluctuations and gravitational waves. Since these modes are produced by the same population of gauge modes, they are correlated. We study this correlator in section 3.2.

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As we will see, due to the smallness of the amplitude of the gauge field — and therefore, of the sourced scalar fluctuations and gravitational waves — at CMB scales, the former effect is generally dominant over the latter, and leads to a normalized correlator of the order of $10^{-4} \div 10^{-2}$.

The correlator studied in this work is the one between scalar perturbations at CMB scales, corresponding to modes that left the horizon early during inflation and gravitational waves at interferometer scales, which correspond to modes that left the horizon later during inflation. Even though these gravitational waves have relatively short (i.e., non cosmological) wavelengths, their *anisotropies* are at large, cosmological scales.

During the last stages of axion inflation the large amplitude acquired by the gauge modes implies that they can have strong backreaction effects on the inflating background. The nonperturbative inflaton-gauge field dynamics, studied in numerous papers including [20–30], is rich, complicated, and not yet fully understood. The production of gravitational waves, although generated during the phase of strong backreaction, is treated at the perturbative level. Reference [28] derived spectra of gravitational waves produced during this stage keeping into account the nonperturbative dynamics of the inflaton-gauge field system, even if it ignored inflaton inhomogeneities. Reference [31] performed an analogous study for the case of an SU(2) gauge sector. The results of [28] suggest that, even though strong backreaction effects complicate significantly the dynamics of the inflaton and of the gauge quanta, if the inflaton evolution $\phi(t)$ is known, then the resulting gravitational wave spectra reflect quite accurately the shape of the function $\dot{\phi}(t)$. For the scope of our calculation, since we will formulate our results in terms of $\dot{\phi}(t)$ without referring to the specific dynamics that led to that expression, our results should be valid even in the strong backreaction regime, at least as long as the inflaton inhomogeneities are ignored. Moreover, there are reasons to expect that our results will not change even once inflaton gradients are accounted for, since causality will prevent the late strong dynamics from affecting physics at scales that have left the horizon at much earlier times.

This paper is organized as follows. Section 2 contains a review of the amplification process that quanta of gauge field undergo as the inflaton rolls down its potential, together with the generation of curvature perturbations and of gravitational waves. Then, in section 3, we calculate the two contributions to the correlator between scalar fluctuations and the energy density of the gravitational waves: in subsection 3.1 we study the correlation of gravitational waves with the amplified vacuum scalar fluctuations and in subsection 3.2 the correlation of gravitational waves with sourced scalar fluctuations. In section 4 we discuss our results and we conclude. Appendix A contains the details of the calculation leading to the results in section 3.2.

2 Review of scalar and tensor perturbations from axion inflation

Our system consists of a pseudoscalar inflaton ϕ and a U(1) gauge field A_μ in interaction with each other and with gravity through the action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} - \frac{\phi}{8\,f}\frac{\epsilon^{\mu\nu\rho\lambda}}{\sqrt{-g}}F_{\mu\nu}\,F_{\rho\lambda}\right], \qquad (2.1)$$

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where $g = \det(g_{\mu\nu})$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, f is a constant with dimensions of mass, R is the Ricci scalar, and $\epsilon^{\mu\nu\rho\lambda}$ is the totally antisymmetric object defined by $\epsilon^{0123} = +1$. We will not make any assumption about the shape of the potential $V(\phi)$, other than it is flat enough to be able to support inflation.

Concerning the metric, we will assume that it is of the form of de Sitter space in flat slicing plus tensor perturbations (repeated latin indices are understood to be summed upon)

$$ds^2 = a^2(\tau) \left[-d\tau^2 + (\delta_{ij} + h_{ij}(\mathbf{x},\, \tau))\, dx^i\, dx^j\right] ,$$
$$a(\tau) = -\frac{1}{H\,\tau} , \qquad h_{ii} = \partial_i h_{ij} = 0 . \tag{2.2}$$

We perturb the inflaton as

$$\phi(\mathbf{x},\, \tau) \equiv \phi_0(\tau) + \delta\phi(\mathbf{x},\, \tau) , \tag{2.3}$$

so that the curvature perturbation is given by $\zeta \equiv -\frac{H}{\dot\phi_0}\delta\phi$. We will denote the derivative with respect to conformal time τ by a prime and that with respect to the cosmic time t, defined through $dt = a(\tau)\, d\tau$, by an overdot. We set the scale factor to be equal to unity at the end of inflation, i.e., inflation will end at $\tau_{\rm e} = -1/H$.

We treat the homogeneous inflaton $\phi_0(\tau)$ and the scale factor $a(\tau)$ as background quantities, and we work with the following canonically normalized perturbations

$$A_\mu(\mathbf{x},\, \tau) \qquad \text{with} \qquad A_0(\mathbf{x},\, \tau) = 0 , \quad \partial_i A_i(\mathbf{x},\, \tau) = 0 ,$$
$$\Phi(\mathbf{x},\, \tau) \equiv a(\tau)\, \delta\phi(\mathbf{x},\, \tau) ,$$
$$H_{ij}(\mathbf{x},\, \tau) \equiv \frac{M_P}{2}\, a(\tau)\, h_{ij}(\mathbf{x},\, \tau) . \tag{2.4}$$

Neglecting the mass of the inflaton, our perturbed Lagrangian takes the form

$$\begin{aligned}\mathcal{L} = &\left(\frac{1}{2}\Phi'^2 - \frac{1}{2}\partial_k\Phi\, \partial_k\Phi + \frac{a''}{2\,a}\Phi^2\right) + \left(\frac{1}{2}H'_{ij}\, H'_{ij} - \frac{1}{2}\partial_k H_{ij}\, \partial_k H_{ij} + \frac{a''}{2\,a}H_{ij}\, H_{ij}\right)\\ &+ \left(\frac{1}{2}A'_i\, A'_i - \frac{1}{2}\partial_k A_i\, \partial_k A_i - \frac{\dot\phi_0}{f}\epsilon^{ijk}\, A'_i\, \partial_j A_k\right)\\ &- \frac{H_{ij}}{a\, M_P}\left[A'_i\, A'_j - (\partial_i A_k - \partial_k A_i)\, (\partial_j A_k - \partial_k A_j)\right] - \frac{\Phi}{f\, a}\epsilon^{ijk}\, A'_i\, \partial_j A_k ,\end{aligned} \tag{2.5}$$

where the first line describes the free scalar and free tensor perturbations, the second line describes the free gauge field modes, and the last line contains the interactions that lead to processes of the form $A_i A_j \to H_{ij}$ and $A_i A_j \to \Phi$.

By varying the Lagrangian (2.5) with respect to Φ, H_{ij} and A_i, we obtain the equations of motion

$$\Phi'' - \frac{a''}{a}\Phi - \nabla^2\Phi + \frac{1}{f\, a}\epsilon^{ijk}\, A'_i\, \partial_j A_k = 0 , \tag{2.6}$$

$$H''_{ij} - \frac{a''}{a}H_{ij} - \nabla^2 H_{ij} + \frac{1}{a\, M_P}\left[A'_i\, A'_j - (\partial_i A_k - \partial_k A_i)\, (\partial_j A_k - \partial_k A_j)\right] = 0 , \tag{2.7}$$

$$A''_i - \nabla^2 A_i - \frac{\phi'_0}{f}\epsilon^{ijk}\, \partial_j A_k = 0 . \tag{2.8}$$

The solution of eq. (2.6) splits into two parts: the solution of the homogeneous equation, denoted as $\Phi_{\rm V}$, and the particular solution, denoted as $\Phi_{\rm S}$. The solution of the homogeneous equation represents the usual vacuum fluctuations generated during inflation due to the accelerated expansion of the background, while the particular solution is induced by the inverse decay of the gauge fields. The homogeneous solution can be quantized through the standard quantization of the free Lagrangian, using the first line of eq. (2.5), as

$$\begin{aligned}\Phi_{\rm V}(\mathbf{x},\,\tau) &= \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}} \left[\Phi_{\rm V}(k,\,\tau)\,\hat{a}(\mathbf{k}) + \Phi_{\rm V}^*(k,\,\tau)\,\hat{a}^\dagger(-\mathbf{k})\right]\,,\\ \Phi_{\rm V}(k,\,\tau) &\equiv \frac{1}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)\,e^{-ik\tau}\,,\end{aligned} \tag{2.9}$$

where the creation/annihilation operators $\hat{a}^\dagger(\mathbf{k})/\hat{a}(\mathbf{k})$ satisfy the usual commutation relations $\left[\hat{a}(\mathbf{k}),\,\hat{a}^\dagger(\mathbf{q})\right] = \delta(\mathbf{k}-\mathbf{q})$, $[\hat{a}(\mathbf{k}),\,\hat{a}(\mathbf{q})] = \left[\hat{a}^\dagger(\mathbf{k}),\,\hat{a}^\dagger(\mathbf{q})\right] = 0$.

The power spectrum of the curvature perturbation, $\mathcal{P}_\zeta$, defined through the two point function

$$\langle\zeta(\mathbf{k})\,\zeta(\mathbf{q})\rangle \equiv \frac{2\pi^2}{k^3}\,\mathcal{P}_\zeta(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{q})\,, \tag{2.10}$$

results in the sum of the power spectra corresponding to the homogeneous and the particular solutions, denoted as $\mathcal{P}_{\zeta,\,\rm V}$ and $\mathcal{P}_{\zeta,\,\rm S}$, respectively.

Specifically, the homogeneous solution, corresponding to the scalar perturbations associated to the mode functions (2.9), yields, at the end of inflation and for large scales,

$$\mathcal{P}_{\zeta,\,\rm V} = \frac{k^3}{2\pi^2}\frac{H^2}{\dot\phi_0^2}\,|\Phi_{\rm V}(k,\,\tau_{\rm e})|^2 \xrightarrow[k\ll H]{} \frac{H^4}{4\pi^2\,\dot\phi_0^2}\,. \tag{2.11}$$

An analogous discussion holds also for the tensor perturbations $H_{ij}(\mathbf{x},\,\tau)$, whose vacuum component gives rise to $\mathcal{P}_{h,\,\rm V} = \frac{2}{\pi^2}\frac{H^2}{M_P^2}$.

In order to find the sourced components of the scalar and tensor power spectra we need to take into account the generation of the electromagnetic field by the rolling pseudoscalar. In order to do that, we start with the quantization of the vector field $A_i(\mathbf{x},\,\tau)$:

$$A_i(\mathbf{x},\,\tau) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=\pm} e_i^\lambda(\widehat{\mathbf{k}})\,e^{i\mathbf{k}\mathbf{x}}\left[A_\lambda(k,\,\tau)\,\hat{a}_\lambda(\mathbf{k}) + A_\lambda^*(k,\,\tau)\,\hat{a}_\lambda^\dagger(-\mathbf{k})\right]\,, \tag{2.12}$$

where the helicity projectors $e_i^\pm(\widehat{\mathbf{k}})$ satisfy the relations

$$\begin{aligned} &k_i\,e_i^\lambda(\widehat{\mathbf{k}}) = 0\,, && e_i^\lambda(\widehat{\mathbf{k}})^* = e_i^{-\lambda}(\widehat{\mathbf{k}}) = e_i^\lambda(-\widehat{\mathbf{k}})\,,\\ &i\epsilon_{ijk}k_j e_k^\lambda(\widehat{\mathbf{k}}) = \lambda\,k\,e_i^\lambda(\widehat{\mathbf{k}})\,, && e_i^\lambda(\widehat{\mathbf{k}})e_i^{\lambda'}(\widehat{\mathbf{k}}) = \delta_{\lambda,\,-\lambda'}\,.\end{aligned} \tag{2.13}$$

Inserting the decomposition (2.12) into eq. (2.8) we obtain the equation of motion for the mode functions $A_\lambda(k,\,\tau)$,

$$A_\lambda''(k,\,\tau) + \left(k^2 - \lambda\frac{\phi_0'}{f}k\right)\,A_\lambda(k,\,\tau) = 0\,, \tag{2.14}$$

which can be solved explicitly in terms of special functions if $\dot\phi_0 =$ constant. However, we do not need the exact solution. Defining

$$\xi \equiv \frac{\dot\phi_0}{2\, f\, H}\,, \tag{2.15}$$

we can rewrite eq. (2.14) as

$$\frac{d^2 A_\lambda}{d(k\,\tau)^2} + \left(1 + 2\,\lambda\, \frac{\xi}{k\,\tau}\right) A_\lambda = 0\,, \tag{2.16}$$

so that, assuming $\xi > 0$, the helicity $\lambda = -1$ in eq. (2.16) has always real frequencies that are adiabatically evolving (remember that $\tau < 0$). As a consequence, the mode A_- stays in its vacuum and we will neglect it from now on. On the other hand, the positive helicity mode A_+ has imaginary frequencies for a range of values of $k\tau$ and is therefore exponentially amplified.

In the WKB approximation, the leading term in the solution of the tachyonic modes of A_+ reads [9]

$$A_+(k,\,\tau) \simeq \frac{1}{\sqrt{2\,k}}\,\left(-\frac{k\,\tau}{2\,\xi}\right)^{1/4}\, e^{-2\,\sqrt{-2\xi k\tau}+\pi\,\xi}\,, \tag{2.17}$$

which is strictly speaking valid only in the range [2] $\frac{1}{8\,\xi} \lesssim |k\,\tau| \lesssim 2\,\xi$ (we will assume $\xi \gtrsim O(1)$ throughout this paper). However, since the momenta in this range dominate the contributions to the observables we will be interested in, we will apply the expression (2.17) to the entire range $0 < |k\,\tau| < \infty$. Eq. (2.17) shows that the $\lambda = +$ helicity of the gauge field is amplified by a factor $e^{\pi\xi}$, which can be very large even for moderate values of ξ.

We are now in position to compute the leading order contribution of the amplified gauge field to the curvature perturbation ζ. Taking the Fourier of eq. (2.6), we obtain the equation

$$\Phi''(\mathbf{q},\,\tau) + q^2\Phi(\mathbf{q},\,\tau) - \frac{2}{\tau^2}\Phi(\mathbf{q},\,\tau) - i\,\frac{H\tau}{f}\,\epsilon^{ijk}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\,A_i'(\mathbf{p},\,\tau)\,(\mathbf{q}-\mathbf{p})_j A_k(\mathbf{q}-\mathbf{p},\,\tau) = 0\,. \tag{2.18}$$

The particular solution of this equation, $\Phi_{\rm S}$, which corresponds to the sourced component of scalar fluctuations, can be found using the retarded propagator

$$\Phi_{\rm S}(\mathbf{q},\,\tau) \equiv i\int d\tau'\,G_q(\tau,\,\tau')\frac{H\tau'}{f}\,\epsilon^{ijk}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\,A_i'(\mathbf{p},\,\tau')\,(\mathbf{q}-\mathbf{p})_j A_k(\mathbf{q}-\mathbf{p},\,\tau')\,. \tag{2.19}$$

Given that we are assuming an exact de Sitter background, the retarded propagator can be written explicitly as

$$G_k(\tau,\tau') = \frac{(1+k^2\,\tau\,\tau')\,\sin(k\,(\tau-\tau')) + k\,(\tau'-\tau)\,\cos(k\,(\tau-\tau'))}{k^3\,\tau\,\tau'}\,\Theta(\tau-\tau')\,, \tag{2.20}$$

where Θ denotes the Heaviside step function.

The sourced component of the scalar fluctuations induces an additional contribution to the power spectrum of the curvature perturbation, that for $\xi \gtrsim 3$, is well approximated by the formula [2]

$$\mathcal{P}_{\zeta,\rm S} = \frac{k^3}{2\pi^2}\,\frac{H^2}{\dot\phi_0^2}\,|\Phi_{\rm S}(k,\,\tau_{\rm e})|^2 \xrightarrow[k\ll H]{} 4.8\times 10^{-8}\,\frac{H^8}{\dot\phi_0^4}\,\frac{e^{4\pi\xi}}{\xi^6}\,. \tag{2.21}$$

A commonly used measure of nongaussianity is the parameter $f_{\rm NL}$, which measures the amplitude of the bispectrum of the curvature perturbation and is defined via

$$\langle\zeta(\mathbf{k}_1)\,\zeta(\mathbf{k}_2)\,\zeta(\mathbf{k}_3)\rangle = \frac{3}{10}\,(2\pi)^{5/2}\,\,f_{\rm NL}(k_1,\,k_2,\,k_3)\,\mathcal{P}_\zeta^2\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,\frac{k_1^3+k_2^3+k_3^3}{k_1^3\,k_2^3\,k_3^3}\,. \tag{2.22}$$

For single field, slow-roll inflation, the bispectrum has a small amplitude, and $f_{\rm NL}$ is of the order of the slow-roll parameters [32]. On the other hand, the sourced component of the curvature perturbation, since it results from a $2\to 1$ process, obeys an intrinsically nongaussian statistics. Since such nongaussianities originate from some sub-horizon dynamics, the bispectrum is peaked on equilateral configurations, i.e., for $k_1=k_2=k_3$, with [2]

$$f_{\rm NL}^{\rm equil}\simeq 7.1\times 10^5\,\frac{H^{12}}{\dot\phi^6}\,\frac{e^{6\pi\xi}}{\xi^9}\,, \tag{2.23}$$

for $\xi\gtrsim 3$ and in the regime $\mathcal{P}_{\zeta,\,\rm S}\ll\mathcal{P}_{\zeta,\,\rm V}$. In the regime of large ξ, where $\mathcal{P}_{\zeta,\,\rm S}\gg\mathcal{P}_{\zeta,\,\rm V}$, $f_{\rm NL}^{\rm equil}$ converges to a value of the order of 10^4, which exceeds by a $O(10^3)$ factor the constraints from Planck. This limits severely the value $\xi_{\rm CMB}$ taken by ξ when Cosmic Microwave Background scales are leaving the horizon, leading to $\xi_{\rm CMB}\lesssim 2.5$ [33, 34].

The excited modes of the vector field are also a source of gravitational waves. To leading order, production of gravitational waves via this process is described by the equation

$$\begin{aligned}&H_{ij}''(\mathbf{q},\,\tau)+q^2H_{ij}(\mathbf{q},\,\tau)-\frac{2}{\tau^2}H_{ij}(\mathbf{q},\,\tau)\\&=\frac{H\,\tau}{M_P}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\left(A_i'(\mathbf{p},\,\tau)\,A_j'(\mathbf{q}-\mathbf{p},\,\tau)-F_{ik}(\mathbf{p},\,\tau)F_{jk}(\mathbf{q}-\mathbf{p},\,\tau)\right)\,,\end{aligned} \tag{2.24}$$

where $F_{ij}(\mathbf{p},\,\tau)\equiv i\,\mathbf{p}_iA_j(\mathbf{p},\tau)-i\,\mathbf{p}_jA_i(\mathbf{p},\tau)$. As a consequence of the functional dependence of A_+ on $k\,\tau$ and on ξ, the electric field is stronger than the magnetic field by a factor $\sim\xi\gtrsim 1$. For this reason we will neglect the term $F_{ik}(\mathbf{p},\,\tau)F_{jk}(\mathbf{q}-\mathbf{p},\,\tau)$ in eq. (2.24). Using again the Green's function (2.20) we eventually obtain

$$H_{ij,\,\rm S}(\mathbf{q},\,\tau)\equiv\int d\tau'\,G_q(\tau,\tau')\,\frac{H\,\tau'}{M_P}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\,A_i'(\mathbf{p},\,\tau')\,A_j'(\mathbf{q}-\mathbf{p},\,\tau')\,. \tag{2.25}$$

The resulting power spectrum for the tensor modes reads [5]

$$\mathcal{P}_h=\mathcal{P}_{h,\,\rm V}+\mathcal{P}_{h,\,\rm S}\simeq\frac{2\,H^2}{\pi^2\,M_P^2}+8.7\times 10^{-8}\,\frac{H^4}{M_P^4}\,\frac{e^{4\pi\xi}}{\xi^6}\,. \tag{2.26}$$

It is worth stressing that the sourced component of the gravitational waves is almost fully chiral, as a consequence of the fact that only the + helicity of the gauge field is excited. While this fact can lead to a rich and interesting phenomenology, we will not be concerned with it here.

The constraint on the parameter ξ coming from the limits on nongaussianities implies that $\mathcal{P}_{h,\,\rm V}\gg\mathcal{P}_{h,\,\rm S}$. This constraint, however, holds only for the value $\xi_{\rm CMB}$ taken by ξ when CMB scales left the horizon. The quantity $\xi\propto\dot\phi_0/H$ remains approximately constant in a slow-roll inflationary background, but it shows small time variations at higher orders in the slow-roll parameters. Therefore, we consider it as an adiabatically evolving quantity, i.e. we treat it as constant when studying the production of gauge fields at a particular moment during inflation (eq. (2.17)), but we must take into account its variation, typically an increase, when comparing two distinct stages of inflation. Since the sourced component of the gravitational wave spectrum has an exponential dependence on ξ, it is possible that at later times $\mathcal{P}_{h,\,\mathrm{V}}$ is actually overwhelmed by $\mathcal{P}_{h,\,\mathrm{S}}$. We will denote by $\xi_{\mathrm{INT}} > \xi_{\mathrm{CMB}}$ the value taken by ξ at this later stage, where the subscript $_{\mathrm{INT}}$ refers to the fact that we are thinking of frequencies probed by gravitational interferometers. In particular, this leads to the possibility that gravitational waves sourced by the vector field have such large amplitude to be directly detectable by current or future gravitational detectors [6].

In the next section we will describe two mechanisms that induce correlation between the curvature perturbation and the gravitational waves produced in axion inflation.

3 The correlator between scalar fluctuations and gravitational waves

We define the normalized correlator of scalar fluctuations and gravitational waves as

$$\begin{aligned} \mathcal{C}_{\Omega\zeta}(\mathbf{k},\,t_0) &\equiv \frac{1}{\Omega_{GW}^{\mathrm{INT}}\sqrt{\mathcal{P}_\zeta^{\mathrm{CMB}}}}\frac{k^3}{2\pi^2}\int d\mathbf{y}\,e^{-i\mathbf{k}\mathbf{y}}\langle\Omega_{GW}(\mathbf{x}+\mathbf{y},\,t_0)\,\zeta(\mathbf{x},\,t_0)\rangle \\ &= \frac{1}{\Omega_{GW}^{\mathrm{INT}}\sqrt{\mathcal{P}_\zeta^{\mathrm{CMB}}}}\frac{k^3}{2\pi^2}\langle\Omega_{GW}(\mathbf{k},\,t_0)\,\zeta(-\mathbf{k},\,t_0)\rangle'\,, \end{aligned} \tag{3.1}$$

where the symbol $\langle\ldots\rangle'$ denotes the correlator stripped of the Dirac delta associated to momentum conservation and t_0 indicates the present value of cosmic time. Moreover, $\Omega_{GW}^{\mathrm{INT}}$ denotes the fractional energy in gravitational waves at interferometer frequencies, whereas $\mathcal{P}_\zeta^{\mathrm{CMB}}$ denotes the amplitude of scalar perturbations at CMB scales. Given the weak scale dependence of $\mathcal{P}_\zeta^{\mathrm{CMB}}$, from now on we will drop the index $^{\mathrm{CMB}}$ from $\mathcal{P}_\zeta$, and will treat this quantity as constant. On the other hand, axion inflation can lead to a strong scale dependence of the energy in gravitational waves, which cannot be ignored in our analysis.

To proceed we observe that $\Omega_{GW}(\mathbf{k}) = \frac{1}{12\,H_0^2}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\,|\mathbf{k}-\mathbf{p}|\,p\,h_{ij}(\mathbf{k}-\mathbf{p},\,t_0)h_{ij}(\mathbf{p},\,t_0)$. The current amplitude $h_{ij}(k,\,t_0)$ is related to the primordial amplitude calculated at the end of inflation $h_{ij}(k,\,t_e)$ through the transfer function $\mathcal{T}(k)$, which is proportional to k^{-1} for modes that have re-entered the horizon during radiation domination, and to k^{-2} for modes that have re-entered the horizon during matter domination. Putting everything together, we have

$$\begin{aligned} \mathcal{C}_{\Omega\zeta}(\mathbf{k},\,t_0) = \frac{1}{12\,H_0^2\,\Omega_{GW}^{\mathrm{INT}}\sqrt{\mathcal{P}_\zeta}}\frac{k^3}{2\pi^2}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\hat{\mathcal{T}}(|\mathbf{k}-\mathbf{p}|)\,\hat{\mathcal{T}}(p) \\ \times\,\langle h_{ij}(\mathbf{k}-\mathbf{p},\,t_e)\,h_{ij}(\mathbf{p},\,t_e)\,\zeta(-\mathbf{k},\,t_e)\rangle'\,, \end{aligned} \tag{3.2}$$

where we have defined $\hat{\mathcal{T}}(p) \equiv p\,\mathcal{T}(p)$ and we have replaced the amplitude of the scalar perturbations with its value at the end of inflation.

The correlator $\mathcal{C}_{\Omega\zeta}(\mathbf{k},\,t_0)$ receives two different contributions: the first is the result of the correlation of gravitational waves with the amplified vacuum scalar fluctuations; the second is due to the correlation of gravitational waves with the sourced scalar fluctuations. Below we will examine the two cases separately.

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3.1 Correlation with amplified vacuum scalar fluctuations

The spectrum $\mathcal{P}_{h,\mathrm{S}}$ of gravitational waves sourced by the gauge field depends on the values of ϕ and $\dot{\phi}$ evaluated approximately at the time when the tensor modes under consideration left the horizon, and where, in slow-roll approximation, $\dot{\phi}$ is a function of ϕ. As a consequence, long wavelength perturbations in the values of ϕ will lead to correlated long wavelength perturbations in the spectrum of gravitational waves.

To first order in the vacuum-amplified fluctuation $\delta\phi_{\mathrm{V}}$ of the inflaton, and in the limit in which the wavelength of $\delta\phi_{\mathrm{V}}$ is much larger than that of $h_{ij,\mathrm{S}}$, we have

$$h_{ij,\mathrm{S}}(\mathbf{x},\,\phi(\mathbf{x})) = h_{ij,\mathrm{S}}(\mathbf{x},\,\phi_0) + \frac{\partial h_{ij,\mathrm{S}}(\mathbf{x},\,\phi_0)}{\partial\phi_0}\,\delta\phi_{\mathrm{V}}(\mathbf{x})\,, \tag{3.3}$$

where the first term does not contribute to $\mathcal{C}_{\Omega\zeta}$. Since $h_{ij,\mathrm{S}}(\mathbf{x},\,\phi_0) \propto e^{2\pi\xi}$, we can also write

$$h_{ij,\mathrm{S}}(\mathbf{x},\,\phi(\mathbf{x})) = h_{ij,\mathrm{S}}(\mathbf{x},\,\phi_0)\left(1 - 2\pi\frac{d\,\xi}{d\phi_0}\,\frac{\dot{\phi}_0}{H}\,\zeta_{\mathrm{V}}(\mathbf{x})\right)\,, \tag{3.4}$$

where we used $\delta\phi = -\dot{\phi}_0\,\zeta/H$. We thus obtain the first contribution to the correlator between Ω_{GW} and ζ_{V}, that we denote as $(\mathcal{C}_{\Omega\zeta})_{\mathrm{V}}$, and which reads

$$\begin{aligned}(\mathcal{C}_{\Omega\zeta})_{\mathrm{V}} = &-\frac{1}{12\,H_0^2\,\Omega_{GW}^{\mathrm{INT}}\,\sqrt{\mathcal{P}_\zeta}}\frac{k^3}{2\pi^2}\int\frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^3}\hat{\mathcal{T}}(|\mathbf{k}-\mathbf{p}|)\,\hat{\mathcal{T}}(|\mathbf{p}-\mathbf{q}|)\\ &\times 4\pi\,\frac{\dot{\phi}_0}{H}\,\frac{d\,\xi}{d\phi_0}\langle h_{ij,\mathrm{S}}(\mathbf{k}-\mathbf{p},\,t_e)\,h_{ij,\mathrm{S}}(\mathbf{p}-\mathbf{q},\,t_e)\,\zeta_{\mathrm{V}}(\mathbf{q},\,t_e)\,\zeta_{\mathrm{V}}(-\mathbf{k},\,t_e)\rangle'\,.\end{aligned} \tag{3.5}$$

Assuming $\dot{\phi}_0 > 0$, $V' < 0$, we have

$$\xi \equiv \frac{\dot{\phi}_0}{2\,fH} \simeq -\frac{V'}{6\,fH^2} = -\frac{M_P^2}{2\,f}\frac{V'}{V}\,, \tag{3.6}$$

so that

$$\frac{d\xi}{d\phi_0} = -\frac{M_P^2}{2\,f}\left(\frac{V''}{V} - \frac{V'^2}{V^2}\right) = \left(\epsilon - \frac{\eta}{2}\right)\frac{1}{f}\,, \tag{3.7}$$

where we have defined as usual the slow-roll parameters as

$$\epsilon = \frac{M_P^2}{2}\,\frac{V'^2}{V^2}\,, \qquad\qquad \eta = M_P^2\frac{V''}{V}\,. \tag{3.8}$$

The correlator therefore becomes

$$(\mathcal{C}_{\Omega\zeta})_{\mathrm{V}} = -\frac{\sqrt{\mathcal{P}_\zeta}}{12\,H_0^2\,\Omega_{GW}^{\mathrm{INT}}}\int\frac{d\mathbf{p}}{p^3}\,\xi\,\left(2\epsilon - \eta\right)\,\hat{\mathcal{T}}(p)^2\,\mathcal{P}_{h,\mathrm{S}}(p)\,. \tag{3.9}$$

To proceed we note that, since typically the amplitude of the induced tensor modes increases as inflation progresses, the integral in eq. (3.5) is dominated by the largest frequencies, that are typically close to those probed by the interferometers. For those wavelengths, that re-entered the horizon well into the radiation dominated regime, we have

$$\frac{\hat{\mathcal{T}}(p)^2\,\mathcal{P}_{h,\mathrm{S}}(p)}{12\,H_0^2\,\Omega_{GW}^{\mathrm{INT}}} = \frac{\mathcal{P}_{h,\mathrm{S}}(p)}{\mathcal{P}_{h,\mathrm{S}}(p^{\mathrm{INT}})}\,. \tag{3.10}$$

Using again the fact that the integral in eq. (3.5) is dominated by values of p of the order of $p^{\rm INT}$, we can estimate

$$(\mathcal{C}_{\Omega\zeta})_{\rm V} \simeq -4\pi\,\xi\,\Delta\mathcal{N}_*\,(2\epsilon-\eta)\,\sqrt{\mathcal{P}_\zeta}\,, \tag{3.11}$$

where both ξ and the slow-roll parameters ϵ and η are evaluated at the time when the scales probed by interferometers have left the horizon. In eq. (3.11) the parameter $\Delta\mathcal{N}_*$ accounts for the number of efoldings during which the tensor power spectrum is approximately constant. Numerical simulations indicate that this is the case in the strong backreaction regime, which usually lasts $\Delta\mathcal{N}_* \simeq 10 \div 30$ efoldings. At this stage the parameter ξ takes values that are typically of the order of $5 \div 10$. The quantity $(2\epsilon-\eta)$ has to be smaller than unity and is typically of the order of $10^{-2} \div 10^{-1}$. So by putting everything together we obtain that $(\mathcal{C}_{\Omega\zeta})_{\rm V}$ is typically of the order of $10^{-4} \div 10^{-2}$.

3.2 Correlation with sourced scalar fluctuations

In order to calculate the correlator between the sourced scalar and tensor fluctuations, that we denote as $(\mathcal{C}_{\Omega\zeta})_{\rm S}$, we use eqs. (2.4), (2.19) and (2.25) to find $\langle h_{ab,\,\rm S}(\mathbf{k}_1,\tau)\,h_{ab,\,\rm S}(\mathbf{k}_2,\tau)\,\zeta_{\rm S}(\mathbf{k}_3,\tau)\rangle$ in terms of the canonically normalized perturbations as

$$\begin{aligned}
&\langle h_{ab,\,\rm S}(\mathbf{k}_1,\tau)\,h_{ab,\,\rm S}(\mathbf{k}_2,\tau)\,\zeta_{\rm S}(\mathbf{k}_3,\tau)\rangle\\
&= -\frac{4\,H(\tau)}{M_P^2\,\dot\phi_0(\tau)\,a^3(\tau)}\langle H_{ab,\,\rm S}(\mathbf{k}_1,\tau)\,H_{ab,\,\rm S}(\mathbf{k}_2,\tau)\,\Phi_{\rm S}(\mathbf{k}_3,\tau)\rangle\\
&= \frac{4\,H(\tau)}{M_P^4\,\dot\phi_0(\tau)\,a^3(\tau)\,f}\int_{-\infty}^{\tau}\frac{d\tau_1}{a(\tau_1)}\,\frac{d\tau_2}{a(\tau_2)}\,\frac{d\tau_3}{a(\tau_3)}\,G_{k_1}(\tau,\tau_1)\,G_{k_2}(\tau,\tau_2)\,G_{k_3}(\tau,\tau_3)\\
&\quad\times\int\frac{d\mathbf{q}_1\,d\mathbf{q}_2\,d\mathbf{q}_3}{(2\pi)^{9/2}}\,e_a^+(\widehat{\mathbf{q}_1})\,e_b^+(\widehat{\mathbf{k}_1-\mathbf{q}_1})\,e_a^+(\widehat{\mathbf{q}_2})\,e_b^+(\widehat{\mathbf{k}_2-\mathbf{q}_2})\,e_i^+(\widehat{\mathbf{q}_3})\,e_i^+(\widehat{\mathbf{k}_3-\mathbf{q}_3})\,|\mathbf{k}_3-\mathbf{q}_3|\\
&\quad\times\langle A_+'(q_1,\tau_1)\,A_+'(|\mathbf{k}_1-\mathbf{q}_1|,\tau_1)\,A_+'(q_2,\tau_2)\,A_+'(|\mathbf{k}_2-\mathbf{q}_2|,\tau_2)\,A_+'(q_3,\tau_3)\,A_+(|\mathbf{k}_3-\mathbf{q}_3|,\tau_3)\rangle\,,
\end{aligned} \tag{3.12}$$

where we have assumed that only the positive helicity photons contribute because, from eq. (2.14), A_+ is the only helicity that is amplified.

Using Wick's theorem to decompose the last line of eq. (3.12) and inserting it back into (3.2) we obtain

$$\begin{aligned}
(\mathcal{C}_{\Omega\zeta})_{\rm S} = {}& \frac{k^3\,H(\tau)}{6\,H_0^2\,\pi^2\,M_P^4\,\dot\phi_0(\tau)\,a^3(\tau)\,f\,\Omega_{GW}^{\rm INT}\,\sqrt{\mathcal{P}_\zeta}}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\int_{-\infty}^{\tau}\frac{d\tau_1}{a(\tau_1)}\,\frac{d\tau_2}{a(\tau_2)}\,\frac{d\tau_3}{a(\tau_3)}\\
&\times G_{k_1}(\tau,\tau_1)\,G_{k_2}(\tau,\tau_2)\,G_{k_3}(\tau,\tau_3)\int\frac{d\mathbf{q}}{(2\pi)^{9/2}}\,\hat{\mathcal{T}}(|\mathbf{k}-\mathbf{p}|)\,\hat{\mathcal{T}}(p)\,\mathcal{A}(\mathbf{q},\mathbf{k}_1-\mathbf{q},\mathbf{k}_2+\mathbf{q})\\
&\times\Big(|\mathbf{k}_2+\mathbf{q}|\,A_+'(q,\tau_1)\,A_+'(|\mathbf{k}_1-\mathbf{q}|,\tau_1)\,A_+'(q,\tau_2)\,A_+'(|\mathbf{k}_1-\mathbf{q}|,\tau_3)\\
&\times A_+'(|\mathbf{k}_2+\mathbf{q}|,\tau_2)\,A_+(|\mathbf{k}_2+\mathbf{q}|,\tau_3)\\
&+|\mathbf{k}_1-\mathbf{q}|\,A_+'(q,\tau_1)\,A_+'(q,\tau_2)\,A_+'(|\mathbf{k}_1-\mathbf{q}|,\tau_1)\,A_+(|\mathbf{k}_1-\mathbf{q}|,\tau_3)\\
&\times A_+'(|\mathbf{k}_2+\mathbf{q}|,\tau_2)\,A_+'(|\mathbf{k}_2+\mathbf{q}|,\tau_3)\Big)\,,
\end{aligned} \tag{3.13}$$

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where $\mathbf{k}_1 = \mathbf{k} - \mathbf{p}$, $\mathbf{k}_2 = \mathbf{p}$ and $\mathbf{k}_3 = -\mathbf{k}$, and where we have collected the angular part into the expression $\mathcal{A}$:

$$\mathcal{A}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \\ \delta_{ac}\, \delta_{bd} \left(\left(e_a^+(\widehat{\mathbf{k}_1})\, e_c^+(\widehat{-\mathbf{k}_1})\, e_b^+(\widehat{\mathbf{k}_2})\, e_i^+(\widehat{-\mathbf{k}_2})\, e_d^+(\widehat{\mathbf{k}_3})\, e_i^+(\widehat{-\mathbf{k}_3}) + (a \leftrightarrow b) \right) + (c \leftrightarrow d) \right).$$

Using the explicit form of the gauge field (2.17), the expression (3.13) becomes

$$\begin{aligned}
&(\mathcal{C}_{\Omega\zeta})_{\rm S} = \qquad (3.14)\\
&-\frac{k^3\, H^4(\tau)}{3 \times 2^9\, \pi^8\, H_0^2\, M_P^4\, \dot\phi_0(\tau)\, f\, a^3(\tau)\, \Omega_{GW}^{\rm INT}\, \sqrt{\mathcal{P}_\zeta}} \int d\mathbf{p}\, \hat{\mathcal{T}}(|\mathbf{k}-\mathbf{p}|)\, \hat{\mathcal{T}}(p)\\
&\times \int_{-\infty}^{\tau} d\tau_1\, d\tau_2\, d\tau_3\, \xi_1^{1/2}\, \xi_2^{1/2}\, \sqrt{\tau_1\, \tau_2}\, \tau_3\, G_{|\mathbf{k}-\mathbf{p}|}(\tau, \tau_1)\, G_p(\tau, \tau_2)\, G_k(\tau, \tau_3) e^{2\pi(\xi_1+\xi_2+\xi_3)}\\
&\times \int d\mathbf{q}\, \mathcal{A}(\mathbf{q}, \mathbf{k}-\mathbf{p}-\mathbf{q}, \mathbf{p}+\mathbf{q}) q^{1/2}\, |\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2}\, |\mathbf{p}+\mathbf{q}|^{1/2} (|\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2} + |\mathbf{p}+\mathbf{q}|^{1/2})\\
&\times e^{-2\sqrt{-2\,\xi_1\, q\, \tau_1} - 2\sqrt{-2\,\xi_1\, |\mathbf{k}-\mathbf{p}-\mathbf{q}|\, \tau_1} - 2\sqrt{-2\,\xi_2\, q\, \tau_2} - 2\sqrt{-2\,\xi_2\, |\mathbf{p}+\mathbf{q}|\, \tau_2} - 2\sqrt{-2\,\xi_3\, |\mathbf{p}+\mathbf{q}|\, \tau_3} - 2\sqrt{-2\,\xi_3\, |\mathbf{k}-\mathbf{p}-\mathbf{q}|\, \tau_3}},
\end{aligned}$$

where we have also accounted for the adiabatic time variation of the parameter ξ, as we are considering the entire inflationary stage, and we have denoted $\xi_i \equiv \xi(\tau_i)$. In order to perform the calculation we set the time at the end of inflation to be $\tau_{\rm e} = -1/H$. Since we are interested in modes that are well outside of the horizon at the end of inflation, we will assume $k/H \to 0$. The dependence of the integrand on $e^{-2\sqrt{-2\,\xi_1\,\tau_1}\left(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|}\right)}$ with $\xi_1 \gg 1$ implies that we can set $|\mathbf{k}-\mathbf{p}|\,|\tau_1| \ll 1$ in the propagator, and we can approximate $G_{|\mathbf{k}-\mathbf{p}|}(\tau, \tau_1) \simeq -\tau_1^2/(3\,\tau)$. A similar argument applies to the other two propagators which are approximated as $G_p(\tau, \tau_2) \simeq -\tau_2^2/(3\,\tau)$ and $G_k(\tau, \tau_3) \simeq -\tau_3^2/(3\,\tau)$. As a consequence, the dependence of the integrand on τ_1, τ_2 and τ_3 takes the form $\tau_1^{5/2}\, e^{-2\sqrt{-2\,\xi_1\,\tau_1}\left(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|}\right)+2\pi\xi_1}$, $\tau_2^{5/2}\, e^{-2\sqrt{-2\,\xi_2\,\tau_2}\left(\sqrt{q}+\sqrt{|\mathbf{p}+\mathbf{q}|}\right)+2\pi\xi_2}$ and $\tau_3^3\, e^{-2\sqrt{-2\,\xi_3\,\tau_3}\left(\sqrt{|\mathbf{p}+\mathbf{q}|}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|}\right)+2\pi\xi_3}$ respectively. To proceed with the calculation, we need to know the explicit form of the model-dependent function $\xi(\tau)$. Without choosing a particular model, we can still estimate the integral by assuming that ξ has a weak dependence on τ. In this case we see that the integral is dominated by values of $|\tau_1|$, $|\tau_2|$ and $|\tau_3|$ belonging respectively to a relatively narrow window around $(\sqrt{q} + \sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|})^{-2}$, $(\sqrt{q} + \sqrt{|\mathbf{p}+\mathbf{q}|})^{-2}$ and $(\sqrt{|\mathbf{p}+\mathbf{q}|} + \sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|})^{-2}$. We can therefore approximate

$$\begin{aligned}
\xi_1 &= \xi\left(\tau \simeq -\left(\sqrt{q} + \sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|}\right)^{-2}\right),\\
\xi_2 &= \xi\left(\tau \simeq -\left(\sqrt{q} + \sqrt{|\mathbf{p}+\mathbf{q}|}\right)^{-2}\right),\\
\xi_3 &= \xi\left(\tau \simeq -\left(\sqrt{|\mathbf{p}+\mathbf{q}|} + \sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|}\right)^{-2}\right),
\end{aligned} \qquad (3.15)$$

which are now momentum-dependent. Using the expression

$$\int_0^\infty dx\, x^{n-1}\, e^{-a\,\sqrt{x}} = \frac{2}{a^{2n}} \Gamma(2n)\,, \qquad (3.16)$$

we obtain

$$
\begin{aligned}
(\mathcal{C}_{\Omega\zeta})_{\rm S} = & \frac{k^3\, H^7\, \Gamma(7)^2\, \Gamma(8)}{2^{39}\times 3^4\, H_0^2\, \pi^8\, M_P^4\, \dot\phi_0(\tau)\, f\, \Omega_{GW}^{\rm INT}\, \sqrt{\mathcal{P}_\zeta}} \int d\mathbf{p}\, d\mathbf{q} \frac{e^{2\pi(\xi_1+\xi_2+\xi_3)}}{\xi_1^3\, \xi_2^3\, \xi_3^4} \\
& \times \hat{\mathcal{T}}(|\mathbf{k}-\mathbf{p}|)\, \hat{\mathcal{T}}(p) \\
& \times \frac{\mathcal{A}(\mathbf{q},\mathbf{k}-\mathbf{p}-\mathbf{q},\mathbf{p}+\mathbf{q})\, q^{1/2}\, |\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2}\, |\mathbf{p}+\mathbf{q}|^{1/2}}{(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|}\,)^7\, (\sqrt{q}+\sqrt{|\mathbf{p}+\mathbf{q}|}\,)^7\, (\sqrt{|\mathbf{p}+\mathbf{q}|}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|}\,)^7}\,.
\end{aligned}
\tag{3.17}
$$

The computation of the remaining six-dimensional integral is complicated, again, by the fact that the function $\xi(\tau)$ is model dependent. Even if the time dependence is weak (i.e., slow-roll implies that $d\xi/dt \ll H\,\xi$), we cannot neglect it, because ξ appears in exponents. Moreover, ξ is in general increasing during inflation. The time- (and therefore $\mathbf{p}$- and $\mathbf{q}$-) dependence in the exponent leads the coefficient $e^{2\pi(\xi_1+\xi_2+\xi_3)}$ to be an increasing function of the integration variables. On the other hand, the factors $(\ldots)^{-7}\times(\ldots)^{-7}\times(\ldots)^{-7}$ in the denominator of eq. (3.17) give a contribution that is peaked at small values of $|\mathbf{p}|$ and $|\mathbf{q}|$, i.e. $|\mathbf{p}|\approx|\mathbf{q}|\approx k$. The result of the integral will thus depend on whether it is dominated by $|\mathbf{p}|\approx|\mathbf{q}|\approx k$ or by the largest values of $|\mathbf{p}|$ and $|\mathbf{q}|$.

To proceed with our estimates, we assume that the function $\xi(\tau)$ is monotonically increasing, which, as we said, is what typically happens. It will take a value $\xi(\tau=-1/k)\equiv\xi_k$ when scales with comoving wavenumber k, leave the horizon, N_k efoldings before the end of inflation. In particular, we have in mind the case where $k\simeq k_{\rm CMB}$, with $N_{\rm CMB}\simeq 60$ (as noted above, observations constrain $\xi_{\rm CMB}\lesssim 2.5$ [34]). At a later time, denoted by $\tau_{\rm BR}$, i.e. $N_{\rm BR}=\log(-H\tau_{\rm BR})$ efoldings before the end of inflation, the system gets into the strong backreaction regime, and ξ takes the value $\xi=\xi_{\rm BR}$. The behavior of the system in this regime is still object of active research, but it is reasonable to assume that ξ will be approximately constant for $\tau>\tau_{\rm BR}$, so that the integral does not receive significant contributions by the values of $\mathbf{p}$ and $\mathbf{q}$ corresponding to scales that left the horizon after $\tau_{\rm BR}$.

As we show in the appendix, the integral is dominated by $|\mathbf{p}|\approx|\mathbf{q}|\approx k$ if $\xi_{\rm BR}-\xi_k\lesssim(N_k-N_{\rm BR})/(2\pi)$, and by $|\mathbf{p}|\approx|\mathbf{q}|\approx -1/\tau_{\rm BR}$ otherwise. Let us examine these two cases separately.

3.2.1 $\xi_{\rm BR}-\xi_k\lesssim\frac{N_k-N_{\rm BR}}{2\pi}$

In this case the integral is dominated by $|\mathbf{p}|\approx|\mathbf{q}|\approx k$, so that we can set $\xi_1\simeq\xi_2\simeq\xi_3\equiv\xi_k$ everywhere. Moreover, since we are assuming that k is at CMB scales, it corresponds to wavenumbers that reentered the horizon during matter domination, so that we can assume $\hat{\mathcal{T}}(k)\simeq\bar{k}^2/k$, where $\bar{k}^2\equiv\frac{3}{4\sqrt{2}}\,k_{\rm eq}\,H_0\,\sqrt{\Omega_{\rm rad}}\simeq(.5\,H_0)^2$, with $k_{\rm eq}$ being the scale that reentered the horizon during matter-radiation equality [35]. We are thus left with

$$
\begin{aligned}
(\mathcal{C}_{\Omega\zeta})_{\rm S} = & \frac{H^7\, \Gamma(7)^2\, \Gamma(8)\, \bar{k}^4}{2^{39}\times 3^4\, \pi^8\, H_0^2\, M_P^4\, \dot\phi_0\, f\, \Omega_{GW}^{\rm INT}\, \sqrt{\mathcal{P}_\zeta}} \frac{e^{6\pi\xi_k}}{\xi_k^{10}} \\
& \times k^3 \int \frac{d\mathbf{p}\, d\mathbf{q}}{p\,|\mathbf{k}-\mathbf{p}|} \frac{\mathcal{A}(\mathbf{q},\mathbf{k}-\mathbf{p}-\mathbf{q},\mathbf{p}+\mathbf{q})\, q^{1/2}\, |\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2}\, |\mathbf{p}+\mathbf{q}|^{1/2}}{(\sqrt{q}+\sqrt{\mathbf{k}-\mathbf{p}-\mathbf{q}})^7\, (\sqrt{q}+\sqrt{\mathbf{p}+\mathbf{q}})^7\, (\sqrt{|\mathbf{p}+\mathbf{q}|}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}|})^7}\,,
\end{aligned}
\tag{3.18}
$$

where the integral on the second line can be computed numerically, using

$$\begin{aligned}\mathcal{A}(\mathbf{k}_1,\,\mathbf{k}_2,\,\mathbf{k}_3) = \frac{1}{4}\Big(&2+3\,(\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3})^2-5\,\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3}+(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3})^2+(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2})^2\\ &-\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3}+\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2}-(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2})-(\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2})\\ &+(\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3})-(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3})-i\,(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2}-\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3}\\ &-\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3}+1)\,\epsilon_{dil}\,\widehat{\mathbf{k}}_{1d}\,\widehat{\mathbf{k}}_{2i}\,\widehat{\mathbf{k}}_{3l}\Big)\,. \end{aligned}\tag{3.19}$$

One thus obtains

$$\left(\mathcal{C}_{\Omega\zeta}\right)_{\rm S}(k) \simeq 6\times10^{-12}\,\frac{H^7\,\bar{k}^2}{H_0^2\,M_P^4\,\dot{\phi}_0\,f\,\Omega_{GW}^{\rm INT}\,\sqrt{\mathcal{P}_\zeta}}\,\frac{e^{6\pi\,\xi_k}}{\xi_k^{10}}\,\frac{\bar{k}^2}{k^2}\,. \tag{3.20}$$

After substituting $\sqrt{\mathcal{P}_\zeta}\simeq\sqrt{\mathcal{P}_{\zeta,\rm V}}=H^2/(2\pi\,\dot{\phi}_0)$ and $\Omega_{GW}^{\rm INT}\simeq\frac{\Omega^0_{\rm rad}}{24}\,\mathcal{P}_{h,\rm S}(k_{\rm INT})$ [35], with $\Omega^0_{\rm rad}\simeq 8.2\times10^{-5}$ and $\mathcal{P}_{h,\rm S}(k_{\rm INT})$ from (2.26), we obtain the simple form

$$\left(\mathcal{C}_{\Omega\zeta}\right)_{\rm S}\simeq 8\,\frac{H_0^2}{k^2}\,\frac{H}{f}\,e^{6\pi\,\xi_k-4\pi\,\xi_{\rm INT}}\,\frac{\xi_{\rm INT}^6}{\xi_k^{10}}\,. \tag{3.21}$$

Finally, if k is at CMB scales, we use eq. (2.23) together with the measured amplitude of the scalar perturbations $\mathcal{P}_{\zeta,\rm V}\simeq 2\times10^{-9}$ to obtain

$$\left(\mathcal{C}_{\Omega\zeta}\right)_{\rm S}\simeq 600\,\frac{H_0^2}{k^2}\,(f_{\rm NL}^{\rm equil})^{1/3}\,e^{-4\pi(\xi_{\rm INT}-\xi_k)}\,\frac{\xi_{\rm INT}^6}{\xi_k^6}\,, \tag{3.22}$$

which despite the $\mathcal{O}(10^3)$ coefficient in front, and assuming the factor $\frac{\bar{k}^2}{k^2}\,(f_{\rm NL}^{\rm equil})^{1/3}$ to be of the order of the unity, is exponentially small. For instance, assuming $\xi_k\simeq 2.5$ (which is the largest value of ξ_k allowed by non-observation of nongaussianities in the CMB) and $\xi_{\rm INT}\simeq 5$, which is on the lower end of the values found in numerical studies for ξ in the strong backreaction regime, the factor $e^{-4\pi(\xi_{\rm INT}-\xi_k)}\,\frac{\xi^6_{\rm INT}}{\xi^6_k}$ evaluates to approximately 10^{-11}, making this dimensionless, normalized correlator tiny.

3.2.2 $\xi_{\rm BR}-\xi_k\gtrsim\frac{N_k-N_{\rm BR}}{2\pi}$

In this case the integral is dominated by the scales that left the horizon when ξ attained its largest value at the beginning of the strong backreaction regime. Since we are interested in largest value of the momenta, we consider only wavenumbers that reentered the horizon during radiation domination. The integral

$$\int d\mathbf{p}\,d\mathbf{q}\,\frac{\mathcal{A}(\mathbf{q},\mathbf{k}-\mathbf{p}-\mathbf{q},\mathbf{p}+\mathbf{q})\,q^{1/2}\,|\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2}\,|\mathbf{p}+\mathbf{q}|^{1/2}}{(\sqrt{q}+\sqrt{\mathbf{k}-\mathbf{p}-\mathbf{q}})^7\,(\sqrt{q}+\sqrt{\mathbf{p}+\mathbf{q}})^7\,(\sqrt{|\mathbf{p}+\mathbf{q}|}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}})^7}\,, \tag{3.23}$$

is estimated in the appendix, and it evaluates to $\mathcal{O}(10^{-2})\,e^{6\pi\,\xi_{\rm BR}}/k^3_{\rm BR}$. As a consequence we obtain the result

$$\left(\mathcal{C}_{\Omega\zeta}\right)_{\rm S}(k)\simeq\mathcal{O}(10^{-2})\frac{k^3}{k^3_{\rm BR}}\,\frac{H}{f}\,\frac{e^{2\pi\,\xi_{\rm BR}}}{\xi^4_{\rm BR}}. \tag{3.24}$$

In this case the correlator contains an exponentially large factor (for typical values of $\xi_{\rm BR} \approx 5$, one has $e^{2\pi\,\xi_{\rm BR}} = \mathcal{O}(10^{13})$) that is however suppressed by a volume factor $k^3/k_{\rm BR}^3$ equal to the inverse of the number of patches of size $\sim k^{-1}$. Given that typically strong backreaction kicks in only ≈ 10 efoldings before the end of inflation (see however [28], where this occurs as early as ≈ 40 efoldings before the end of inflation), the suppression factor is typically of the order of $e^{-150} \approx 10^{-65}$ (!), making this correlator, also in this regime, tiny.

4 Discussion and conclusions

An important component of current and future gravitational wave research is the detection and characterization of the stochastic gravitational wave background. This background may originate from astrophysical sources or have a cosmological origin. Specifically, identifying a cosmological gravitational wave background will provide important information about the very early universe.

A powerful approach to distinguish between astrophysical and cosmological backgrounds involves studying their anisotropies. Notably, it has been shown that these anisotropies are correlated with the anisotropies in the CMB [36, 37]. The exploration of such correlations can significantly contribute to the interpretation of the CMB and SGWB measurements.

In the present paper we have investigated the correlator between the curvature perturbation and the energy density of the gravitational waves, computed today, within the axion inflation model. In this model, scalar fluctuations are generated through two distinct mechanisms: first, from the vacuum via the standard amplification process, and second, as a consequence of the production of gauge fields through a process of inverse decay. Consequently, the correlator exhibits two distinct components.

Our analysis shows that the dominant contribution is provided by the correlator with the amplified vacuum fluctuations of the inflaton, that we examined in section 3.1. Our main result, eq. (3.11), shows that the normalized correlator between $\Omega_{\rm GW}$ and ζ could be as large as $\mathcal{O}(10^{-2})$. The formalism of [38–40] can then be applied to derive potentially observable quantities. The actual observability of such correlators, subject to instrumental noise as well as to the intrinsic variance of the isotropic component [41, 42], will depend on the amplitude of the anisotropies in the gravitational wave spectra. Such an amplitude is encoded in the correlator $\langle \Omega_{\rm GW}(\mathbf{x})\, \Omega_{\rm GW}(\mathbf{y})\rangle$, whose calculation, in the model of axion inflation, includes the evaluation of the gauge field's eight-point function — a calculation that we leave to future work (see however [40] for work along this direction). Anisotropies might be large. For instance, the lattice study of [43] showed that the spectrum of gravitational waves induced by preheating at the end of inflation display anisotropies with an amplitude of the order of $\sim 10^{-2}$.

Acknowledgments

We thank Gabriele Franciolini, Marco Peloso and especially Valerie Domcke for very useful discussions. This work is partially supported by the US-NSF grant PHY-2112800.

A Finding the dominant contribution to the integral in eq. (3.17)

In this appendix we discuss how to evaluate the integral in eq. (3.17)

$$\mathcal{I}(\mathbf{k},\,\tau) = \int d\mathbf{p}\, d\mathbf{q}\, e^{2\pi(\xi_1+\xi_2+\xi_3)} \times \frac{\mathcal{A}(\mathbf{q}, \mathbf{k}-\mathbf{p}-\mathbf{q}, \mathbf{p}+\mathbf{q})\, q^{1/2}\, |\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2}\, |\mathbf{p}+\mathbf{q}|^{1/2}}{(\sqrt{q}+\sqrt{\mathbf{k}-\mathbf{p}-\mathbf{q}})^7\, (\sqrt{q}+\sqrt{\mathbf{p}+\mathbf{q}})^7\, (\sqrt{|\mathbf{p}+\mathbf{q}|}+\sqrt{|\mathbf{k}-\mathbf{p}-\mathbf{q}})^7}\,, \tag{A.1}$$

where the quantities ξ_1, ξ_2 and ξ_3 are given in eq. (3.15).

As discussed in the main body of the paper, the integral $\mathcal{I}(\mathbf{k},\,\tau)$ includes a factor (containing inverse powers of p and q) that decreases as p and q increase, and a factor $\propto e^{2\pi(\xi_1+\xi_2+\xi_3)}$ that is, on the other hand, an increasing function of those variables. To estimate which contribution dominates the integral we model the function $\xi(\tau)$ as

$$\xi(\tau) = \begin{cases} \xi_{\rm BR} + \delta \log(\tau_{\rm BR}/\tau)\,, & \tau < \tau_{\rm BR}\,, \\ \xi_{\rm BR}\,, & \tau > \tau_{\rm BR}\,, \end{cases} \tag{A.2}$$

where $\tau_{\rm BR} < 0$ corresponds to the time when the produced quanta of gauge field start to backreact strongly on the inflating background. This rough modeling of the function $\xi(\tau)$ has the sole purpose of indicating which range of values of p and q dominate the integral in eq. (3.17). Given that in this parameterization ξ is constant for $\tau > \tau_{\rm BR}$, the integral will receive a subdominant contribution from momenta satisfying $|p\,\tau_{\rm BR}| \gtrsim 1$, $|q\,\tau_{\rm BR}| \gtrsim 1$, so we will limit our integrations to p, $q \lesssim 1/|\tau_{\rm BR}| \equiv k_{\rm BR}$. Moreover, since the strong backreaction regime will kick in relatively late during inflation, when the scales that reenter during radiation domination are leaving the horizon, we can set $\hat{\mathcal{T}} =$ constant in this regime, and thus ignore the effects of the transfer function in this analysis.

We present here only an analysis of the contribution to $\mathcal{I}(\mathbf{k},\,\tau)$ given by the range of momenta where $p \gtrsim k$. We have checked that the contribution from $p \lesssim k$ has no significant effect. To start with, we estimate the integral in $d\mathbf{q}$ which is composed by three different relevant momentum intervals

$$\int d\mathbf{q} = \left(\int_0^k + \int_k^p + \int_p^{k_{\rm BR}}\right) dq\, q^2 \int d\Omega_q\,, \tag{A.3}$$

and we subsequently estimate the integrals in $d\mathbf{p}$, using

$$\int d\mathbf{p} = \int_k^{k_{\rm BR}} dp\, p^2 \int d\Omega_p\,. \tag{A.4}$$

After performing the integrals in dq and on the solid angles $d\Omega_p$, $d\Omega_q$, we obtain

$$\mathcal{I}(\mathbf{k},\,\tau) \simeq \int_k^{k_{\rm BR}} dp\, p^2 (A_1 + A_2 + A_3)\,, \tag{A.5}$$

with

$$
\begin{aligned}
A_1 &\simeq .9 \times \frac{e^{6\pi\xi_{BR}}}{k_{\rm BR}^{6\pi\delta}}\, p^{6\pi\delta-\frac{19}{2}}\, k^{7/2}\,, \\
A_2 &\simeq .9 \times \frac{e^{6\pi\xi_{BR}}}{k_{\rm BR}^{6\pi\delta}}\, p^{6\pi\delta-\frac{19}{2}}\, (p^{7/2}-k^{7/2}) \sim \frac{e^{6\pi\xi_{BR}}}{k_{\rm BR}^{6\pi\delta}}\, p^{6\pi\delta-\frac{19}{2}}\, p^{7/2}\,, \\
A_3 &\simeq \frac{8\times 10^{-6}}{\delta-1/\pi}\,\frac{e^{6\pi\xi_{BR}}}{k_{\rm BR}^{6\pi\delta}}\,(-p^{6\pi\delta-6}+{k_{\rm BR}}^{6\pi\delta-6}) \\
&\sim \frac{8\times 10^{-6}}{|\delta-1/\pi|}\,\frac{e^{6\pi\xi_{BR}}}{k_{\rm BR}^{6\pi\delta}} \times \begin{cases} {k_{\rm BR}}^{6\pi\delta-6}\,, & \text{if } \delta > 1/\pi\,, \\ p^{6\pi\delta-6}\,, & \text{if } \delta < 1/\pi\,. \end{cases}
\end{aligned} \tag{A.6}
$$

Finally, performing the integral on p we have $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$, with

$$
\mathcal{I}_1 \simeq \frac{5\times 10^{-2}}{|\delta-13/(12\pi)|}\,\frac{e^{6\pi\,\xi_{\rm BR}}}{k^3} \times \begin{cases} (k/k_{\rm BR})^{13/2}\,, & \text{if } \delta > 13/(12\pi)\,, \\ (k/k_{\rm BR})^{6\pi\delta}\,, & \text{if } \delta < 13/(12\pi)\,, \end{cases}
$$

$$
\mathcal{I}_2 \simeq \frac{5\times 10^{-2}}{|\delta-1/(2\pi)|}\,\frac{e^{6\pi\,\xi_{\rm BR}}}{k^3} \times \begin{cases} (k/k_{\rm BR})^{3}\,, & \text{if } \delta > 1/(2\pi), \\ (k/k_{\rm BR})^{6\pi\,\delta}\,, & \text{if } \delta < 1/(2\pi)\,, \end{cases}
$$

$$
\mathcal{I}_3 \simeq \frac{3\times 10^{-6}}{|\delta-1/\pi|}\,\frac{e^{6\pi\,\xi_{\rm BR}}}{k^3} \times \begin{cases} (k/k_{\rm BR})^{3}\,, & \text{if } \delta > 1/\pi, \\ \frac{0.2}{|\delta-1/(2\pi)|}\,(k/k_{\rm BR})^{3}\,, & \text{if } 1/(2\pi) < \delta < 1/\pi, \\ \frac{0.2}{|\delta-1/(2\pi)|}\,(k/k_{\rm BR})^{6\pi\,\delta}\,, & \text{if } \delta < 1/(2\pi)\,. \end{cases} \tag{A.7}
$$

In particular, we find that for $\delta < \frac{1}{2\pi}$ the correlator is proportional to the sixth power of the amplitude of the gauge field when the scale k left the horizon, i.e. $e^{\pi(\xi_{\rm BR}-\delta\,\log(k_{\rm BR}/k)}$. On the other hand, for $\delta > \frac{1}{2\pi}$, the result is proportional to the sixth power of the gauge field at the beginning of the strong backreaction regime. From the definition (A.2) we deduce that the integral is dominated by the value of ξ when scales k leave the horizon if $\xi_{\rm BR} - \xi_k \lesssim \frac{N_k - N_{\rm BR}}{2\pi}$, it is dominated by the scales that left the horizon at the beginning of the strong backreaction regime. While this result is based on the parameterization (A.2), we expect it to be generally valid as long as $\xi(\tau)$ monotonically increases during inflation.

References

[1] K. Freese, J.A. Frieman and A.V. Olinto, *Natural inflation with pseudo-Nambu-Goldstone bosons*, *Phys. Rev. Lett.* **65** (1990) 3233 [INSPIRE].

[2] N. Barnaby and M. Peloso, *Large Nongaussianity in Axion Inflation*, *Phys. Rev. Lett.* **106** (2011) 181301 [arXiv:1011.1500] [INSPIRE].

[3] R. Namba et al., *Scale-dependent gravitational waves from a rolling axion*, *JCAP* **01** (2016) 041 [arXiv:1509.07521] [INSPIRE].

[4] A. Linde, S. Mooij and E. Pajer, *Gauge field production in supergravity inflation: Local non-Gaussianity and primordial black holes*, *Phys. Rev. D* **87** (2013) 103506 [arXiv:1212.1693] [INSPIRE].

[5] L. Sorbo, *Parity violation in the Cosmic Microwave Background from a pseudoscalar inflaton*, *JCAP* **06** (2011) 003 [arXiv:1101.1525] [INSPIRE].

JCAP10(2024)024

[6] J.L. Cook and L. Sorbo, *Particle production during inflation and gravitational waves detectable by ground-based interferometers*, *Phys. Rev. D* **85** (2012) 023534 [*Erratum ibid.* **86** (2012) 069901] [arXiv:1109.0022] [INSPIRE].

[7] M.M. Anber and E. Sabancilar, *Hypermagnetic Fields and Baryon Asymmetry from Pseudoscalar Inflation*, *Phys. Rev. D* **92** (2015) 101501 [arXiv:1507.00744] [INSPIRE].

[8] W.D. Garretson, G.B. Field and S.M. Carroll, *Primordial magnetic fields from pseudoGoldstone bosons*, *Phys. Rev. D* **46** (1992) 5346 [hep-ph/9209238] [INSPIRE].

[9] M.M. Anber and L. Sorbo, *N-flationary magnetic fields*, *JCAP* **10** (2006) 018 [astro-ph/0606534] [INSPIRE].

[10] E. Pajer and M. Peloso, *A review of Axion Inflation in the era of Planck*, *Class. Quant. Grav.* **30** (2013) 214002 [arXiv:1305.3557] [INSPIRE].

[11] J. Garcia-Bellido, M. Peloso and C. Unal, *Gravitational waves at interferometer scales and primordial black holes in axion inflation*, *JCAP* **12** (2016) 031 [arXiv:1610.03763] [INSPIRE].

[12] NANOGrav collaboration, *The NANOGrav 15 yr Data Set: Evidence for a Gravitational-wave Background*, *Astrophys. J. Lett.* **951** (2023) L8 [arXiv:2306.16213] [INSPIRE].

[13] EPTA and InPTA: collaborations, *The second data release from the European Pulsar Timing Array — III. Search for gravitational wave signals*, *Astron. Astrophys.* **678** (2023) A50 [arXiv:2306.16214] [INSPIRE].

[14] D.J. Reardon et al., *Search for an Isotropic Gravitational-wave Background with the Parkes Pulsar Timing Array*, *Astrophys. J. Lett.* **951** (2023) L6 [arXiv:2306.16215] [INSPIRE].

[15] KAGRA et al. collaborations, *Search for anisotropic gravitational-wave backgrounds using data from Advanced LIGO and Advanced Virgo's first three observing runs*, *Phys. Rev. D* **104** (2021) 022005 [arXiv:2103.08520] [INSPIRE].

[16] LISA Cosmology Working Group collaboration, *Probing anisotropies of the Stochastic Gravitational Wave Background with LISA*, *JCAP* **11** (2022) 009 [arXiv:2201.08782] [INSPIRE].

[17] M. Geller, A. Hook, R. Sundrum and Y. Tsai, *Primordial Anisotropies in the Gravitational Wave Background from Cosmological Phase Transitions*, *Phys. Rev. Lett.* **121** (2018) 201303 [arXiv:1803.10780] [INSPIRE].

[18] F. Schulze et al., *GW_CLASS: Cosmological Gravitational Wave Background in the cosmic linear anisotropy solving system*, *JCAP* **10** (2023) 025 [arXiv:2305.01602] [INSPIRE].

[19] P. Adshead et al., *Multimessenger cosmology: Correlating cosmic microwave background and stochastic gravitational wave background measurements*, *Phys. Rev. D* **103** (2021) 023532 [arXiv:2004.06619] [INSPIRE].

[20] S.-L. Cheng, W. Lee and K.-W. Ng, *Numerical study of pseudoscalar inflation with an axion-gauge field coupling*, *Phys. Rev. D* **93** (2016) 063510 [arXiv:1508.00251] [INSPIRE].

[21] A. Notari and K. Tywoniuk, *Dissipative Axial Inflation*, *JCAP* **12** (2016) 038 [arXiv:1608.06223] [INSPIRE].

[22] O.O. Sobol, E.V. Gorbar and S.I. Vilchinskii, *Backreaction of electromagnetic fields and the Schwinger effect in pseudoscalar inflation magnetogenesis*, *Phys. Rev. D* **100** (2019) 063523 [arXiv:1907.10443] [INSPIRE].

[23] G. Dall'Agata, S. González-Martín, A. Papageorgiou and M. Peloso, *Warm dark energy*, *JCAP* **08** (2020) 032 [arXiv:1912.09950] [INSPIRE].

[24] V. Domcke, V. Guidetti, Y. Welling and A. Westphal, *Resonant backreaction in axion inflation*, *JCAP* **09** (2020) 009 [arXiv:2002.02952] [INSPIRE].

[25] A. Caravano, E. Komatsu, K.D. Lozanov and J. Weller, *Lattice simulations of axion-U(1) inflation*, *Phys. Rev. D* **108** (2023) 043504 [arXiv:2204.12874] [INSPIRE].

[26] M. Peloso and L. Sorbo, *Instability in axion inflation with strong backreaction from gauge modes*, *JCAP* **01** (2023) 038 [arXiv:2209.08131] [INSPIRE].

[27] D.G. Figueroa, J. Lizarraga, A. Urio and J. Urrestilla, *Strong Backreaction Regime in Axion Inflation*, *Phys. Rev. Lett.* **131** (2023) 151003 [arXiv:2303.17436] [INSPIRE].

[28] J. Garcia-Bellido, A. Papageorgiou, M. Peloso and L. Sorbo, *A flashing beacon in axion inflation: recurring bursts of gravitational waves in the strong backreaction regime*, *JCAP* **01** (2024) 034 [arXiv:2303.13425] [INSPIRE].

[29] R. von Eckardstein et al., *Axion inflation in the strong-backreaction regime: decay of the Anber-Sorbo solution*, *JHEP* **11** (2023) 183 [arXiv:2309.04254] [INSPIRE].

[30] A. Caravano and M. Peloso, *Unveiling the nonlinear dynamics of a rolling axion during inflation*, arXiv:2407.13405 [INSPIRE].

[31] O. Iarygina, E.I. Sfakianakis, R. Sharma and A. Brandenburg, *Backreaction of axion-SU(2) dynamics during inflation*, *JCAP* **04** (2024) 018 [arXiv:2311.07557] [INSPIRE].

[32] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [astro-ph/0210603] [INSPIRE].

[33] N. Barnaby, R. Namba and M. Peloso, *Phenomenology of a Pseudo-Scalar Inflaton: Naturally Large Nongaussianity*, *JCAP* **04** (2011) 009 [arXiv:1102.4333] [INSPIRE].

[34] Planck collaboration, *Planck 2015 results. XX. Constraints on inflation*, *Astron. Astrophys.* **594** (2016) A20 [arXiv:1502.02114] [INSPIRE].

[35] C. Caprini and D.G. Figueroa, *Cosmological Backgrounds of Gravitational Waves*, *Class. Quant. Grav.* **35** (2018) 163001 [arXiv:1801.04268] [INSPIRE].

[36] A. Ricciardone et al., *Cross-Correlating Astrophysical and Cosmological Gravitational Wave Backgrounds with the Cosmic Microwave Background*, *Phys. Rev. Lett.* **127** (2021) 271301 [arXiv:2106.02591] [INSPIRE].

[37] M. Braglia and S. Kuroyanagi, *Probing prerecombination physics by the cross-correlation of stochastic gravitational waves and CMB anisotropies*, *Phys. Rev. D* **104** (2021) 123547 [arXiv:2106.03786] [INSPIRE].

[38] V. Alba and J. Maldacena, *Primordial gravity wave background anisotropies*, *JHEP* **03** (2016) 115 [arXiv:1512.01531] [INSPIRE].

[39] C.R. Contaldi, *Anisotropies of Gravitational Wave Backgrounds: A Line Of Sight Approach*, *Phys. Lett. B* **771** (2017) 9 [arXiv:1609.08168] [INSPIRE].

[40] N. Bartolo et al., *Characterizing the cosmological gravitational wave background: Anisotropies and non-Gaussianity*, *Phys. Rev. D* **102** (2020) 023527 [arXiv:1912.09433] [INSPIRE].

[41] G. Mentasti, C.R. Contaldi and M. Peloso, *Intrinsic Limits on the Detection of the Anisotropies of the Stochastic Gravitational Wave Background*, *Phys. Rev. Lett.* **131** (2023) 221403 [arXiv:2301.08074] [INSPIRE].

[42] Y. Cui, S. Kumar, R. Sundrum and Y. Tsai, *Unraveling cosmological anisotropies within stochastic gravitational wave backgrounds*, *JCAP* **10** (2023) 064 [arXiv:2307.10360] [INSPIRE].

[43] L. Bethke, D.G. Figueroa and A. Rajantie, *Anisotropies in the Gravitational Wave Background from Preheating*, *Phys. Rev. Lett.* **111** (2013) 011301 [arXiv:1304.2657] [INSPIRE].

Prepared for submission to JCAP

Gravitational wave anisotropies from axion inflation

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Abstract. An important prediction of inflation is the production of a primordial stochastic gravitational wave background. Observing this background is challenging due to the weakness of the signal and the simultaneous presence of an astrophysical background generated by many unresolved late-time sources. One possible way to distinguish between the two is to examine their anisotropies. In this paper we calculate the primordial correlation function of gravitational wave anisotropies in the cosmological background generated by axion inflation, where the inflaton is a pseudo-Nambu–Goldstone boson coupled to gauge fields. In this scenario, tensor modes arise not only from the standard amplification of vacuum fluctuations present in any inflationary model, but also from the inverse decay process of the produced gauge fields. The correlator of gravitational wave anisotropies consists therefore of two main components: the contribution from vacuum tensor modes and the contribution from tensor modes sourced by the gauge fields. Our analysis shows that, while the former, previously studied in the literature, is negligible, the one arising from the sourced tensor modes, normalized by the fractional energy density at interferometer frequencies, can reach values as large as $\mathcal{O}(10^{-1})$. This result shows that axion inflation can generate large anisotropies with the potential to be observed by gravitational wave detectors within a reasonable time frame.

Contents

1 Introduction

Gravitational waves (GW) have recently received a lot of attention, especially after their first detection in September 2015 by the LIGO/Virgo collaboration [1] and the more recent evidence for a stochastic gravitational wave background (SGWB) reported by pulsar timing array (PTA) measurements [2–4]. This background could be either astrophysical (AGWB), generated from unresolved astrophysical sources in later epochs, or cosmological (CGWB), which originates from phenomena in the early Universe such as inflation, reheating, phase transitions, primordial black holes, or topological defects [5–7]. Investigating the CGWB provides information about the dynamics prevalent at the time of generation of the primordial GWs, offering a unique window into the physics of the early Universe.

In this paper we focus on cosmological gravitational waves originated from a particular inflationary model known as axion inflation [8]. In axion inflation, the inflaton is a pseudo-Nambu-Goldstone boson exhibiting a broken shift symmetry, i.e., a symmetry under the transformation $\phi \to \phi + \text{const.}$, which protects the flatness of the potential against large radiative corrections. In this model, proposed for the first time in 1990 as natural inflation [9], the inflaton interacts with gauge fields through the coupling $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}/f$, where f is the axion decay constant. As a consequence, the gauge field quanta get amplified and in turn produce scalar and tensor fluctuations through a process of inverse decay. Therefore, in axion inflation, both scalar fluctuations and gravitational waves are generated through two distinct mechanisms: first, from the vacuum, via the standard amplification process, and second, as a consequence of the production of gauge fields, through an inverse decay process. Remarkably, because of the parity-violating nature of the system, only photons of a given helicity are produced [10], implying that the sourced gravitational waves of different helicities have different amplitudes.

The phenomenological predictions of axion inflation are multiple, including nongaussianities [11], deviations from scale invariance [12], formation of primordial black holes [13], baryogenesis [14], generation of cosmologically relevant magnetic fields [15, 16], as well as generation of primordial chiral gravitational waves at CMB [10] or interferometer [17] frequencies. In particular, we expect these gravitational waves to generate an SGWB of cosmological origin, the characterization of which is essential for distinguishing it from its astrophysical counterpart.

One method for characterizing the SGWB involves examining its anisotropies. In fact, the SGWB is expected to present small spatial fluctuations analogous to the temperature fluctuations of the CMB, the detection of which is a major challenge for the next generation of gravitational wave detectors [18, 19]. More importantly, these anisotropies may correlate with those of the CMB and the study of this cross-correlation provides a powerful way to distinguish between astrophysical and cosmological origins of the background [20–25].

In the specific context of axion inflation, reference [26] analyzed the correlation between the curvature perturbation $\zeta(\mathbf{x})$ and the gravitational energy density $\Omega_{GW}(\mathbf{x}) = \dot{h}_{ij}(\mathbf{x})\,\dot{h}_{ij}(\mathbf{x})/(12\,H_0^2)$. In axion inflation, both scalar fluctuations and gravitational waves have vacuum and sourced contributions. At the same time, the expansion of the Universe induces vacuum fluctuations in the inflaton, leading to spatial variations in the gauge field population, which in turn generate spatial fluctuations in the sourced gravitational waves. As a result, the sourced gravitational waves consist of two components: one that we denote as the *homogeneous* component, and the other as the component of *fluctuations*. The homogeneous component arises from the gauge field and depends on the zero mode of the rolling inflaton. In contrast, the fluctuations originate from the gauge field's inhomogeneities, which are, in turn, imprinted by the inflaton's fluctuations.

The correlator studied in [26] receives two contributions: one from the correlation of the sourced gravitational waves with the vacuum scalar fluctuations, and the other from the correlation of the sourced gravitational waves with the sourced scalar fluctuations. The former effect is generally dominant and the correlator, normalized by the amplitude of ζ and by the fractional energy in sourced gravitational waves at interferometer frequencies, turned out to be of the order of $10^{-4} \div 10^{-2}$. The observability of this correlation, influenced by the intrinsic variance of the isotropic component and instrumental noise [27, 28], depends not only on the overall gravitational wave energy density, but also on the amplitude of anisotropies in the gravitational wave spectra. Studies on preheating at the end of inflation and on baryogenesis suggest that these anisotropies may be large [29, 30].

In this work, we investigate the anisotropies in the gravitational wave spectra produced during axion inflation by computing the correlator $\langle\Omega_{\rm GW}(\mathbf{x})\Omega_{\rm GW}(\mathbf{y})\rangle$ of the gravitational wave energy densities. This correlator consists of two main contributions: one arising from the correlation of gravitational wave energy densities generated by the vacuum tensor modes, which we refer to as the *vacuum correlator*, and the other from the correlation of energy densities associated with the sourced tensor modes, called *sourced correlator*. Since the vacuum correlator has already been studied in the literature [19, 31, 32], we will only present it briefly using an analytical approach, and instead focus primarily on the sourced correlator.

The sourced correlator arises from three distinct contributions, reflecting the fact that the sourced gravitational waves are composed of a homogeneous component and fluctuations. The first contribution comes from the correlation of the homogeneous components, resulting in the *intrinsic correlator*. The second contribution comes from the correlation between the homogeneous components and the fluctuations, referred to as the *extrinsic correlator*. Finally, the third contribution arises from the correlation of the fluctuations and represents a higher-order contribution in the perturbations.

Our analysis shows that the sourced extrinsic correlator, normalized by the square of the fractional energy in sourced gravitational waves at interferometer frequencies, lies in the range $\mathcal{O}(10^{-5} - 10^{-1})$. In contrast, the sourced intrinsic correlator is significantly smaller, while the correlator of the sourced fluctuations is negligible. The vacuum correlator is also found to be small and unobservable. The relatively large value of the sourced extrinsic correlator, which is the main result of this paper, is particularly significant, as it implies that the resulting anisotropies lie within the observational reach of GW detectors.

The sourced gravitational waves studied in the sourced correlator are produced towards the end of axion inflation, when the amplitude of the gauge fields becomes large and they significantly backreact on the background inflationary evolution. Although the inflaton-gauge field dynamics is nonperturbative [33–43], the production of gravitational waves can be considered at the perturbative level. In [41], the authors showed that although backreaction can modify the dynamics of the system, the behavior of the sourced gravitational waves depends only on the velocity of the inflaton field, assuming inflaton inhomogeneities are neglected. Since our results are expressed entirely in terms of $\dot{\phi}(t)$, we assume them to remain valid even in the strong backreaction regime.

This paper is organized as follows. In Section 2, we review the model of axion inflation, explaining the mechanism of gauge field amplification and the resulting production of scalar fluctuations (Subsection 2.1) and gravitational waves (Subsection 2.2). In Section 3, we define the correlator of the gravitational wave energy densities. In Section 4, we present the sourced correlator, while Section 5 provides a brief overview of the vacuum correlator. Finally, in Section 6 we discuss our results and we conclude.

2 Overview of the axion inflation model

The action which describes our model of axion inflation is that of a pseudoscalar inflaton field ϕ minimally coupled to gravity and to a $U(1)$ gauge field A_μ

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} - \frac{\phi}{8\,f}\frac{\epsilon^{\mu\nu\rho\lambda}}{\sqrt{-g}}F_{\mu\nu}\,F_{\rho\lambda}\right], \tag{2.1}$$

where $g = \det(g_{\mu\nu})$, R is the Ricci scalar, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength, $\epsilon^{\mu\nu\rho\lambda}$ is the totally antisymmetric tensor defined by $\epsilon^{0123} = +1$, f is the axion decay constant and $V(\phi)$ is a generic inflationary potential.

The quantum scalar and tensor fluctuations produced during inflation are obtained by adding spatially varying perturbations to the inflaton and the metric, respectively. In particular, the curvature perturbation $\zeta \equiv -\frac{H}{\dot{\phi}_0}\delta\phi$, where the overdot denotes the derivative with respect to cosmic time t (in contrast to the prime, which denotes the derivative with respect to the conformal time τ), is related to the inflaton perturbations arising from

$$\phi(\mathbf{x},\,\tau) = \phi_0(\tau) + \delta\phi(\mathbf{x},\,\tau)\,. \tag{2.2}$$

Gravitational waves, on the other hand, are obtained by introducing spatially varying perturbations in the form of transverse traceless tensor modes, i.e. $h_{ij}(\mathbf{x},t)$ with $h_{ii} = \partial_i h_{ij} = 0$, to the de Sitter metric

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \left(\delta_{ij} + h_{ij}(\mathbf{x},\,\tau)\right)\,dx^i\,dx^j\right]\,, \tag{2.3}$$

where for repeated latin indices Einstein notation is used. The scale factor is $a(\tau) = -1/(H\,\tau)$, and it is set to be equal to unity at the end of inflation, i.e., at $\tau_e = -1/H$.

To proceed, we expand the Lagrangian density around the background solution, identified by $\phi_0(\tau)$ and $a(\tau)$, and then discard the terms of zeroth and first order in the perturbations. By choosing the Coulomb gauge, i.e. $A_0(\mathbf{x},\,\tau) = 0$ and $\partial_i A_i(\mathbf{x},\,\tau) = 0$, the perturbed Lagrangian takes the form

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\Phi'^2 - \frac{1}{2}\partial_k\Phi\,\partial_k\Phi + \frac{a''}{2\,a}\Phi^2 + \frac{1}{2}H'_{ij}\,H'_{ij} - \frac{1}{2}\partial_k H_{ij}\,\partial_k H_{ij} + \frac{a''}{2\,a}H_{ij}\,H_{ij} + \frac{1}{2}A'_i\,A'_i\\ &- \frac{1}{2}\partial_k A_i\,\partial_k A_i - \frac{\phi_0}{f}\epsilon^{ijk}\,A'_i\,\partial_j A_k - \frac{H_{ij}}{a\,M_P}\,\left[A'_i\,A'_j - \left(\partial_i A_k - \partial_k A_i\right)\left(\partial_j A_k - \partial_k A_j\right)\right]\\ &- \frac{\Phi}{f\,a}\epsilon^{ijk}\,A'_i\,\partial_j A_k\,,\end{aligned} \tag{2.4}$$

where we have expressed the scalar and tensor perturbations $\delta\phi$ and h_{ij} in terms of their canonically normalized versions

$$\begin{aligned}\Phi(\mathbf{x},\,\tau) &= a(\tau)\,\delta\phi(\mathbf{x},\,\tau)\,,\\ H_{ij}(\mathbf{x},\,\tau) &= \frac{M_P}{2}\,a(\tau)\,h_{ij}(\mathbf{x},\,\tau)\,.\end{aligned} \tag{2.5}$$

By varying the Lagrangian (2.4) with respect to A_i, Φ and H_{ij}, we obtain the equations of motion that govern the dynamics of the system

$$A''_i - \nabla^2 A_i - \frac{\phi'_0}{f}\epsilon^{ijk}\,\partial_j A_k = 0\,, \tag{2.6}$$

$$\Phi'' - \frac{a''}{a}\Phi - \nabla^2\Phi + \frac{1}{f\,a}\epsilon^{ijk}\,A'_i\,\partial_j A_k = 0\,, \tag{2.7}$$

$$H''_{ij} - \frac{a''}{a}H_{ij} - \nabla^2 H_{ij} + \frac{1}{a\,M_P}\,\left[A'_i\,A'_j - \left(\partial_i A_k - \partial_k A_i\right)\left(\partial_j A_k - \partial_k A_j\right)\right] = 0\,. \tag{2.8}$$

Equation (2.6) describes the evolution of the gauge fields. To study the amplification of gauge modes due to the rolling inflaton, we promote the classical field to an operator $\hat{A}_i(\mathbf{x},\,\tau)$, which we decompose into creation/annihilation operators $\hat{a}^\dagger_\lambda(\mathbf{k})/\hat{a}_\lambda(\mathbf{k})$, satisfying the usual commutation relations $[\hat{a}_\lambda(\mathbf{k}),\,\hat{a}^\dagger_{\lambda'}(\mathbf{q})] = \delta(\mathbf{k}-\mathbf{q})\,\delta_{\lambda,\lambda'}$, $[\hat{a}_\lambda(\mathbf{k}),\,\hat{a}_{\lambda'}(\mathbf{q})] = [\hat{a}^\dagger_\lambda(\mathbf{k}),\,\hat{a}^\dagger_{\lambda'}(\mathbf{q})] = 0$,

$$\hat{A}_i(\mathbf{x},\,\tau) = \int\frac{d\mathbf{k}}{(2\pi)^{3/2}}\sum_{\lambda=\pm} e^\lambda_i(\widehat{\mathbf{k}})\,e^{i\mathbf{k}\mathbf{x}}\left[A_\lambda(k,\,\tau)\,\hat{a}_\lambda(\mathbf{k}) + A^*_\lambda(k,\,\tau)\,\hat{a}^\dagger_\lambda(-\mathbf{k})\right]\,, \tag{2.9}$$

with the helicity projectors $e_i^\lambda(\widehat{\mathbf{k}})$ following the relations

$$\begin{aligned} &k_i\, e_i^\lambda(\widehat{\mathbf{k}}) = 0\,, && e_i^\lambda(\widehat{\mathbf{k}})^* = e_i^{-\lambda}(\widehat{\mathbf{k}}) = e_i^\lambda(-\widehat{\mathbf{k}})\,, \\ &i\epsilon_{ijk} k_j e_k^\lambda(\widehat{\mathbf{k}}) = \lambda\, k\, e_i^\lambda(\widehat{\mathbf{k}})\,, && e_i^\lambda(\widehat{\mathbf{k}}) e_i^{\lambda'}(\widehat{\mathbf{k}}) = \delta_{\lambda,\,-\lambda'}\,. \end{aligned} \tag{2.10}$$

Inserting eq. (2.9) into eq. (2.6) and defining

$$\xi \equiv \frac{\dot{\phi}_0}{2\, f\, H}\,, \tag{2.11}$$

we obtain the equation that governs the evolution of the mode functions $A_\lambda(k,\, \tau)$

$$\frac{d^2 A_\lambda(k,\, \tau)}{d\tau^2} + \left(k^2 + \lambda \frac{2\,\xi\, k}{\tau}\right) A_\lambda(k,\, \tau) = 0\,. \tag{2.12}$$

Depending on the sign of ξ, one of the two helicities, $\lambda = 1$ or $\lambda = -1$, develops tachyonic instability. Assuming, without loss of generality, $\xi > 0$ and keeping in mind that $\tau < 0$ throughout the entire inflationary phase, the negative helicity mode A_- has real frequencies that evolve adiabatically for all parameter values. As a result, A_- remains in its vacuum state and can therefore be neglected. On the other hand, the positive helicity mode A_+ can acquire imaginary frequencies, leading to exponential amplification.

More precisely, the solution of eq. (2.12) that reduces to positive frequency as $k\tau \to -\infty$ can be explicitly expressed in terms of the regular and irregular Coulomb wave functions, F_0 and G_0, as $A_\pm = \frac{1}{\sqrt{2\,k}}(i\, F_0(\pm\xi, -k\tau) + G_0(\pm\xi, -k\tau))$. Under the WKB approximation, the leading term in the solution for the tachyonic modes of A_+, in the range $\frac{1}{8\xi} \lesssim |k\,\tau| \lesssim 2\,\xi$ [11, 16], assuming $\xi \gtrsim O(1)$ throughout, takes the form

$$A_+(k,\, \tau) \simeq \frac{1}{\sqrt{2\,k}} \left(-\frac{k\,\tau}{2\,\xi}\right)^{1/4} e^{-2\sqrt{-2\xi k\tau} + \pi\,\xi}\,, \tag{2.13}$$

which can be generalized to the entire range $0 < |k\,\tau| < \infty$, as the observables of interest depend only on the range where the approximation is valid. The positive helicity mode of the gauge field is therefore amplified by a factor of $e^{\pi\xi}$ and can become very large even for moderate values of ξ.

The accelerated expansion of the background during axion inflation gives rise to the vacuum components of the scalar and tensor fluctuations, denoted respectively as $\delta\phi_{\rm V}$ and $h_{ij,{\rm V}}$, generated via the standard amplification process present in any inflationary model. The gauge fields amplified by the rolling zero mode of the inflaton are, on the other hand, responsible for the production of sourced scalar and tensor fluctuations through an inverse decay process, schematically represented as $\phi_0 \to A \to \{\delta\phi_{\rm S},\, h^0_{ij,{\rm S}}\}$. Additionally, vacuum scalar fluctuations of the background inflaton induce fluctuations in the population of the produced gauge fields, resulting in fluctuations in the sourced tensor modes, schematically $\delta\phi_{\rm V} \to \delta A \to \delta h_{ij,{\rm S}}$. As a result of these mechanisms, analyzed in the next two Subsections, we obtain the fluctuations

$$\delta\phi = \delta\phi_{\rm V} + \delta\phi_{\rm S} \quad \text{and} \quad h_{ij} = h_{ij,{\rm V}} + h_{ij,{\rm S}}\,, \tag{2.14}$$

with

$$h_{ij,{\rm S}} = h^0_{ij,{\rm S}} + \delta h_{ij,{\rm S}}\,. \tag{2.15}$$

The same decomposition holds also for the normalized versions of the fluctuations given in (2.5).

2.1 Scalar fluctuations

The production of scalar fluctuations in the axion inflation model is described by eq. (2.7). The solution of the homogeneous part of the equation corresponds to the vacuum fluctuations $\Phi_{\rm V}$, generated as a result of the accelerated expansion of the background. This vacuum component can be quantized through the standard quantization of the free Lagrangian as

$$\hat{\Phi}_{\rm V}(\mathbf{x},\,\tau) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}} \left[\Phi_{\rm V}(k,\,\tau)\,\hat{a}(\mathbf{k}) + \Phi^*_{\rm V}(k,\,\tau)\,\hat{a}^\dagger(-\mathbf{k})\right]\,, \tag{2.16}$$

where

$$\Phi_{\rm V}(k,\,\tau) \equiv \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}\,. \tag{2.17}$$

On the other hand, the particular solution of eq. (2.7) corresponds to the sourced fluctuations, $\Phi_{\rm S}$, produced by the amplified gauge fields. This solution, determined using the retarded propagator

$$G_k(\tau,\tau') = \frac{(1 + k^2\,\tau\,\tau')\,\sin(k\,(\tau-\tau')) + k\,(\tau'-\tau)\,\cos(k\,(\tau-\tau'))}{k^3\,\tau\,\tau'}\,\Theta(\tau-\tau')\,, \tag{2.18}$$

where Θ denotes the Heaviside step function, is found to be

$$\Phi_{\rm S}(\mathbf{q},\,\tau) \equiv i\int d\tau'\,G_q(\tau,\,\tau')\frac{H\tau'}{f}\,\epsilon^{ijk}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\,A'_i(\mathbf{p},\,\tau')\,(\mathbf{q}-\mathbf{p})_j A_k(\mathbf{q}-\mathbf{p},\,\tau')\,. \tag{2.19}$$

The complete solution is the sum of the two components, i.e. $\Phi = \Phi_{\rm V} + \Phi_{\rm S}$, which gives rise to the curvature perturbation $\zeta = \zeta_{\rm V} + \zeta_{\rm S}$. The power spectrum of the curvature perturbation, $\mathcal{P}_\zeta$, defined through the two point function

$$\langle\zeta(\mathbf{k})\,\zeta(\mathbf{q})\rangle = \frac{2\pi^2}{k^3}\,\mathcal{P}_\zeta(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{q})\,, \tag{2.20}$$

is the sum of the power spectra corresponding to the vacuum and sourced components, denoted as $\mathcal{P}_{\zeta,\rm V}$ and $\mathcal{P}_{\zeta,\rm S}$, respectively. The vacuum power spectrum associated with the mode functions in (2.17), evaluated at the end of inflation and in the large scale limit, is given by

$$\mathcal{P}_{\zeta,\rm V} = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}_0^2}\,|\Phi_{\rm V}(k,\,\tau_e)|^2 \xrightarrow[k\ll H]{} \frac{H^4}{4\pi^2\,\dot{\phi}_0^2}\,, \tag{2.21}$$

while, the sourced power spectrum corresponding to (2.19), for $\xi \gtrsim 3$, is found to be [11]

$$\mathcal{P}_{\zeta,\rm S} = \frac{k^3}{2\pi^2}\frac{H^2}{\dot{\phi}_0^2}\,|\Phi_{\rm S}(k,\,\tau_e)|^2 \xrightarrow[k\ll H]{} 4.8\times10^{-8}\,\frac{H^8}{\dot{\phi}_0^4}\,\frac{e^{4\pi\xi}}{\xi^6}\,. \tag{2.22}$$

The CMB observations, particularly from the Planck satellite, have placed important constraints on nongaussianities at large scales, which are consistent with the predictions of single-field inflationary models. Specifically, the parameter $f_{\rm NL}$ used to quantify nongaussianity and defined through the three-point correlation function of the curvature perturbation

$$\langle\zeta(\mathbf{k}_1)\,\zeta(\mathbf{k}_2)\,\zeta(\mathbf{k}_3)\rangle = \frac{3}{10}\,(2\pi)^{5/2}\,f_{\rm NL}(k_1,\,k_2,\,k_3)\,\mathcal{P}_\zeta^2\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,\frac{k_1^3+k_2^3+k_3^3}{k_1^3\,k_2^3\,k_3^3}\,, \tag{2.23}$$

in the context of single-field, slow-roll inflation, is predicted to be of the order of the slow-roll parameters [44]. In the model of axion inflation the sourced curvature perturbations can lead to large nongaussianities. These perturbations arise from gauge fields through an inverse decay process, which maximizes the nongaussian effects in the equilateral configuration, i.e., when $k_1 = k_2 = k_3$. In this configuration, the nongaussianity parameter $f_{\rm NL}^{\rm equil}$ is given by [11]

$$f_{\rm NL}^{\rm equil} \simeq 7.1 \times 10^5 \, \frac{H^{12}}{\dot{\phi}^6} \, \frac{e^{6\pi\xi}}{\xi^9} \, . \tag{2.24}$$

For large values of $\xi_{\rm CMB}$, i.e., the value of ξ when CMB scales exit the horizon, the parameter $f_{\rm NL}^{\rm equil}$ exceeds the observational bounds on nongaussianity. In order to reproduce the observations it is necessary that $\xi_{\rm CMB} \lesssim 2.5$ [45]. As a result, the sourced power spectrum (2.22) at this time is significantly suppressed and becomes subdominant compared to the vacuum contribution, i.e. $\mathcal{P}_{\zeta,{\rm S}} \ll \mathcal{P}_{\zeta,{\rm V}}$. The constraint on ξ imposes a restriction on the amplitude of the produced gauge field, which in turn must remain relatively small. Since the gauge fields are responsible for generating the sourced tensor modes, as described in the next Subsection, these modes will also be small at this stage.

2.2 Tensor fluctuations

A similar analysis holds also for tensor fluctuations, which correspond to gravitational waves. In this case, the homogeneous and particular solutions of eq. (2.8) correspond, respectively, to the vacuum fluctuations $H_{ij,{\rm V}}$, generated by the expanding inflationary background, and the sourced fluctuations $H_{ij,{\rm S}}$, produced through the inverse decay of the gauge fields. The vacuum component can be quantized once again through the standard quantization process, starting from the free Lagrangian as

$$\hat{H}_{ij,{\rm V}}(\mathbf{x},\,\tau) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=\pm} e_{ij}^{\lambda}(\widehat{\mathbf{k}})\, e^{i\mathbf{k}\mathbf{x}} \left[H_{\rm V}^{\lambda}(k,\,\tau)\, \hat{a}_{\lambda}(\mathbf{k}) + H_{\rm V}^{\lambda\,*}(k,\,\tau)\, \hat{a}_{\lambda}^{\dagger}(-\mathbf{k}) \right] , \tag{2.25}$$

where

$$H_{\rm V}^{\pm}(k,\,\tau) \equiv \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) e^{-ik\tau} \, , \tag{2.26}$$

and

$$e_{ij}^{\lambda}(\widehat{\mathbf{k}}) = e_i^{\lambda}(\widehat{\mathbf{k}})\, e_j^{\lambda}(\widehat{\mathbf{k}}). \tag{2.27}$$

Using again the retarded propagator (2.18) we find the particular solution as

$$H_{ij,{\rm S}}(\mathbf{q},\,\tau) \equiv \int d\tau'\, G_q(\tau,\tau')\, \frac{H\,\tau'}{M_P} \int \frac{d\mathbf{p}}{(2\pi)^{3/2}}\, A_i'(\mathbf{p},\,\tau')\, A_j'(\mathbf{q}-\mathbf{p},\,\tau') \, , \tag{2.28}$$

which has been simplified by the fact that the electric field dominates over the magnetic field in strength. The complete solution is again the sum of the two components, i.e. $H_{ij} = H_{ij,{\rm V}} + H_{ij,{\rm S}}$, which give rise, respectively, to the vacuum and sourced power spectra [10]

$$\mathcal{P}_h = \mathcal{P}_{h,{\rm V}} + \mathcal{P}_{h,{\rm S}} \simeq \frac{2\,H^2}{\pi^2\,M_P^2} + 8.7 \times 10^{-8}\, \frac{H^4}{M_P^4}\, \frac{e^{4\pi\xi}}{\xi^6} \, . \tag{2.29}$$

The scales relevant to the observation of gravitational waves, i.e., those measured by gravitational wave detectors, exit the horizon much later than the CMB scales, closer to the end of inflation. At these later times, the parameter ξ, which typically increases (slowly) during inflation, reaches values larger than $\xi_{\rm CMB}$. Since the power spectrum of the sourced gravitational waves depends exponentially on ξ, at these later times we can have $\mathcal{P}_{h,{\rm S}} > \mathcal{P}_{h,{\rm V}}$, in contrast to what occurs when the CMB modes leave the horizon. The exponential amplification of the sourced power spectrum makes it possible for the gravitational waves to be directly detectable by current or future GW detectors [17].

The sourced gravitational waves produced through the mechanism described above are decomposed into a homogeneous part, which depends entirely on the background rolling inflaton ϕ_0, and fluctuations, which are caused by the vacuum fluctuations $\delta\phi_{\rm V}$, as described in the paragraph above eq. (2.15). To first order in the vacuum-amplified inflaton fluctuations $\delta\phi_{\rm V}$ and in the limit in which the wavelength of $\delta\phi_{\rm V}$ is much larger than that of $h_{ij,{\rm S}}$, the sourced gravitational field decomposition in homogeneous part and fluctuations (2.15) takes the form

$$h_{ij,{\rm S}}(\mathbf{x},\,\phi(\mathbf{x})) = h^0_{ij,{\rm S}}(\mathbf{x}) + \delta h_{ij,{\rm S}}(\mathbf{x}) \equiv h^0_{ij,{\rm S}}(\mathbf{x},\,\phi_0) + \frac{\partial h^0_{ij,{\rm S}}(\mathbf{x},\,\phi_0)}{\partial\phi_0}\,\delta\phi_{\rm V}(\mathbf{x})\,. \tag{2.30}$$

Since $h^0_{ij,{\rm S}}(\mathbf{x},\,\phi_0) \propto e^{2\pi\xi}$, $\delta h_{ij,{\rm S}}$ becomes

$$\delta h_{ij,{\rm S}}(\mathbf{x}) = -2\pi \frac{d\,\xi}{d\phi_0}\,\frac{\dot\phi_0}{H}\,\zeta_{\rm V}(\mathbf{x})\,h^0_{ij,{\rm S}}(\mathbf{x})\,. \tag{2.31}$$

Assuming $\dot\phi_0 > 0$, $V' < 0$, we have

$$\xi \equiv \frac{\dot\phi_0}{2\,fH} \simeq -\frac{V'}{6\,fH^2} = -\frac{M_P^2}{2\,f}\frac{V'}{V}\,, \tag{2.32}$$

and using the usual slow roll parameters $\epsilon = \frac{M_P^2}{2}\,\frac{V'^2}{V^2}$ and $\eta = M_P^2\frac{V''}{V}$, the derivative of ξ with respect to the background inflaton becomes

$$\frac{d\xi}{d\phi_0} = -\frac{M_P^2}{2\,f}\left(\frac{V''}{V} - \frac{V'^2}{V^2}\right) = \left(\epsilon - \frac{\eta}{2}\right)\frac{1}{f}\,. \tag{2.33}$$

Using again eq. (2.11) and considering that the parameter ξ typically takes the values $2 \div 5$, and that the quantity $(2\epsilon - \eta)$ is of the order of $10^{-2} \div 10^{-1}$, we have

$$2\pi\frac{d\,\xi}{d\phi_0}\,\frac{\dot\phi_0}{H} = 2\pi\,(2\,\epsilon - \eta)\,\xi\, \sim \mathcal{O}\,(.1 \div 3)\;. \tag{2.34}$$

We can now study the anisotropies in the gravitational wave background produced in axion inflation by estimating the correlator of the gravitational wave energy densities. This correlator receives two contributions: the vacuum correlator, which corresponds to the gravitational wave energy densities of the vacuum tensor modes, and the sourced correlator, which corresponds to the energy densities of the sourced tensor modes. The sourced correlator can further be decomposed into three components: the intrinsic correlator, the extrinsic correlator and the correlator of the fluctuations, which however will be neglected being very small. In Section 3, we present the general form of the correlator, while in Sections 4 and 5, we analyze the sourced and vacuum correlators, respectively.

3 The correlator of the gravitational wave energy densities

The normalized correlator of the gravitational wave energy densities is defined as

$$\begin{aligned}\mathcal{C}_{\Omega\Omega}(\mathbf{k}) &= \frac{1}{\Omega_{GW}^2}\frac{k^3}{2\pi^2}\int d\mathbf{y}\, e^{-i\mathbf{k}\mathbf{y}}\,\langle \Omega_{GW}(\mathbf{x}+\mathbf{y},\, t_0)\,\Omega_{GW}(\mathbf{x},\, t_0)\rangle \\ &= \frac{1}{\Omega_{GW}^2}\frac{k^3}{2\pi^2}\,\langle \Omega_{GW}(\mathbf{k},\, t_0)\,\Omega_{GW}(-\mathbf{k},\, t_0)\rangle' ,\end{aligned} \tag{3.1}$$

where t_0 is the present value of the cosmic time, $\Omega_{GW} \simeq \frac{\Omega^0_{\rm rad}}{24}\,\mathcal{P}_h(k_{\rm INT})$ with $\Omega^0_{\rm rad} \simeq 8.2\times 10^{-5}$ is the fractional energy in gravitational waves at interferometer frequencies [6] and $\langle \dots\rangle'$ represents the correlator stripped of the Dirac delta. Considering the explicit expression $\Omega_{GW}(\mathbf{k}, t_0) = \frac{1}{12\,H_0^2}\int \frac{d\mathbf{p}}{(2\pi)^{3/2}}\,|\mathbf{k}-\mathbf{p}|\,p\,h_{ab}(\mathbf{k}-\mathbf{p},\, t_0)h_{ab}(\mathbf{p},\, t_0)$ for the gravitational wave energy density, and defining $\Omega = 12\,H_0^2\,\Omega_{GW}$, the correlator becomes

$$\begin{aligned}\mathcal{C}_{\Omega\Omega}(\mathbf{k}) = \frac{1}{\Omega^2}\frac{k^3}{2\pi^2}\int &\frac{d\mathbf{p}_1\,d\mathbf{p}_2}{(2\pi)^3}\,|\mathbf{k}-\mathbf{p}_1|\,p_1\,|\mathbf{k}+\mathbf{p}_2|\,p_2 \\ &\times \langle h_{ab}(\mathbf{k}-\mathbf{p}_1,\, t_0)\,h_{ab}(\mathbf{p}_1,\, t_0)\,h_{cd}(-\mathbf{k}-\mathbf{p}_2,\, t_0)\,h_{cd}(\mathbf{p}_2,\, t_0)\rangle' .\end{aligned} \tag{3.2}$$

The current gravitational wave amplitude is related to its primordial value, calculated at the time t_e when inflation ends, through the transfer function: $h_{ab}(\mathbf{k},\, t_0) = T(k)\,h_{ab}(\mathbf{k},\, t_e)$. For simplicity, from now on we will write $h_{ab}(\mathbf{k},\, t_e)$ simply as $h_{ab}(\mathbf{k})$, with the understanding that it refers to the value the tensor mode takes at the end of inflation. If we further define $\hat{T}(k) = k\,T(k)$, we can eventually express the correlator as

$$\mathcal{C}_{\Omega\Omega}(\mathbf{k}) = \frac{1}{\Omega^2}\frac{k^3}{2\pi^2}\int \frac{d\mathbf{p}_1\,d\mathbf{p}_2}{(2\pi)^3}\,\hat{T}(k_1)\,\hat{T}(k_2)\,\hat{T}(k_3)\,\hat{T}(k_4)\,\langle h_{ab}(\mathbf{k}_1)\,h_{ab}(\mathbf{k}_2)\,h_{cd}(\mathbf{k}_3)\,h_{cd}(\mathbf{k}_4)\rangle' , \tag{3.3}$$

with $\mathbf{k}_1 = \mathbf{k}-\mathbf{p}_1$, $\mathbf{k}_2 = \mathbf{p}_1$, $\mathbf{k}_3 = -\mathbf{k}-\mathbf{p}_2$ and $\mathbf{k}_4 = \mathbf{p}_2$. The integration must be performed in the regime of large momenta, i.e. $p \gg k_{eq}$, where k_{eq} is the scale that reentered the horizon at matter-radiation equality [6], since these are the momenta to which gravitational wave detectors are sensitive. For these modes, which exited the horizon towards the end of inflation and reentered during radiation domination, the transfer function takes the form

$$\hat{T}(k) = \hat{T}_r = \frac{3\,H_0\,\sqrt{\Omega^0_{\rm rad}}}{4\sqrt{2}} . \tag{3.4}$$

In the following, when we explicitly evaluate the integrals in the large-momentum regime, we will denote the corresponding correlator with the subscript *l.m.*. Finally, since we are interested in large scales, the momentum k at which we evaluate the correlator is very small compared to the momenta over which we integrate, and is of the order of the scalar large-scale perturbation scale, i.e. $k \sim k_{\rm CMB}$.

4 Sourced correlator

For the normalized sourced correlator eq. (3.1) takes the form

$$\mathcal{C}^{\rm S}_{\Omega\Omega}(\mathbf{k}) = \frac{1}{\Omega_{GW,{\rm S}}^2}\frac{k^3}{2\pi^2}\int d\mathbf{y}\, e^{-i\mathbf{k}\mathbf{y}}\,\langle \Omega_{GW,{\rm S}}(\mathbf{x}+\mathbf{y},\, t_0)\,\Omega_{GW,{\rm S}}(\mathbf{x},\, t_0)\rangle , \tag{4.1}$$

with the fractional energy at interferometer scales for the sourced component being

$$\Omega_{GW,\mathrm{S}} \simeq \frac{\Omega^0_{\mathrm{rad}}}{24}\,\mathcal{P}_{h,\mathrm{S}}(k_{\mathrm{INT}}) \quad \text{with} \quad \mathcal{P}_{h,\mathrm{S}}(k_{\mathrm{INT}}) = 8.7\times 10^{-8}\,\frac{H^4}{M_P^4}\,\frac{e^{4\pi\xi_{\mathrm{INT}}}}{\xi^6_{\mathrm{INT}}}, \tag{4.2}$$

and

$$\Omega_{\mathrm{S}} = 12\, H_0^2\,\Omega_{GW,\mathrm{S}}\,. \tag{4.3}$$

Expression (3.3) for the sourced correlator takes the form

$$\mathcal{C}^{\mathrm{S}}_{\Omega\Omega} = \frac{1}{\Omega^2_{\mathrm{S}}}\frac{k^3}{2\pi^2}\int \frac{d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^3}\,\hat{T}(k_1)\,\hat{T}(k_2)\,\hat{T}(k_3)\,\hat{T}(k_4)\,\langle h_{ab,\mathrm{S}}(\mathbf{k}_1)\,h_{ab,\mathrm{S}}(\mathbf{k}_2)\,h_{cd,\mathrm{S}}(\mathbf{k}_3)\,h_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle'\,, \tag{4.4}$$

with $\mathbf{k}_1 = \mathbf{k}-\mathbf{p}_1$, $\mathbf{k}_2 = \mathbf{p}_1$, $\mathbf{k}_3 = -\mathbf{k}-\mathbf{p}_2$ and $\mathbf{k}_4 = \mathbf{p}_2$. By substituting the decomposition of the sourced tensor modes in homogeneous part and fluctuations defined in (2.30)-(2.31) into the four-point function present in (4.4) we obtain

$$\begin{aligned}
&\langle h_{ab,\mathrm{S}}(\mathbf{k}_1)\,h_{ab,\mathrm{S}}(\mathbf{k}_2)\,h_{cd,\mathrm{S}}(\mathbf{k}_3)\,h_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle = \langle h^0_{ab,\mathrm{S}}(\mathbf{k}_1)\,h^0_{ab,\mathrm{S}}(\mathbf{k}_2)\,h^0_{cd,\mathrm{S}}(\mathbf{k}_3)\,h^0_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle\\
&+\langle h^0_{ab,\mathrm{S}}(\mathbf{k}_1)\,h^0_{ab,\mathrm{S}}(\mathbf{k}_2)\,\delta h_{cd,\mathrm{S}}(\mathbf{k}_3)\,\delta h_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle + \langle \delta h_{ab,\mathrm{S}}(\mathbf{k}_1)\,\delta h_{ab,\mathrm{S}}(\mathbf{k}_2)\,h^0_{cd,\mathrm{S}}(\mathbf{k}_3)\,h^0_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle\\
&+4\langle h^0_{ab,\mathrm{S}}(\mathbf{k}_1)\,\delta h_{ab,\mathrm{S}}(\mathbf{k}_2)\,h^0_{cd,\mathrm{S}}(\mathbf{k}_3)\,\delta h_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle + \langle \delta h_{ab,\mathrm{S}}(\mathbf{k}_1)\,\delta h_{ab,\mathrm{S}}(\mathbf{k}_2)\,\delta h_{cd,\mathrm{S}}(\mathbf{k}_3)\,\delta h_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle\,.
\end{aligned} \tag{4.5}$$

Plugging eq. (4.5) back into the correlator (4.4), we identify three contributions: the intrinsic correlator, $\mathcal{C}^{\mathrm{I}}_{\Omega\Omega}$, which includes the first term in the r.h.s. of eq. (4.5) and involves only the homogeneous components; the extrinsic correlator, $\mathcal{C}^{\mathrm{E}}_{\Omega\Omega}$, which includes the sum of the next three terms in the r.h.s. of eq. (4.5), containing both homogeneous components and fluctuations; and the correlator of the fluctuations, $\mathcal{C}^{\mathrm{F}}_{\Omega\Omega}$, arising from the fifth term in the r.h.s. of eq. (4.5). The total sourced correlator of eq. (4.4) is therefore decomposed as

$$\mathcal{C}^{\mathrm{S}}_{\Omega\Omega} = \mathcal{C}^{\mathrm{I}}_{\Omega\Omega} + \mathcal{C}^{\mathrm{E}}_{\Omega\Omega} + \mathcal{C}^{\mathrm{F}}_{\Omega\Omega}. \tag{4.6}$$

To proceed, we substitute the Fourier transform of eq. (2.31)

$$\delta h_{ij,\mathrm{S}}(\mathbf{p}) = -2\pi\frac{d\xi}{d\phi_0}\frac{\dot{\phi}_0}{H}\int\frac{d\mathbf{q}}{(2\pi)^{3/2}}\,h_{ij,\mathrm{S}}(\mathbf{p}-\mathbf{q})\,\zeta_{\mathrm{V}}(\mathbf{q})\,, \tag{4.7}$$

which is valid, strictly speaking, when $\mathbf{p} \gg \mathbf{q}$. This condition is generally satisfied, as scalar fluctuations are evaluated at CMB scales, which are much larger than the interferometer scales at which gravitational waves are measured. Then, using eq. (2.20) for the vacuum curvature perturbations we have

$$\mathcal{C}^{\mathrm{I}}_{\Omega\Omega} = \frac{k^3}{2\pi^2\Omega^2_{\mathrm{S}}}\int \frac{d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^3}\,\hat{T}(k_1)\,\hat{T}(k_2)\,\hat{T}(k_3)\,\hat{T}(k_4)\langle h^0_{ab,\mathrm{S}}(\mathbf{k}_1)\,h^0_{ab,\mathrm{S}}(\mathbf{k}_2)\,h^0_{cd,\mathrm{S}}(\mathbf{k}_3)\,h^0_{cd,\mathrm{S}}(\mathbf{k}_4)\rangle'\,, \tag{4.8}$$

$$
\mathcal{C}^{\rm E}_{\Omega\Omega} = \frac{k^3}{2\pi^2\Omega_{\rm S}^2}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot\phi_0}{H}\right)^2 \int \frac{d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3}{(2\pi)^6}\,\hat T(k_1)\,\hat T(k_2)\,\hat T(k_3)\,\hat T(k_4)\frac{2\,\pi^2}{p_3^3}\,\mathcal{P}_{\zeta,{\rm V}}
$$
$$
\times\Bigg(\langle h^0_{ab,{\rm S}}(\mathbf{k}_1)\,h^0_{ab,{\rm S}}(\mathbf{k}_2)\,h^0_{cd,{\rm S}}(\mathbf{k}_3-\mathbf{p}_3)\,h^0_{cd,{\rm S}}(\mathbf{k}_4+\mathbf{p}_3)\rangle'
$$
$$
+\ \langle h^0_{ab,{\rm S}}(\mathbf{k}_1-\mathbf{p}_3)\,h^0_{ab,{\rm S}}(\mathbf{k}_2+\mathbf{p}_3)\,h^0_{cd,{\rm S}}(\mathbf{k}_3)\,h^0_{cd,{\rm S}}(\mathbf{k}_4)\rangle'
$$
$$
+\ 4\,\langle h^0_{ab,{\rm S}}(\mathbf{k}_1)\,h^0_{ab,{\rm S}}(\mathbf{k}_2-\mathbf{p}_3)\,h^0_{cd,{\rm S}}(\mathbf{k}_3)\,h^0_{cd,{\rm S}}(\mathbf{k}_4+\mathbf{p}_3)\rangle'\Bigg)\,, \tag{4.9}
$$

$$
\mathcal{C}^{\rm F}_{\Omega\Omega} = \frac{k^3}{2\pi^2\Omega_{\rm S}^2}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot\phi_0}{H}\right)^4 \int \frac{d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3\,d\mathbf{p}_4}{(2\pi)^9}\,\hat T(k_1)\,\hat T(k_2)\,\hat T(k_3)\,\hat T(k_4)\frac{(2\,\pi^2)^2}{p_3^3\,p_4^3}\,\mathcal{P}^2_{\zeta,{\rm V}}
$$
$$
\times\Bigg(\langle h^0_{ab,{\rm S}}(\mathbf{k}_1-\mathbf{p}_3)\,h^0_{ab,{\rm S}}(\mathbf{k}_2+\mathbf{p}_3)\,h^0_{cd,{\rm S}}(\mathbf{k}_3-\mathbf{p}_4)\,h^0_{cd,{\rm S}}(\mathbf{k}_4+\mathbf{p}_4)\rangle'
$$
$$
+\ 2\,\langle h^0_{ab,{\rm S}}(\mathbf{k}_1-\mathbf{p}_3)\,h^0_{ab,{\rm S}}(\mathbf{k}_2-\mathbf{p}_4)\,h^0_{cd,{\rm S}}(\mathbf{k}_3+\mathbf{p}_3)\,h^0_{cd,{\rm S}}(\mathbf{k}_4+\mathbf{p}_4)\rangle'\Bigg)\,, \tag{4.10}
$$

with $\mathbf{k}_1 = \mathbf{k}-\mathbf{p}_1$, $\mathbf{k}_2 = \mathbf{p}_1$, $\mathbf{k}_3 = -\mathbf{k}-\mathbf{p}_2$ and $\mathbf{k}_4 = \mathbf{p}_2$. The last correlator, which corresponds to the correlator of the fluctuations, is found to be much smaller than the other two and will therefore be neglected from now on. In order to compute the intrinsic and extrinsic sourced correlators, we use eqs. (2.5) and (2.28) to calculate the four-point function of the homogeneous components of the sourced tensor modes, denoted as $\mathbb{C}$

$$
\mathbb{C}(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)
$$
$$
= \langle h^0_{ab,{\rm S}}(\boldsymbol{\kappa}_1)\,h^0_{ab,{\rm S}}(\boldsymbol{\kappa}_2)\,h^0_{cd,{\rm S}}(\boldsymbol{\kappa}_3)\,h^0_{cd,{\rm S}}(\boldsymbol{\kappa}_4)\rangle = \frac{16}{M_P^4}\langle H^0_{ab,{\rm S}}(\boldsymbol{\kappa}_1)\,H^0_{ab,{\rm S}}(\boldsymbol{\kappa}_2)\,H^0_{cd,{\rm S}}(\boldsymbol{\kappa}_3)\,H^0_{cd,{\rm S}}(\boldsymbol{\kappa}_4)\rangle
$$
$$
= \frac{16}{M_P^8}\int_{-\infty}^{\tau}\frac{d\tau_1}{a(\tau_1)}\,\frac{d\tau_2}{a(\tau_2)}\,\frac{d\tau_3}{a(\tau_3)}\,\frac{d\tau_4}{a(\tau_4)}\,G_{\kappa_1}(\tau,\tau_1)\,G_{\kappa_2}(\tau,\tau_2)\,G_{\kappa_3}(\tau,\tau_3)\,G_{\kappa_4}(\tau,\tau_4)\times\mathcal{I}\,, \tag{4.11}
$$

with $a(\tau) = -1/(H\,\tau)$, and

$$
\mathcal{I} = \int\frac{d\mathbf{q}_1\,d\mathbf{q}_2\,d\mathbf{q}_3\,d\mathbf{q}_4}{(2\pi)^6}
$$
$$
\times\ e^+_a(\widehat{\mathbf{q}_1})\,e^+_b(\widehat{\boldsymbol{\kappa}_1-\mathbf{q}_1})\ e^+_a(\widehat{\mathbf{q}_2})\,e^+_b(\widehat{\boldsymbol{\kappa}_2-\mathbf{q}_2})\ e^+_c(\widehat{\mathbf{q}_3})\,e^+_d(\widehat{\boldsymbol{\kappa}_3-\mathbf{q}_3})\ e^+_c(\widehat{\mathbf{q}_4})\,e^+_d(\widehat{\boldsymbol{\kappa}_4-\mathbf{q}_4})
$$
$$
\times\ \langle A'_+(q_1,\tau_1)\,A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}_1|,\tau_1)A'_+(q_2,\tau_2)\,A'_+(|\boldsymbol{\kappa}_2-\mathbf{q}_2|,\tau_2)\,A'_+(q_3,\tau_3)\,A'_+(|\boldsymbol{\kappa}_3-\mathbf{q}_3|,\tau_3)
$$
$$
\times\ A'_+(q_4,\tau_4)\,A'_+(|\boldsymbol{\kappa}_4-\mathbf{q}_4|,\tau_4)\rangle\,. \tag{4.12}
$$

Expression (4.12) has been simplified by neglecting the negative-helicity photons, as, according to eq. (2.12), A_+ is the only helicity that undergoes amplification. Using Wick's theorem to decompose the eight-point function of gauge fields appearing in the last two lines of eq. (4.12), we find that the integral $\mathcal{I}$ can be written as the sum of six distinct integrals, i.e. $\mathcal{I} = \mathcal{I}_A + \mathcal{I}_B + \mathcal{I}_C + \mathcal{I}_D + \mathcal{I}_E + \mathcal{I}_F$, the explicit expressions of which are provided in (A.1). Substituting these integrals back into Eq. (4.11), we obtain the four-point function of the homogeneous components of the sourced gravitational waves, expressed as the sum of six terms

$$
\mathbb{C} = \mathbb{C}_A + \mathbb{C}_B + \mathbb{C}_C + \mathbb{C}_D + \mathbb{C}_E + \mathbb{C}_F\,, \tag{4.13}
$$

which can be calculated using the explicit form of the gauge field (2.13). Starting from $\mathbb{C}_A$ we have

$$\begin{aligned}\mathbb{C}_A =& \frac{4\,H^4}{M_P^8}\int_{-\infty}^{\tau} d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4\,(\tau_1\,\tau_2\,\tau_3\,\tau_4)^{1/2}\, G_{\kappa_1}(\tau,\tau_1)\, G_{\kappa_2}(\tau,\tau_2)\, G_{\kappa_3}(\tau,\tau_3)\; G_{\kappa_4}(\tau,\tau_4)\\ &\times \int \frac{d\mathbf{q}}{(2\pi)^6}(\xi_1\,\xi_2\,\xi_3\,\xi_4)^{1/2}\, e^{2\pi(\xi_1+\xi_2+\xi_3+\xi_4)}\, q^{1/2}\,|\boldsymbol{\kappa}_1-\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_2+\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|^{1/2}\\ &\times \mathcal{A}_A(\mathbf{q},\,\boldsymbol{\kappa}_1-\mathbf{q},\,\boldsymbol{\kappa}_2+\mathbf{q},\,\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q})\; e^{-2\sqrt{-2\,\xi_1\,\tau_1}(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}\,)}\, e^{-2\sqrt{-2\,\xi_2\,\tau_2}(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}\,)}\\ &\times e^{-2\sqrt{-2\,\xi_3\,\tau_3}(\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)}\, e^{-2\sqrt{-2\,\xi_4\,\tau_4}(\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)}\;\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,,\end{aligned} \tag{4.14}$$

where $\xi_i\equiv\xi(\tau_i)$ are the slowly growing ξ parameters satisfying $\xi_i\gg 1$. $\mathcal{A}_A$, the explicit expression of which is given in (A.2), represents the angular part arising from the product of the helicity projectors. We simplify the expression by recalling that the fields are calculated at the end of inflation, i.e., at $\tau_e=-1/H$. Furthermore, since we are interested in modes that are well outside the horizon at the end of inflation, we can take the limit $k/H\to 0$. The presence of exponential terms such as $e^{-2\sqrt{-2\,\xi_1\,\tau_1}(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}\,)}$, with $\xi_1\gg 1$, implies that $\kappa_1\,\tau_1\ll 1$. In the same way $\kappa_2\,\tau_2\ll 1$, $\kappa_3\,\tau_3\ll 1$ and $\kappa_4\,\tau_4\ll 1$. By Taylor expanding the propagator (2.18) we have $G_{\kappa_i}(\tau,\,\tau_i)\simeq -H\,\tau_i^2/3$.

To obtain the most general result, we will not assume a specific form for the ξ parameter, but instead estimate the integral under the assumption of weak time dependence. In this case, we observe that the integral is dominated by the values of $|\tau_1|$, $|\tau_2|$, $|\tau_3|$ and $|\tau_4|$ belonging to relatively narrow windows around the following expressions, respectively: $(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}\,)^{-2}$, $(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}\,)^{-2}$, $(\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)^{-2}$ and $(\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)^{-2}$. We can therefore approximate the ξ parameters present in expression $\mathbb{C}_A$ as

$$\begin{aligned}\xi_1^A &= \xi(\tau_1^A \simeq -(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}\,)^{-2})\,,\\ \xi_2^A &= \xi(\tau_2^A \simeq -(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}\,)^{-2})\,,\\ \xi_3^A &= \xi(\tau_3^A \simeq -(\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)^{-2})\,,\\ \xi_4^A &= \xi(\tau_4^A \simeq -(\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)^{-2})\,.\end{aligned} \tag{4.15}$$

Finally, using the expression

$$\int_0^{\infty} dx\, x^{n-1}\, e^{-a\,\sqrt{x}} = \frac{2}{a^{2n}}\Gamma(2n)\,, \tag{4.16}$$

we obtain

$$\begin{aligned}\mathbb{C}_A =& \frac{H^8\,\Gamma(7)^4}{3^4\,2^{36}\,M_P^8}\int \frac{d\mathbf{q}}{(2\pi)^6}\frac{e^{2\pi(\xi_1^A+\xi_2^A+\xi_3^A+\xi_4^A)}}{(\xi_1^A\,\xi_2^A\,\xi_3^A\,\xi_4^A)^3}\,\mathcal{A}_A(\mathbf{q},\,\boldsymbol{\kappa}_1-\mathbf{q},\,\boldsymbol{\kappa}_2+\mathbf{q},\,\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q})\\ &\times q^{1/2}\,|\boldsymbol{\kappa}_1-\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_2+\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|^{1/2}\,(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}\,)^{-7}\,(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}\,)^{-7}\\ &\times (\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)^{-7}(\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|}\,)^{-7}\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,.\end{aligned} \tag{4.17}$$

In the same way we calculate all the other contributions to the eight-point function $\mathbb{C}$ in eq. (4.13), which are given in (A.3). By substituting these expressions into eqs. (4.8) and

(4.9), we can now obtain the intrinsic and extrinsic correlators of the sourced gravitational wave energy densities, whose computation is described in the next two Subsections.

Before proceeding with the calculation, we first parameterize the weak time dependence of the monotonically increasing function $\xi(\tau)$. We define $\xi(\tau = -1/k) \equiv \xi_k$ as the value of ξ when a mode with comoving wavenumber k exits the horizon, N_k e-foldings before the end of inflation. For large momenta, i.e. $p \gg k_{eq}$, the parameter ξ is close to its maximum value, $\xi_{\rm INT} \simeq 5$, evaluated at the time when interferometer-scale modes exit the horizon. This time is typically close to $\tau_{\rm BR}$, the time when the system enters the strong backreaction regime. Usually, this happens around $N_{\rm BR} \simeq 10$ e-foldings before the end of inflation [37], although there are models in which it can happen earlier, i.e. $N_{\rm BR} \simeq 40$ in [41].

While the behavior of ξ during the regime of strong backreaction is still object of research, lattice studies seem to suggest that this quantity stabilizes and evolves relatively slowly [40]. For this reason, we assume that ξ becomes approximately constant for $\tau > \tau_{\rm BR}$, implying $\xi_{\rm BR} \simeq \xi_{\rm INT}$. In this case, contributions to the integrals from momenta larger than $k_{\rm BR}$ are negligible and can be safely ignored. For large momenta, the ξ parameters in the denominators of the integrals can therefore be approximated as constants, contributing an overall factor of $\xi_{\rm BR}^{12}$. Meanwhile, the ξ appearing in the exponents are approximated as

$$\xi(\tau) = \begin{cases} \xi_{\rm BR} + \delta \log(\tau_{\rm BR}/\tau) \ , & \tau < \tau_{\rm BR} \, , \\ \xi_{\rm BR} & , \ \tau > \tau_{\rm BR} \, , \end{cases} \tag{4.18}$$

with $\tau_{BR} = -1/k_{BR}$, accounting for contributions to the integrals from lower momenta. The parameter δ depends on the specific model under consideration, and more precisely on the number of e-foldings before the end of inflation at which backreaction becomes significant. In the cases discussed above, its value lies in the range between 0.06 [37] and 0.2 [41]. Therefore, the exponential terms $e^{2\pi\xi_i^L}$, with $L = A, B, \ldots, F$ and $i = 1, 2, 3, 4$, appearing in expressions $\mathbb{C}_A$ to $\mathbb{C}_F$, transform as

$$e^{2\pi\xi_i^L} = e^{2\pi\xi(\tau_i^L)} = e^{2\pi\xi_{\rm BR}} \left(\frac{\tau_{\rm BR}}{\tau_i^L}\right)^{2\pi\delta} . \tag{4.19}$$

4.1 Sourced intrinsic correlator

In order to find the intrinsic correlator of the sourced gravitational wave energy densities we start by substituting eq. (4.11) with $\boldsymbol{\kappa}_1 = \mathbf{k}_1 = \mathbf{k} - \mathbf{p}_1$, $\boldsymbol{\kappa}_2 = \mathbf{k}_2 = \mathbf{p}_1$, $\boldsymbol{\kappa}_3 = \mathbf{k}_3 = -\mathbf{k} - \mathbf{p}_2$, $\boldsymbol{\kappa}_4 = \mathbf{k}_4 = \mathbf{p}_2$ into eq. (4.8)

$$\mathcal{C}^{\rm I}_{\Omega\Omega}(\mathbf{k}) = \frac{1}{\Omega_{\rm S}^2} \frac{k^3}{2\pi^2} \int \frac{d\mathbf{p}_1 \, d\mathbf{p}_2}{(2\pi)^3} \, \hat{T}(k_1) \, \hat{T}(k_2) \, \hat{T}(k_3) \, \hat{T}(k_4) \, \mathbb{C}(\mathbf{k} - \mathbf{p}_1, \mathbf{p}_1, -\mathbf{k} - \mathbf{p}_2, \mathbf{p}_2) \, , \tag{4.20}$$

and we expand $\mathbb{C}$ using eqs. (4.13), (4.17) and (A.3). Term $\mathbb{C}_D$ contains a tadpole, as $\delta(\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2) = \delta(\boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_4) = \delta(\mathbf{k})$, and can therefore be neglected. In addition, the terms $\mathbb{C}_E$ and $\mathbb{C}_F$ become identical after resolving the delta functions. For large values of the momenta we use (4.18) and we factor out of the integrals the ξ_{BR}^{12} from the denominators, the $e^{8\pi\xi_{BR}}$ from (4.19) and the constant transfer functions (3.4). The intrinsic correlator turns out to be

$$\left(\mathcal{C}^{\rm I}_{\Omega\Omega}\right)_{\rm l.m.} = \frac{k^3 \, H^8 \, \Gamma(7)^4 \, e^{8\,\pi\,\xi_{\rm BR}} \, \hat{T}_r^4}{\Omega_{\rm S}^2 \, \pi^2 \, 3^4 \, 2^{37} \, M_P^8 \, (2\pi)^9 \, \xi_{BR}^{12}} \times I_{\rm I} \simeq 9.8 \times 10^6 \left(\frac{k}{k_{BR}}\right)^3 , \tag{4.21}$$

where in the last equality we have used (4.2), (4.3) and the integral $I_{\rm I}$ explicitly evaluated in Appendix B, eq. (B.7). For typical values $k \sim k_{\rm CMB}$ the factor $(k/k_{BR})^3$ is of the order of e^{-150}, which makes the intrinsic correlator very small.

4.2 Sourced extrinsic correlator

In order to find the extrinsic correlator of the sourced gravitational wave energy densities we substitute eq. (4.11) into the three four-point functions of eq. (4.9), obtaining

$$\begin{aligned}
\mathcal{C}^{\rm E}_{\Omega\Omega}(\mathbf{k}) &= \frac{k^3\,\mathcal{P}_{\zeta,{\rm V}}}{\Omega_{\rm S}^2}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot\phi_0}{H}\right)^2\int\frac{d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3}{(2\pi)^6}\,\hat T(|\mathbf{k}-\mathbf{p}_1|)\,\hat T(p_1)\,\hat T(|\mathbf{k}+\mathbf{p}_2)\,\hat T(p_2)\,\frac{1}{p_3^3}\\
&\times\left(\mathbb{C}(\mathbf{k}-\mathbf{p}_1,\mathbf{p}_1,-\mathbf{k}-\mathbf{p}_2-\mathbf{p}_3,\mathbf{p}_2+\mathbf{p}_3)+\mathbb{C}(\mathbf{k}-\mathbf{p}_1-\mathbf{p}_3,\mathbf{p}_1+\mathbf{p}_3,-\mathbf{k}-\mathbf{p}_2,\mathbf{p}_2)\right.\\
&+\left.\mathbb{C}(\mathbf{k}-\mathbf{p}_1,\mathbf{p}_1-\mathbf{p}_3,-\mathbf{k}-\mathbf{p}_2,\mathbf{p}_2+\mathbf{p}_3)\right)\\
&= \mathcal{C}^{\rm E}_{\Omega\Omega,1}(\mathbf{k})+\mathcal{C}^{\rm E}_{\Omega\Omega,2}(\mathbf{k})+\mathcal{C}^{\rm E}_{\Omega\Omega,3}(\mathbf{k})\,,
\end{aligned}\tag{4.22}$$

where we have considered the scalar vacuum power spectrum $\mathcal{P}_{\zeta,{\rm V}}\simeq 2\times10^{-9}$ as a constant.

The sourced extrinsic correlator is therefore composed by three terms. In both $\mathcal{C}^{\rm E}_{\Omega\Omega,1}(\mathbf{k})$ and $\mathcal{C}^{\rm E}_{\Omega\Omega,2}(\mathbf{k})$, the term $\mathbb{C}_D$ corresponds to a tadpole, leading to effects similar to those found in the intrinsic correlator. These correlators turn out to be very small due to the presence of the factor $(k/k_{BR})^3$. In the third correlator $\mathcal{C}^{\rm E}_{\Omega\Omega,3}(\mathbf{k})$, however, $\mathbb{C}_D$ is no longer a tadpole, but instead generates a significant scale-invariant term. Consequently, $\mathcal{C}^{\rm E}_{\Omega\Omega,3}(\mathbf{k})$ contains, besides the small contributions similar to those in the other two cases, denoted collectively as $\mathcal{C}^{\rm E}_{\Omega\Omega,3}(\mathbf{k})'$, a dominant component, which we denote as $\mathcal{C}^{\rm S.I.}_{\Omega\Omega}(\mathbf{k})$. This component constitutes the main result of this paper and for this reason we present its calculation separately in the next Subsection. In Subsection 4.2.2, we show for completeness all the other small contributions.

4.2.1 Extrinsic correlator: Scale-invariant term $\mathcal{C}^{\rm S.I.}_{\Omega\Omega}(\mathbf{k})$

We now focus on calculating the scale-invariant contribution $\mathcal{C}^{\rm S.I.}_{\Omega\Omega}(\mathbf{k})$, which comes from term $\mathbb{C}_D$ of $\mathcal{C}^{\rm E}_{\Omega\Omega,3}$ with $\boldsymbol{\kappa}_1=\mathbf{k}-\mathbf{p}_1$, $\boldsymbol{\kappa}_2=\mathbf{p}_1-\mathbf{p}_3$, $\boldsymbol{\kappa}_3=-\mathbf{k}-\mathbf{p}_2$, and $\boldsymbol{\kappa}_4=\mathbf{p}_2+\mathbf{p}_3$. This term is particularly important because it contains the quantity $\delta(\mathbf{k}-\mathbf{p}_3)$, which cancels the k^3 element in the prefactor, giving rise to a significantly large contribution. More specifically,

$$\begin{aligned}
\mathcal{C}^{\rm S.I}_{\Omega\Omega}(\mathbf{k}) &= \frac{H^8\,\Gamma(7)^4}{\Omega_{\rm S}^2\,3^4\,2^{34}\,M_P^8\,(2\pi)^{12}}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot\phi_0}{H}\right)^2\mathcal{P}_{\zeta,{\rm V}}\int d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{q}_1\,d\mathbf{q}_2\\
&\times\hat T(|\mathbf{k}-\mathbf{p}_1|)\,\hat T(p_1)\,\hat T(|\mathbf{k}+\mathbf{p}_2|)\,\hat T(p_2)\,\mathcal{A}_D(\mathbf{q}_1,\mathbf{k}-\mathbf{p}_1-\mathbf{q}_1,\mathbf{q}_2,-\mathbf{k}-\mathbf{p}_2-\mathbf{q}_2)\\
&\times\frac{q_1^{1/2}\,|\mathbf{k}-\mathbf{p}_1-\mathbf{q}_1|^{1/2}\,q_2^{1/2}\,|\mathbf{k}+\mathbf{p}_2+\mathbf{q}_2|^{1/2}\,e^{2\pi(\xi_1^{\rm S.I.}+\xi_2^{\rm S.I.}+\xi_3^{\rm S.I.}+\xi_4^{\rm S.I.})}}{(\sqrt{q_1}+\sqrt{|\mathbf{k}-\mathbf{p}_1-\mathbf{q}_1|}\,)^{14}\,(\sqrt{q_2}+\sqrt{|\mathbf{k}+\mathbf{p}_2+\mathbf{q}_2|}\,)^{14}(\xi_1^{\rm S.I.}\,\xi_2^{\rm S.I.}\,\xi_3^{\rm S.I.}\,\xi_4^{\rm S.I.})^3}\,,
\end{aligned}\tag{4.23}$$

with

$$\begin{aligned}
\xi_1^{\rm S.I.}=\xi_2^{\rm S.I.}&=\xi(\tau_1^{\rm S.I.}\simeq-(\sqrt{q_1}+\sqrt{|\mathbf{k}-\mathbf{p}_1-\mathbf{q}_1|}\,)^{-2})\,,\\
\xi_3^{\rm S.I.}=\xi_4^{\rm S.I.}&=\xi(\tau_3^{\rm S.I.}\simeq-(\sqrt{q_2}+\sqrt{|\mathbf{k}+\mathbf{p}_2+\mathbf{q}_2|}\,)^{-2})\,.
\end{aligned}\tag{4.24}$$

The angular part is

$$\mathcal{A}_D(\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s})=\frac{1}{16}(5+2\,\widehat{\mathbf{p}}\,\widehat{\mathbf{q}}+(\widehat{\mathbf{p}}\,\widehat{\mathbf{q}})^2)(5+2\,\widehat{\mathbf{r}}\,\widehat{\mathbf{s}}+(\widehat{\mathbf{r}}\,\widehat{\mathbf{s}})^2)\,,\tag{4.25}$$

which is calculated using

$$e_i^\lambda(\widehat{\mathbf{k}})\,e_j^\lambda(-\widehat{\mathbf{k}})=\frac{1}{2}\,(\delta_{ij}-\widehat{k}_i\,\widehat{k}_j-i\,\lambda\,\epsilon_{ijk}\,\widehat{k}_k)\,.\tag{4.26}$$

For large momenta, we use the parametrization (4.18) for the ξ functions in the exponents, while in the denominators we approximate them as simply ξ_{BR}. The transfer function takes the form (3.4), and we simplify the integral by neglecting the contribution of the small k, wherever it appears. The scale-invariant correlator then takes the form

$$\left(\mathcal{C}^{\rm S.I.}_{\Omega\Omega}(\mathbf{k})\right)_{l.m.} = \frac{H^8\,\Gamma(7)^4\,e^{8\,\pi\,\xi_{\rm BR}}\,\hat{T}_r^4}{\Omega_{\rm S}^2\,3^4\,2^{34}\,M_P^8\,(2\pi)^{12}\,\xi_{BR}^{12}}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot{\phi}_0}{H}\right)^2\,\mathcal{P}_{\zeta,{\rm V}}\,\times I_{\rm S.I.}\,, \tag{4.27}$$

with the integral $I_{\rm S.I.}$ given in (C.3). Using $\mathcal{P}_{\zeta,{\rm V}} \simeq 2\times 10^{-9}$ and equations (3.4), (4.2) and (4.3), we eventually obtain the correlator

$$\left(\mathcal{C}^{\rm S.I.}_{\Omega\Omega}(\mathbf{k})\right)_{l.m.} \simeq \frac{9.8\times 10^{-5}}{\delta^2}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot{\phi}_0}{H}\right)^2. \tag{4.28}$$

Considering (2.34) and the fact that the parameter δ takes values in the interval $0.06 \div 0.2$, the sourced scale-invariant extrinsic correlator is found to lie within the range

$$\left(\mathcal{C}^{\rm S.I.}_{\Omega\Omega}(\mathbf{k})\right)_{l.m.} \simeq 2.4\times 10^{-5} \div 2.4\times 10^{-1}\,. \tag{4.29}$$

This result will constitute the only relevant component of the sourced correlator, as it is many orders of magnitude larger than the intrinsic correlator, studied in Subsection 4.1, and all other contributions to the extrinsic correlator, which we present for completeness in the next Subsection.

4.2.2 Extrinsic correlator: Terms $\mathcal{C}^{\rm E}_{\Omega\Omega,1}(\mathbf{k})$, $\mathcal{C}^{\rm E}_{\Omega\Omega,2}(\mathbf{k})$, $\mathcal{C}^{\rm E}_{\Omega\Omega,3}(\mathbf{k})'$

In order to find all the other terms contributing to the extrinsic correlator, which we anticipated to be very small and unobservable, we start by expanding the terms $\mathbb{C}$ in (4.22) using eqs. (4.13), (4.17) and (A.3). Using again the transfer function (3.4) and the parametrization (4.18) we find

$$\left(\mathcal{C}^{\rm E}_{\Omega\Omega,1}(\mathbf{k})\right)_{\rm l.m.} = \frac{k^3\,H^8\,\Gamma(7)^4\,e^{8\,\pi\,\xi_{\rm BR}}\,\hat{T}_r^4\,\mathcal{P}_{\zeta,{\rm V}}}{\Omega_{\rm S}^2\,3^4\,2^{36}\,M_P^8\,(2\pi)^{12}\,\xi_{BR}^{12}}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot{\phi}_0}{H}\right)^2\times I_{\rm E,1} \simeq 5.5\times 10^{-1}\left(\frac{k}{k_{BR}}\right)^3, \tag{4.30}$$

$$\left(\mathcal{C}^{\rm E}_{\Omega\Omega,2}(\mathbf{k})\right)_{\rm l.m.} = \frac{k^3\,H^8\,\Gamma(7)^4\,e^{8\,\pi\,\xi_{\rm BR}}\,\hat{T}_r^4\,\mathcal{P}_{\zeta,{\rm V}}}{\Omega_{\rm S}^2\,3^4\,2^{36}\,M_P^8\,(2\pi)^{12}\,\xi_{BR}^{12}}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot{\phi}_0}{H}\right)^2\times I_{\rm E,2} \simeq 4.9\times 10^{-1}\left(\frac{k}{k_{BR}}\right)^3, \tag{4.31}$$

$$\left(\mathcal{C}^{\rm E}_{\Omega\Omega,3}(\mathbf{k})'\right)_{\rm l.m.} = \frac{k^3\,H^8\,\Gamma(7)^4\,e^{8\,\pi\,\xi_{\rm BR}}\,\hat{T}_r^4\,\mathcal{P}_{\zeta,{\rm V}}}{\Omega_{\rm S}^2\,3^4\,2^{34}\,M_P^8\,(2\pi)^{12}\,\xi_{BR}^{12}}\left(2\pi\frac{d\xi}{d\phi_0}\frac{\dot{\phi}_0}{H}\right)^2\times I_{\rm E,3} \simeq 1.7\left(\frac{k}{k_{BR}}\right)^3, \tag{4.32}$$

where in the final expressions we have used (2.34), (4.2), (4.3) and the integrals $I_{\rm E,1}$, $I_{\rm E,2}$ and $I_{\rm E,3}$ evaluated, respectively, in (C.6), (C.9), (C.13). These correlators are all very small because of the presence of $(k/k_{BR})^3$, as in the case of the intrinsic correlator.

5 Vacuum correlator

For the normalized vacuum correlator eq. (3.1) becomes

$$\mathcal{C}^{\rm V}_{\Omega\Omega}(\mathbf{k}) = \frac{1}{\Omega^2_{GW,{\rm V}}}\frac{k^3}{2\pi^2}\int d\mathbf{y}\, e^{-i\mathbf{k}\mathbf{y}}\,\langle \Omega_{GW,{\rm V}}(\mathbf{x}+\mathbf{y},\,t_0)\,\Omega_{GW,{\rm V}}(\mathbf{x},\,t_0)\rangle\,, \tag{5.1}$$

with the fractional energy at interferometer scales for the vacuum component being

$$\Omega_{GW,{\rm V}} \simeq \frac{\Omega^0_{\rm rad}}{24}\,\mathcal{P}_{h,{\rm V}}(k_{\rm INT})\,, \qquad \text{with} \qquad \mathcal{P}_{h,{\rm V}}(k_{\rm INT}) = \frac{2\,H^2}{\pi^2\,M_P^2}\,. \tag{5.2}$$

Defining

$$\Omega_{\rm V} = 12\,H_0^2\,\Omega_{GW,{\rm V}}\,, \tag{5.3}$$

expression (3.3) for the vacuum correlator takes the form

$$\mathcal{C}^{\rm V}_{\Omega\Omega} = \frac{1}{\Omega^2_{\rm V}}\frac{k^3}{2\pi^2}\int\frac{d\mathbf{p}_1\,d\mathbf{p}_2}{(2\pi)^3}\,\hat{T}(k_1)\,\hat{T}(k_2)\,\hat{T}(k_3)\,\hat{T}(k_4)\,\langle h_{ab,{\rm V}}(\mathbf{k}_1)\,h_{ab,{\rm V}}(\mathbf{k}_2)\,h_{cd,{\rm V}}(\mathbf{k}_3)\,h_{cd,{\rm V}}(\mathbf{k}_4)\rangle'\,, \tag{5.4}$$

with $\mathbf{k}_1 = \mathbf{k}-\mathbf{p}_1$, $\mathbf{k}_2 = \mathbf{p}_1$, $\mathbf{k}_3 = -\mathbf{k}-\mathbf{p}_2$ and $\mathbf{k}_4 = \mathbf{p}_2$ and $\hat{T}(k) = k\,T(k)$. The four-point function in equation (5.4) is decomposed using Wick's theorem, and, up to a tadpole term, is

$$\langle h_{ab,{\rm V}}(\mathbf{k}_1)\,h_{ab,{\rm V}}(\mathbf{k}_2)\,h_{cd,{\rm V}}(\mathbf{k}_3)\,h_{cd,{\rm V}}(\mathbf{k}_4)\rangle = 2\,\langle h_{ab,{\rm V}}(\mathbf{k}_1)\,h_{cd,{\rm V}}(\mathbf{k}_3)\rangle\,\langle h_{ab,{\rm V}}(\mathbf{k}_2)\;h_{cd,{\rm V}}(\mathbf{k}_4)\rangle\,. \tag{5.5}$$

Using

$$\langle h_{ab,{\rm V}}(\mathbf{k}_1)\,h_{cd,{\rm V}}(\mathbf{k}_3)\rangle = \sum_{\lambda=\pm} e^{\lambda}_{ab}(\widehat{\mathbf{k}}_1)\,e^{\lambda}_{cd}(-\widehat{\mathbf{k}}_1)\,\delta(\mathbf{k}_1+\mathbf{k}_3)\,|h^{\lambda}_{\rm V}(k_1)|^2\,, \tag{5.6}$$

and the definition $|h^{\lambda}_{\rm V}(k)|^2 = \frac{2\pi^2}{k^3}\,\mathcal{P}^{\lambda}_{h,{\rm V}}$, with $\mathcal{P}^{\pm}_{h,{\rm V}} = \frac{H^2}{\pi^2\,M_P^2}$, we eventually have

$$\mathcal{C}^{\rm V}_{\Omega\Omega}(\mathbf{k}) = \frac{k^3\,H^4}{2\pi^5\,M_P^4\,\Omega^2_{\rm V}}\int d\mathbf{p}\,\hat{T}(|\mathbf{k}-\mathbf{p}|)^2\,\hat{T}(p)^2\frac{1}{|\mathbf{k}-\mathbf{p}|^3\,p^3}\times\mathcal{A}(\mathbf{k}-\mathbf{p},\mathbf{p})\,. \tag{5.7}$$

The angular part, calculated using (4.26), is

$$\begin{aligned}\mathcal{A}(\mathbf{k}-\mathbf{p},\mathbf{p}) &= \sum_{\lambda,\sigma} e^{\lambda}_{ab}(\widehat{\mathbf{k}-\mathbf{p}})\,e^{\lambda}_{cd}(-(\widehat{\mathbf{k}-\mathbf{p}}))\,e^{\sigma}_{ab}(\widehat{\mathbf{p}})\,e^{\sigma}_{cd}(-\widehat{\mathbf{p}})\\ &= \frac{1}{4}\left(1+6((\widehat{\mathbf{k}-\mathbf{p}})\widehat{\mathbf{p}})^2+((\widehat{\mathbf{k}-\mathbf{p}})\widehat{\mathbf{p}})^4\right).\end{aligned} \tag{5.8}$$

The integral in (5.7) must be evaluated over the sensitivity range of gravitational wave detectors, i.e., between momenta p_{min} and p_{max} that correspond to the limits of the momentum interval measurable by a given detector. As previously discussed, these momenta are much larger than the value of $k \sim k_{\rm CMB}$ under consideration, which can therefore be neglected. With this in mind, and using (3.4) and (5.2)-(5.3), the correlator takes the form

$$\left(\mathcal{C}^{\rm V}_{\Omega\Omega}\right)_{\rm l.m.}(\mathbf{k}) \simeq \frac{k^3\,3^4\,24^2}{2\pi\,4^6\,144}\int_{p_{min}}^{p_{max}} dp\,p^2\int d\Omega\,\frac{1}{p^6}\,\mathcal{A}(-\mathbf{p},\mathbf{p})\,. \tag{5.9}$$

After performing the integrals $\int d\Omega\, \mathcal{A}(-\mathbf{p},\mathbf{p}) = 8\pi$ and $\int_{p_{min}}^{p_{max}} dp\, \frac{1}{p^4} \simeq \frac{1}{3\,p_{min}^3}$, the correlator eventually becomes

$$\left(\mathcal{C}^{\rm V}_{\Omega\Omega}\right)_{\rm l.m.} = 0.1 \times \left(\frac{k}{p_{min}}\right)^3 . \tag{5.10}$$

This result, similarly to the intrinsic sourced correlator and the subdominant contributions of the extrinsic sourced correlator, is subject to a strong suppression due to the factor $(k/p_{min})^3$, and is therefore very small and unobservable for any gravitational wave detector.

6 Discussion and conclusions

The recent evidence of a stochastic gravitational wave background reported by PTA measurements has opened a promising new observational window in modern cosmology. Such a background can arise either from the combined signals of unresolved late-time astrophysical sources, such as supermassive binary black hole mergers, or from a cosmological origin. In particular, a cosmological SGWB can reveal new details about the very early Universe in ways that previous observations could not, as the gravitational waves that make it up are produced before photon decoupling and propagate almost freely throughout the Universe after their generation.

To extract information from the cosmological gravitational wave background, we must distinguish it from its astrophysical counterpart. One approach is to analyze their anisotropies and, in particular, the correlation of these anisotropies with CMB anisotropies, which is expected to differ between the two backgrounds. In this context, the authors of [26] computed the correlator between the curvature perturbation and the energy density of gravitational waves within the axion inflation model.

In this paper we studied the amplitude of gravitational wave anisotropies in the axion inflation model, by computing the correlator $\langle \Omega_{\rm GW}(\mathbf{x})\Omega_{\rm GW}(\mathbf{y})\rangle$, which provides a measure of the observability of the correlation between scalar and tensor fluctuations. In axion inflation, the coupling of the inflaton to gauge fields implies that fluctuations arise both from the vacuum, through the standard amplification process, and from gauge fields via an inverse decay process. As a result, the correlator consists of two contributions: the correlation of the vacuum gravitational waves and the correlation of the sourced gravitational waves. Moreover, since the sourced gravitational waves consist of one part that depends only on the zero mode of the inflaton and another that depends on its fluctuations, the sourced correlator can be further decomposed into three distinct contributions: the intrinsic part, the extrinsic part, and a part containing only fluctuations.

Our analysis shows that the only relevant contribution to the correlator arises from the scale-invariant part of the extrinsic sourced component, while all other terms are negligible. For typical parameter values, the normalized sourced correlator is found to lie in the range $\mathcal{O}(10^{-5} - 10^{-1})$ and in particular, it can reach values as large as 2.4×10^{-1}. According to [19, 27, 28], anisotropies must be relatively large to be detectable within a reasonable time frame. Our study shows that axion inflation can indeed produce observable anisotropies. Combined with the increased sensitivity of future gravitational wave detectors, this result motivates further study of angular correlations in the GW background as well as cross-correlations with CMB anisotropies in axion inflation models.

Acknowledgments

I thank Lorenzo Sorbo for very useful discussions. This work is partially supported by the US-NSF grants PHY-2112800 and PHY-2412570.

A Sourced correlator: Full expressions

The integrals that compose $\mathcal{I} = \mathcal{I}_A + \mathcal{I}_B + \mathcal{I}_C + \mathcal{I}_D + \mathcal{I}_E + \mathcal{I}_F$ in eq. (4.12) are given by the expressions

$$\begin{aligned}
\mathcal{I}_A = & \int \frac{d\mathbf{q}}{(2\pi)^6} A'_+(q,\tau_1)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}|,\tau_1) A'_+(q,\tau_2)\, A'_+(|\boldsymbol{\kappa}_2+\mathbf{q}|,\tau_2)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}|,\tau_3) \\
& \times\, A'_+(|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|,\tau_3)\, A'_+(|\boldsymbol{\kappa}_2+\mathbf{q}|,\tau_4)\, A'_+(|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q}|,\tau_4) \\
& \times\, \mathcal{A}_A(\mathbf{q},\, \boldsymbol{\kappa}_1-\mathbf{q},\, \boldsymbol{\kappa}_2+\mathbf{q},\, \boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4+\mathbf{q})\, \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,, \\[2ex]
\mathcal{I}_B = & \int \frac{d\mathbf{q}}{(2\pi)^6} A'_+(q,\tau_1)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}|,\tau_1) A'_+(q,\tau_2)\, A'_+(|\boldsymbol{\kappa}_2+\mathbf{q}|,\tau_2)\, A'_+(|\boldsymbol{\kappa}_2+\mathbf{q}|,\tau_3) \\
& \times\, A'_+(|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|,\tau_3)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}|,\tau_4)\, A'_+(|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|,\tau_4) \\
& \times\, \mathcal{A}_B(\mathbf{q},\, \boldsymbol{\kappa}_1-\mathbf{q},\, \boldsymbol{\kappa}_2+\mathbf{q},\, \boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q})\, \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,, \\[2ex]
\mathcal{I}_C = & \int \frac{d\mathbf{q}}{(2\pi)^6} A'_+(q,\tau_1)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}|,\tau_1) A'_+(|\boldsymbol{\kappa}_3+\mathbf{q}|,\tau_2)\, A'_+(|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|,\tau_2)\, A'_+(q,\tau_3) \\
& \times\, A'_+(|\boldsymbol{\kappa}_3+\mathbf{q}|,\tau_3)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}|,\tau_4)\, A'_+(|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|,\tau_4) \\
& \times\, \mathcal{A}_C(\mathbf{q},\, \boldsymbol{\kappa}_1-\mathbf{q},\, \boldsymbol{\kappa}_3+\mathbf{q},\, \boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q})\, \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,, \\[2ex]
\mathcal{I}_D = & \int \frac{d\mathbf{q}_1\, d\mathbf{q}_2}{(2\pi)^6} A'_+(q_1,\tau_1)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}_1|,\tau_1)\, A'_+(q_1,\tau_2)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}_1|,\tau_2)\, A'_+(q_2,\tau_3) \\
& \times\, A'_+(|\boldsymbol{\kappa}_3-\mathbf{q}_2|,\tau_3)\, A'_+(q_2,\tau_4)\, A'_+(|\boldsymbol{\kappa}_3-\mathbf{q}_2|,\tau_4) \\
& \times\, \mathcal{A}_D(\mathbf{q}_1,\, \boldsymbol{\kappa}_1-\mathbf{q}_1,\, \mathbf{q}_2,\, \boldsymbol{\kappa}_3-\mathbf{q}_2)\, \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\, \delta(\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,, \\[2ex]
\mathcal{I}_E = & \int \frac{d\mathbf{q}_1\, d\mathbf{q}_2}{(2\pi)^6} A'_+(q_1,\tau_1)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}_1|,\tau_1)\, A'_+(q_2,\tau_2)\, A'_+(|\boldsymbol{\kappa}_2-\mathbf{q}_2|,\tau_2)\, A'_+(q_1,\tau_3) \\
& \times\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}_1|,\tau_3)\, A'_+(q_2,\tau_4)\, A'_+(|\boldsymbol{\kappa}_2-\mathbf{q}_2|,\tau_4) \\
& \times\, \mathcal{A}_E(\mathbf{q}_1,\, \boldsymbol{\kappa}_1-\mathbf{q}_1,\, \mathbf{q}_2,\, \boldsymbol{\kappa}_2-\mathbf{q}_2)\, \delta(\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4)\, \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3)\,, \\[2ex]
\mathcal{I}_F = & \int \frac{d\mathbf{q}_1\, d\mathbf{q}_2}{(2\pi)^6} A'_+(q_1,\tau_1)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}_1|,\tau_1)\, A'_+(q_2,\tau_2)\, A'_+(|\boldsymbol{\kappa}_2-\mathbf{q}_2|,\tau_2)\, A'_+(q_2,\tau_3) \\
& \times\, A'_+(|\boldsymbol{\kappa}_2-\mathbf{q}_2|,\tau_3)\, A'_+(q_1,\tau_4)\, A'_+(|\boldsymbol{\kappa}_1-\mathbf{q}_1|,\tau_4) \\
& \times\, \mathcal{A}_F(\mathbf{q}_1,\, \boldsymbol{\kappa}_1-\mathbf{q}_1,\, \mathbf{q}_2,\, \boldsymbol{\kappa}_2-\mathbf{q}_2)\, \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_4)\, \delta(\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)\,,
\end{aligned} \tag{A.1}$$

where we have collected the angular parts inside the functions $\mathcal{A}$:

$$\begin{aligned}
\mathcal{A}_A(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = \mathcal{A}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = & ((((e^+_a(\widehat{\mathbf{p}_1})\, e^+_\beta(\widehat{-\mathbf{p}_1})\, e^+_b(\widehat{\mathbf{p}_2})\, e^+_c(\widehat{-\mathbf{p}_2})\, e^+_\alpha(\widehat{\mathbf{p}_3}) \\
\times\, e^+_\gamma(\widehat{-\mathbf{p}_3})\, e^+_\delta(\widehat{\mathbf{p}_4})\, e^+_d(\widehat{-\mathbf{p}_4}) + \, (a \leftrightarrow b)) + (\alpha \leftrightarrow \beta)) + (c \leftrightarrow d)) + (\gamma \leftrightarrow \delta))\delta_{a\alpha}\, \delta_{b\beta}\, \delta_{c\gamma}\, \delta_{d\delta}\,,
\end{aligned}$$

$$\mathcal{A}_C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = ((((e^+_a(\widehat{\mathbf{p}_1})\, e^+_c(\widehat{-\mathbf{p}_1})\, e^+_b(\widehat{\mathbf{p}_2})\, e^+_\gamma(\widehat{-\mathbf{p}_2})\, e^+_d(\widehat{\mathbf{p}_3})\, e^+_\alpha(\widehat{-\mathbf{p}_3})\, e^+_\beta(\widehat{\mathbf{p}_4})\, e^+_\delta(\widehat{-\mathbf{p}_4})$$

$$+\ (a\leftrightarrow b))+(\alpha\leftrightarrow\beta))+(c\leftrightarrow d))+(\gamma\leftrightarrow\delta))\delta_{a\alpha}\,\delta_{b\beta}\,\delta_{c\gamma}\,\delta_{d\delta}\,,$$

$$\mathcal{A}_D(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=((e^+_a(\widehat{\mathbf{p}_1})\,e^+_\alpha(\widehat{-\mathbf{p}_1})\,e^+_b(\widehat{\mathbf{p}_2})\,e^+_\beta(\widehat{-\mathbf{p}_2})\,e^+_c(\widehat{\mathbf{p}_3})\,e^+_\gamma(\widehat{-\mathbf{p}_3})\,e^+_d(\widehat{\mathbf{p}_4})e^+_\delta(\widehat{-\mathbf{p}_4})$$
$$+(\alpha\leftrightarrow\beta))+(\gamma\leftrightarrow\delta))\delta_{a\alpha}\,\delta_{b\beta}\,\delta_{c\gamma}\,\delta_{d\delta}\,,$$

$$\mathcal{A}_E(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=\mathcal{A}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=((e^+_a(\widehat{\mathbf{p}_1})\,e^+_c(\widehat{-\mathbf{p}_1})\,e^+_b(\widehat{\mathbf{p}_2})\,e^+_d(\widehat{-\mathbf{p}_2})\,e^+_\alpha(\widehat{\mathbf{p}_3})$$
$$\times\ e^+_\gamma(\widehat{-\mathbf{p}_3})\,e^+_\beta(\widehat{\mathbf{p}_4})e^+_\delta(\widehat{-\mathbf{p}_4})+(c\leftrightarrow d))+(\gamma\leftrightarrow\delta))\delta_{a\alpha}\,\delta_{b\beta}\,\delta_{c\gamma}\,\delta_{d\delta}\,.\qquad\text{(A.2)}$$

The correlators $\mathbb{C}_B$ to $\mathbb{C}_F$ in eq. (4.13), calculated in a similar way as $\mathbb{C}_A$ in (4.14), take the form

$$\mathbb{C}_B=\frac{H^8\,\Gamma(7)^4}{3^4\,2^{36}\,M_P^8}\int\frac{d\mathbf{q}}{(2\pi)^6}\frac{e^{2\pi(\xi^B_1+\xi^B_2+\xi^B_3+\xi^B_4)}}{(\xi^B_1\,\xi^B_2\,\xi^B_3\,\xi^B_4)^3}\,\mathcal{A}_B(\mathbf{q},\,\boldsymbol{\kappa}_1-\mathbf{q},\,\boldsymbol{\kappa}_2+\mathbf{q},\,\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q})$$
$$\times\ q^{1/2}\,|\boldsymbol{\kappa}_1-\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_2+\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|^{1/2}\,(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}\,)^{-7}\,(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}\,)^{-7}$$
$$\times\,(\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|}\,)^{-7}(\sqrt{|\boldsymbol{\kappa}_2+\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|}\,)^{-7}\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,,$$

$$\mathbb{C}_C=\frac{H^8\,\Gamma(7)^4}{3^4\,2^{36}\,M_P^8}\int\frac{d\mathbf{q}}{(2\pi)^6}\frac{e^{2\pi(\xi^C_1+\xi^C_2+\xi^C_3+\xi^C_4)}}{(\xi^C_1\,\xi^C_2\,\xi^C_3\,\xi^C_4)^3}\,\mathcal{A}_C(\mathbf{q},\,\boldsymbol{\kappa}_1-\mathbf{q},\,\boldsymbol{\kappa}_3+\mathbf{q},\,\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q})$$
$$\times\ q^{1/2}\,|\boldsymbol{\kappa}_1-\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_3+\mathbf{q}|^{1/2}\,|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|^{1/2}\,(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}\,)^{-7}\,(\sqrt{q}+\sqrt{|\boldsymbol{\kappa}_3+\mathbf{q}|}\,)^{-7}$$
$$\times\,(\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|}\,)^{-7}(\sqrt{|\boldsymbol{\kappa}_3+\mathbf{q}|}+\sqrt{|\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\mathbf{q}|}\,)^{-7}\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,,$$

$$\mathbb{C}_D=\frac{H^8\,\Gamma(7)^4}{3^4\,2^{36}\,M_P^8}\int\frac{d\mathbf{q}_1\,d\mathbf{q}_2}{(2\pi)^6}\frac{e^{2\pi(\xi^D_1+\xi^D_2+\xi^D_3+\xi^D_4)}}{(\xi^D_1\,\xi^D_2\,\xi^D_3\,\xi^D_4)^3}\,\mathcal{A}_D(\mathbf{q}_1,\,\boldsymbol{\kappa}_1-\mathbf{q}_1,\,\mathbf{q}_2,\,\boldsymbol{\kappa}_3-\mathbf{q}_2)$$
$$\times\ q_1^{1/2}\,q_2^{1/2}\,|\boldsymbol{\kappa}_1-\mathbf{q}_1|^{1/2}\,|\boldsymbol{\kappa}_3-\mathbf{q}_2|^{1/2}\,(\sqrt{q_1}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}_1|}\,)^{-14}\,(\sqrt{q_2}+\sqrt{|\boldsymbol{\kappa}_3-\mathbf{q}_2|}\,)^{-14}$$
$$\times\ \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\,\delta(\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)\,,$$

$$\mathbb{C}_E=\frac{H^8\,\Gamma(7)^4}{3^4\,2^{36}\,M_P^8}\int\frac{d\mathbf{q}_1\,d\mathbf{q}_2}{(2\pi)^6}\frac{e^{2\pi(\xi^E_1+\xi^E_2+\xi^E_3+\xi^E_4)}}{(\xi^E_1\,\xi^E_2\,\xi^E_3\,\xi^E_4)^3}\,\mathcal{A}_E(\mathbf{q}_1,\,\boldsymbol{\kappa}_1-\mathbf{q}_1,\,\mathbf{q}_2,\,\boldsymbol{\kappa}_2-\mathbf{q}_2)$$
$$\times\ q_1^{1/2}\,q_2^{1/2}\,|\boldsymbol{\kappa}_1-\mathbf{q}_1|^{1/2}\,|\boldsymbol{\kappa}_2-\mathbf{q}_2|^{1/2}\,(\sqrt{q_1}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}_1|}\,)^{-14}\,(\sqrt{q_2}+\sqrt{|\boldsymbol{\kappa}_2-\mathbf{q}_2|}\,)^{-14}$$
$$\times\ \delta(\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_4)\,\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3)\,,$$

$$\mathbb{C}_F=\frac{H^8\,\Gamma(7)^4}{3^4\,2^{36}\,M_P^8}\int\frac{d\mathbf{q}_1\,d\mathbf{q}_2}{(2\pi)^6}\frac{e^{2\pi(\xi^F_1+\xi^F_2+\xi^F_3+\xi^F_4)}}{(\xi^F_1\,\xi^F_2\,\xi^F_3\,\xi^F_4)^3}\,\mathcal{A}_F(\mathbf{q}_1,\,\boldsymbol{\kappa}_1-\mathbf{q}_1,\,\mathbf{q}_2,\,\boldsymbol{\kappa}_2-\mathbf{q}_2)$$
$$\times\ q_1^{1/2}\,q_2^{1/2}\,|\boldsymbol{\kappa}_1-\mathbf{q}_1|^{1/2}\,|\boldsymbol{\kappa}_2-\mathbf{q}_2|^{1/2}\,(\sqrt{q_1}+\sqrt{|\boldsymbol{\kappa}_1-\mathbf{q}_1|}\,)^{-14}\,(\sqrt{q_2}+\sqrt{|\boldsymbol{\kappa}_2-\mathbf{q}_2|}\,)^{-14}$$
$$\times\ \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_4)\,\delta(\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)\,.\qquad\text{(A.3)}$$

The parameters $\xi^L_i=\xi(\tau^L_i)$ have a form similar to (4.15), with the temporal variables evaluated, in each case, at the momenta appearing in the denominators of the respective expressions, i.e.:

$$
\begin{aligned}
&\tau_1^A = \tau_1^B = \tau_1^C \simeq -\left(\sqrt{q} + \sqrt{|\boldsymbol{\kappa}_1 - \mathbf{q}|}\right)^{-2}, &&\tau_2^A = \tau_2^B \simeq -\left(\sqrt{q} + \sqrt{|\boldsymbol{\kappa}_2 + \mathbf{q}|}\right)^{-2},\\
&\tau_3^A \simeq -\left(\sqrt{|\boldsymbol{\kappa}_1 - \mathbf{q}|} + \sqrt{|\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_4 + \mathbf{q}|}\right)^{-2}, &&\tau_4^B \simeq -\left(\sqrt{|\boldsymbol{\kappa}_2 + \mathbf{q}|} + \sqrt{|\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 + \mathbf{q}|}\right)^{-2},\\
&\tau_4^A \simeq -\left(\sqrt{|\boldsymbol{\kappa}_2 + \mathbf{q}|} + \sqrt{|\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_4 + \mathbf{q}|}\right)^{-2}, &&\tau_2^C \simeq -\left(\sqrt{q} + \sqrt{|\boldsymbol{\kappa}_3 + \mathbf{q}|}\right)^{-2},\\
&\tau_3^B = \tau_3^C \simeq -\left(\sqrt{|\boldsymbol{\kappa}_1 - \mathbf{q}|} + \sqrt{|\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 + \mathbf{q}|}\right)^{-2}, &&\tau_4^C \simeq -\left(\sqrt{|\boldsymbol{\kappa}_3 + \mathbf{q}|} + \sqrt{|\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 + \mathbf{q}|}\right)^{-2},\\
&\tau_1^E = \tau_2^E = \tau_1^F = \tau_2^F \simeq -\left(\sqrt{q_1} + \sqrt{|\boldsymbol{\kappa}_1 - \mathbf{q}_1|}\right)^{-2}, &&\tau_1^D = \tau_2^D \simeq -\left(\sqrt{q_1} + \sqrt{|\boldsymbol{\kappa}_1 - \mathbf{q}_1|}\right)^{-2},\\
&\tau_3^E = \tau_4^E = \tau_3^F = \tau_4^F \simeq -\left(\sqrt{q_2} + \sqrt{|\boldsymbol{\kappa}_2 - \mathbf{q}_2|}\right)^{-2}, &&\tau_3^D = \tau_4^D \simeq -\left(\sqrt{q_2} + \sqrt{|\boldsymbol{\kappa}_3 - \mathbf{q}_2|}\right)^{-2}.
\end{aligned}
\tag{A.4}
$$

B Sourced intrinsic correlator: Evaluation of integrals

The integral $I_{\rm I}$ in eq. (4.21) of the sourced intrinsic correlator in the regime of large momenta has the form

$$
\begin{aligned}
I_{\rm I} = &\frac{1}{k_{BR}^{8\pi\delta}} \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{q}\\
&\times \Bigg(\frac{q^{1/2}\,|\mathbf{p}_1 + \mathbf{q}|\,|\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{q}|^{1/2}\,\mathcal{A}_A(\mathbf{q}, -\mathbf{p}_1 - \mathbf{q}, \mathbf{p}_1 + \mathbf{q}, \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{q})}{(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{q}|}\,)^{14}\,(\sqrt{|\mathbf{p}_1 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{q}|}\,)^{14}\,(\tau^A_{\rm I,1}\,\tau^A_{\rm I,2}\,\tau^A_{\rm I,3}\,\tau^A_{\rm I,4})^{2\pi\delta}}\\
&+ \frac{q^{1/2}\,|\mathbf{p}_1 + \mathbf{q}|\,|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|^{1/2}\,\mathcal{A}_B(\mathbf{q}, -\mathbf{p}_1 - \mathbf{q}, \mathbf{p}_1 + \mathbf{q}, \mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2)}{(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{q}|}\,)^{14}\,(\sqrt{|\mathbf{p}_1 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|}\,)^{14}\,(\tau^B_{\rm I,1}\,\tau^B_{\rm I,2}\,\tau^B_{\rm I,3}\,\tau^B_{\rm I,4})^{2\pi\delta}}\\
&+ \frac{q^{1/2}\,|\mathbf{p}_1 + \mathbf{q}|^{1/2}\,|\mathbf{q} - \mathbf{p}_2|^{1/2}\,|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|^{1/2}}{(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{q}|}\,)^{7}\,(\sqrt{q} + \sqrt{|\mathbf{q} - \mathbf{p}_2|}\,)^{7}\,(\sqrt{|\mathbf{q} - \mathbf{p}_2|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|}\,)^{7}}\\
&\times \frac{\mathcal{A}_C(\mathbf{q}, -\mathbf{p}_1 - \mathbf{q}, \mathbf{q} - \mathbf{p}_2, \mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2)}{(\sqrt{|\mathbf{p}_1 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|}\,)^{7}\,(\tau^C_{\rm I,1}\,\tau^C_{\rm I,2}\,\tau^C_{\rm I,3}\,\tau^C_{\rm I,4})^{2\pi\delta}}\\
&+ 2\,\frac{p_2^{1/2}\,q^{1/2}\,|\mathbf{p}_1 + \mathbf{p}_2|^{1/2}\,|\mathbf{p}_1 - \mathbf{q}|^{1/2}\,\mathcal{A}_E(\mathbf{p}_2, -\mathbf{p}_1 - \mathbf{p}_2, \mathbf{q}, \mathbf{p}_1 - \mathbf{q})}{(\sqrt{p_2} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2|}\,)^{14}\,(\sqrt{q} + \sqrt{|\mathbf{p}_1 - \mathbf{q}|}\,)^{14}\,(\tau^E_{\rm I,1}\,\tau^E_{\rm I,2}\,\tau^E_{\rm I,3}\,\tau^E_{\rm I,4})^{2\pi\delta}} \Bigg),
\end{aligned}
\tag{B.1}
$$

where we have used $\tau_{BR} = -1/k_{BR}$ in equation (4.19) and neglected the parameter k wherever it appeared, as it is small compared to the large momenta considered in the integral. The time variables of equation (A.4) in this case take the form

$$
\begin{aligned}
&\tau^A_{I,1} = \tau^A_{I,2} = \tau^B_{I,1} = \tau^B_{I,2} = \tau^C_{I,1} \simeq -\left(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{q}|}\right)^{-2}, &&\tau^C_{I,2} \simeq -\left(\sqrt{q} + \sqrt{|\mathbf{q} - \mathbf{p}_2|}\right)^{-2},\\
&\tau^A_{I,3} = \tau^A_{I,4} \simeq -\left(\sqrt{|\mathbf{p}_1 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{q}|}\right)^{-2}, &&\tau^C_{I,3} \simeq -\left(\sqrt{|\mathbf{q} - \mathbf{p}_2|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|}\right)^{-2},\\
&\tau^B_{I,3} = \tau^B_{I,4} \simeq -\left(\sqrt{|\mathbf{p}_1 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|}\right)^{-2}, &&\tau^E_{I,1} = \tau^E_{I,2} \simeq -\left(\sqrt{p_2} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2|}\right)^{-2},\\
&\tau^C_{I,4} \simeq -\left(\sqrt{|\mathbf{p}_1 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|}\right)^{-2}, &&\tau^E_{I,3} = \tau^E_{I,4} \simeq -\left(\sqrt{q} + \sqrt{|\mathbf{p}_1 - \mathbf{q}|}\right)^{-2}.
\end{aligned}
$$

To compute the integral, we consider all possible orderings of the magnitudes of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{q}$ and integrate over one momentum at a time, keeping only the dominant terms in each expression. For example, in the case $p_1 > p_2 > q$, the expression $(\sqrt{|\mathbf{q} - \mathbf{p}_2|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|})^7$ simplifies to $p_1^{7/2}$, and the integration is carried out as

$$
\int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{q} = \int^{k_{BR}} dp_1\, p_1^2 \int^{p_1} dp_2\, p_2^2 \int^{p_2} dq\, q^2 \int d\Omega\,,
\tag{B.2}
$$

with

$$\int d\Omega = \prod_{i=p_1,p_2,q} \int d\theta_i \, d\phi_i \sin(\theta_i) \,. \tag{B.3}$$

A key simplification arises when the dominant momentum appears in both square roots in a sum. For instance, the expression $(\sqrt{|\mathbf{p}_1 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2|})^7$, when p_1 is the maximum momentum, becomes $2^7 \, p_1^{7/2}$. In such cases, the suppression factor of 2^7 in the denominator makes the corresponding term negligible. Therefore, although we would theoretically need to consider all six possible orderings of the three variables, the number of relevant cases is reduced by focusing only on terms without the large 2^7 suppression.

For the first expression of (B.1) the only ordering that does not lead to a suppression is $p_2 > p_1 > q$, for which the expression becomes

$$\begin{aligned} &\frac{1}{k_{BR}^{8\pi\delta}} \int^{k_{BR}} dp_2 \, p_2^{4\pi\delta - 9/2} \int^{p_2} dp_1 \, p_1^{4\pi\delta - 4} \int^{p_1} dq \, q^{5/2} \int d\Omega \, \mathcal{A}_A(\mathbf{q}, -\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_2) \\ &\quad = \frac{1.4 \times 10^4}{k_{BR}^3} \frac{2}{7\,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}, \quad \text{for } \delta > \frac{3}{8\pi} \,. \end{aligned} \tag{B.4}$$

The number 1.4×10^4 is the result of the numerical integration of the angular part over the solid angle (B.3). The second integral in (B.1) shares the same dependence on the momenta as the first one and therefore gives the same result. For the third integral in (B.1), there is no ordering of the momenta that avoids the significant 2^7 suppression, so it can be neglected. Finally, the last integral in (B.1) receives contributions only from the orderings $p_1 > q > p_2$ and $p_1 > p_2 > q$. Since these two orderings give rise to the same integral, we need to calculate only one of them:

$$\begin{aligned} &\frac{2}{k_{BR}^{8\pi\delta}} \int^{k_{BR}} dp_1 \, p_1^{8\pi\delta - 11} \int^{p_1} dp_2 \, p_2^{5/2} \int^{p_2} dq \, q^{5/2} \int d\Omega \, \mathcal{A}_E(\mathbf{p}_2, -\mathbf{p}_1, \mathbf{q}, \mathbf{p}_1) \\ &\quad = \frac{1.8 \times 10^3}{k_{BR}^3} \frac{4}{49\,(8\pi\delta - 3)}, \quad \text{for } \delta > \frac{3}{8\pi} \,. \end{aligned} \tag{B.5}$$

Summing all contributions, we obtain

$$I_{\mathrm{I}} = \frac{8.1 \times 10^3}{k_{BR}^3\,(4\pi\delta + 1/2)\,(8\pi\delta - 3)} + \frac{2.9 \times 10^2}{k_{BR}^3\,(8\pi\delta - 3)}, \quad \text{for } \delta > \frac{3}{8\pi} \,. \tag{B.6}$$

In practice, the parameter δ takes values in the range $0.06 - 0.2$. However, since this integral contributes to a correlator that turns out to be negligible for any value of δ (due to the suppression factor $(k/k_{\mathrm{BR}})^3$), we can adopt the value $\delta = 0.2$ that allows us to simplify the integrals, while still providing a reasonable estimate. For $\delta = 0.2$, the integral becomes

$$I_{\mathrm{I}} \simeq \frac{1.5 \times 10^3}{k^3} \left(\frac{k}{k_{BR}} \right)^3, \tag{B.7}$$

that gives the result present in (4.21).

C Sourced extrinsic correlator: Evaluation of integrals

C.1 Term: $\mathcal{C}^{\mathrm{S.I.}}_{\Omega\Omega}(\mathbf{k})$

The integral $I_{\mathrm{S.I.}}$ in eq. (4.27) of the scale-invariant part of $\mathcal{C}^{\mathrm{E}}_{\Omega\Omega,3}(\mathbf{k})$ in the regime of large momenta has the form

$$
\begin{aligned}
I_{\mathrm{S.I.}} = \frac{1}{k_{BR}^{8\pi\delta}} \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{q}_1\, d\mathbf{q}_2 \\
\times\, \frac{q_1^{1/2}\, |\mathbf{p}_1+\mathbf{q}_1|^{1/2}\, q_2^{1/2}\, |\mathbf{p}_2+\mathbf{q}_2|^{1/2}\, \mathcal{A}_D(\mathbf{q}_1, -\mathbf{p}_1-\mathbf{q}_1, \mathbf{q}_2, -\mathbf{p}_2-\mathbf{q}_2)}{(\sqrt{q_1}+\sqrt{|\mathbf{p}_1+\mathbf{q}_1|}\,)^{14}\, (\sqrt{q_2}+\sqrt{|\mathbf{p}_2+\mathbf{q}_2|}\,)^{14}\, (\tau_1^{\mathrm{S.I.}}\, \tau_2^{\mathrm{S.I.}}\, \tau_3^{\mathrm{S.I.}}\, \tau_4^{\mathrm{S.I.}})^{2\pi\delta}}\,,
\end{aligned} \tag{C.1}
$$

with

$$
\begin{aligned}
\tau_1^{\mathrm{S.I.}} = \tau_2^{\mathrm{S.I.}} \simeq -(\sqrt{q_1}+\sqrt{|\mathbf{p}_1+\mathbf{q}_1|}\,)^{-2}\,, \\
\tau_3^{\mathrm{S.I.}} = \tau_4^{\mathrm{S.I.}} \simeq -(\sqrt{q_2}+\sqrt{|\mathbf{p}_2+\mathbf{q}_2|}\,)^{-2}\,.
\end{aligned}
$$

The computation is done by integrating over one momentum at a time, accounting for all possible orderings of the four variables, as done in Appendix B. However, when $q_1 > p_1$ or $q_2 > p_2$, a large suppression factor of 2^{14} appears in each of the two parentheses in the denominator. Thus, we can restrict our calculations to the cases where $p_1 > q_1$ and $p_2 > q_2$, for which the integral becomes

$$
I_{\mathrm{S.I.}} = \frac{1}{k_{BR}^{8\pi\delta}} \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{q}_1\, d\mathbf{q}_2\, q_1^{1/2}\, q_2^{1/2}\, p_1^{4\pi\delta-13/2}\, p_2^{4\pi\delta-13/2}\, \mathcal{A}_D(\mathbf{q}_1, -\mathbf{p}_1, \mathbf{q}_2, -\mathbf{p}_2)\,. \tag{C.2}
$$

Finally, due to the symmetry of the integral under the exchanges $\mathbf{p}_1 \leftrightarrow \mathbf{p}_2$ and $\mathbf{q}_1 \leftrightarrow \mathbf{q}_2$, it simplifies to

$$
\begin{aligned}
I_{\mathrm{S.I.}} &= \frac{1}{k_{BR}^{8\pi\delta}} \left(\int^{k_{BR}} dp\, p^{4\pi\delta-9/2} \int^{p} dq\, q^{5/2} \right)^2 \int d\Omega\, \mathcal{A}_D(\mathbf{q}_1, -\mathbf{p}_1, \mathbf{q}_2, -\mathbf{p}_2) \\
&\simeq 4.4\times 10^4 \left(\frac{1}{14\pi\delta} \right)^2,
\end{aligned} \tag{C.3}
$$

where in $d\Omega$ we have collected the angular integrations on the polar angles of all the four variables, i.e. $d\Omega = d\Omega_{p_1}\, d\Omega_{p_2}\, d\Omega_{q_1}\, d\Omega_{q_2}$.

C.2 Term: $\mathcal{C}^{\mathrm{E}}_{\Omega\Omega,1}(\mathbf{k})$

The first contribution to the extrinsic correlator of the gravitational wave energy densities in the regime of large momenta, eq. (4.30), is found by evaluating the integral

$$
\begin{aligned}
I_{\mathrm{E},1} = \frac{1}{k_{BR}^{8\pi\delta}} \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3\, d\mathbf{q}\, \frac{1}{p_3^3} \\
\times \Bigg(\frac{q^{1/2}\, |\mathbf{p}_1+\mathbf{q}|\, |\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{q}|^{1/2}\, \mathcal{A}_A(\mathbf{q}, -\mathbf{p}_1-\mathbf{q}, \mathbf{p}_1+\mathbf{q}, \mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{q})}{(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|}\,)^{14}\, (\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{q}|}\,)^{14}\, (\tau^A_{\mathrm{E1},1}\, \tau^A_{\mathrm{E1},2}\, \tau^A_{\mathrm{E1},3}\, \tau^A_{\mathrm{E1},4})^{2\pi\delta}} \\
+ \frac{q^{1/2}\, |\mathbf{p}_1+\mathbf{q}|\, |\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|^{1/2}\, \mathcal{A}_B(\mathbf{q}, -\mathbf{p}_1-\mathbf{q}, \mathbf{p}_1+\mathbf{q}, \mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3)}{(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|}\,)^{14}\, (\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|}\,)^{14}\, (\tau^B_{\mathrm{E1},1}\, \tau^B_{\mathrm{E1},2}\, \tau^B_{\mathrm{E1},3}\, \tau^B_{\mathrm{E1},4})^{2\pi\delta}}
\end{aligned}
$$

$$+\frac{q^{1/2}\,|\mathbf{p}_1+\mathbf{q}|^{1/2}\,|\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|^{1/2}\,|\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|^{1/2}}{(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|}\,)^7\,(\sqrt{q}+\sqrt{|\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|}\,)^7\,(\sqrt{|\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|}+\sqrt{|\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|}\,)^7}$$
$$\times\frac{\mathcal{A}_C(\mathbf{q},-\mathbf{p}_1-\mathbf{q},\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3,\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3)}{(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|}\,)^7\,(\tau^C_{\mathrm{E}1,1}\,\tau^C_{\mathrm{E}1,2}\,\tau^C_{\mathrm{E}1,3}\,\tau^C_{\mathrm{E}1,4})^{2\pi\delta}}$$
$$+2\,\frac{p_2^{1/2}\,q^{1/2}\,|\mathbf{p}_1+\mathbf{p}_2|^{1/2}\,|\mathbf{p}_1-\mathbf{q}|^{1/2}\,\mathcal{A}_E(\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}_2,\mathbf{q},\mathbf{p}_1-\mathbf{q})}{(\sqrt{p_2}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2|}\,)^{14}\,(\sqrt{q}+\sqrt{|\mathbf{p}_1-\mathbf{q}|}\,)^{14}\,(\tau^E_{\mathrm{E}1,1}\,\tau^E_{\mathrm{E}1,2}\,\tau^E_{\mathrm{E}1,3}\,\tau^E_{\mathrm{E}1,4})^{2\pi\delta}}\Bigg)\,, \tag{C.4}$$

where we have neglected the k terms in the sums and with the time parameters given by

$$\tau^A_{E1,1}=\tau^A_{E1,2}=\tau^B_{E1,1}=\tau^B_{E1,2}=\tau^C_{E1,1}\simeq-(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|})^{-2}\,,$$
$$\tau^A_{E1,3}=\tau^A_{E1,4}\simeq-(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{q}|})^{-2},\qquad \tau^C_{E1,2}\simeq-(\sqrt{q}+\sqrt{|\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|})^{-2},$$
$$\tau^B_{E1,3}=\tau^B_{E1,4}\simeq-(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|})^{-2},\qquad \tau^E_{E1,1}=\tau^E_{E1,2}\simeq-(\sqrt{p_2}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2|})^{-2},$$
$$\tau^C_{E1,3}\simeq-(\sqrt{|\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|}+\sqrt{|\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|})^{-2}\,,\qquad \tau^E_{E1,3}=\tau^E_{E1,4}\simeq-(\sqrt{q}+\sqrt{|\mathbf{p}_1-\mathbf{q}|})^{-2}\,,$$
$$\tau^C_{E1,4}\simeq-(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{q}-\mathbf{p}_2-\mathbf{p}_3|})^{-2}.$$

In order to compute (C.4), we again consider specific orderings of the momenta and integrate over one momentum at a time keeping only the dominant terms, as done in Appendix B. Many orderings give significantly suppressed results due to the 2^7 factors in the denominators whenever the largest momentum appears in both roots of a sum. As a result, the number of contributing cases is greatly reduced. For the first expression of (C.4), there are six cases that contribute significantly:

$p_3>p_2>p_1>q$	$\frac{5.1\times10^4}{k^3_{BR}\,(4\pi\delta+1/2)\,(4\pi\delta+7/2)\,(8\pi\delta-3)}$
$p_2>p_3>p_1>q$	$\frac{5.1\times10^4}{k^3_{BR}\,(4\pi\delta+1/2)^2\,(8\pi\delta-3)}$
$p_3>p_1>p_2>q$	$\frac{7.8\times10^3}{k^3_{BR}\,(4\pi\delta+7/2)\,(8\pi\delta-3)}$
$p_2>p_1>p_3>q$	$\frac{1.4\times10^4}{k^3_{BR}\,(4\pi\delta+1/2)\,(8\pi\delta-3)}$
$p_3>p_1>q>p_2$	$\frac{9.1\times10^3}{k^3_{BR}\,(4\pi\delta+7/2)\,(8\pi\delta-3)}$
$p_2>p_1>q>p_3$	$\frac{1.4\times10^4\,\left(1.2\times10^3\pi^2\delta^2-4.7\times10^2\pi\delta-22.\right)}{k^3_{BR}\,(8\pi\delta-3)^2(8\pi\delta+1)^2}$

valid for $\delta>\frac{3}{8\pi}$. The second expression of (C.4) has the same dependence on momenta as the first one and so gives the same result. For the third expression of (C.4) there are not combinations which do not suffer the suppression. Finally, for the fourth expression, we consider all combinations where $p_1>q$ and $p_1>p_2$. Although there are eight such combinations, the symmetry between p_2 and q in the integral ensures that swapping p_2 and q gives the same result. Therefore, we need to compute only four distinct cases:

$p_1 > p_3 > q > p_2$ (or $p_1 > p_3 > p_2 > q$)	$\dfrac{1.3 \times 10^2}{k_{BR}^3 \, (8\pi\delta - 3)}$
$p_1 > q > p_3 > p_2$ (or $p_1 > p_2 > p_3 > q$)	$\dfrac{2.6 \times 10^2}{k_{BR}^3 \, (8\pi\delta - 3)}$
$p_1 > q > p_2 > p_3$ (or $p_1 > p_2 > q > p_3$)	$\dfrac{1.3 \times 10^2 \, (1.4 \times 10^2 \, \pi\delta - 6.1 \times 10)}{k_{BR}^3 \, (8\pi\delta - 3)^2}$
$p_3 > p_1 > q > p_2$ (or $p_3 > p_1 > p_2 > q$)	$\dfrac{9. \times 10^2}{k_{BR}^3 \, (8\pi\delta - 3)^2}$

valid for $\delta > \frac{3}{8\pi}$. Summing all contributions, we obtain

$$I_{\mathrm{E},1} = \frac{-1.4 \times 10^2 + 1. \times 10^3 \, \delta + 4.3 \times 10^2 \, \delta^2}{k_{BR}^3 \, \left(1.2 \times 10^{-1} - 1. \, \delta\right)^2 \left(4. \times 10^{-2} + 1. \, \delta\right)} \,, \quad \text{for } \delta > \frac{3}{8\pi} \,. \tag{C.5}$$

In particular for $\delta = 0.2$ the integral takes the value

$$I_{\mathrm{E},1} \simeq \frac{5.2 \times 10^4}{k^3} \left(\frac{k}{k_{BR}}\right)^3 , \tag{C.6}$$

used in (4.30).

C.3 Term: $\mathcal{C}^{\mathrm{E}}_{\Omega\Omega,2}(\mathbf{k})$

The second contribution to the extrinsic correlator of the gravitational wave energy densities in the regime of large momenta, eq. (4.31), is found by evaluating the integral

$$\begin{aligned}
I_{\mathrm{E},2} = \; & \frac{1}{k_{BR}^{8\pi\delta}} \int d\mathbf{p}_1 \, d\mathbf{p}_2 \, d\mathbf{p}_3 \, d\mathbf{q} \, \frac{1}{p_3^3} \\
\times & \Bigg(\frac{q^{1/2} \, |\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}| \, |\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{q}|^{1/2}}{(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|}\,)^{14}} \\
\times & \frac{\mathcal{A}_A(\mathbf{q}, -\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{q}, \mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}, \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{q})}{(\sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{q}|}\,)^{14} \, (\tau^A_{\mathrm{E}2,1} \, \tau^A_{\mathrm{E}2,2} \, \tau^A_{\mathrm{E}2,3} \, \tau^A_{\mathrm{E}2,4})^{2\pi\delta}} \\
+ & \frac{q^{1/2} \, |\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}| \, |\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|^{1/2}}{(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|}\,)^{14}} \\
\times & \frac{\mathcal{A}_B(\mathbf{q}, -\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{q}, \mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}, \mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3)}{(\sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|}\,)^{14} \, (\tau^B_{\mathrm{E}2,1} \, \tau^B_{\mathrm{E}2,2} \, \tau^B_{\mathrm{E}2,3} \, \tau^B_{\mathrm{E}2,4})^{2\pi\delta}} \\
+ & \frac{q^{1/2} \, |\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|^{1/2} \, |\mathbf{q} - \mathbf{p}_2|^{1/2} \, |\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|^{1/2}}{(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|}\,)^7 \, (\sqrt{q} + \sqrt{|\mathbf{q} - \mathbf{p}_2|}\,)^7 \, (\sqrt{|\mathbf{q} - \mathbf{p}_2|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|}\,)^7} \\
\times & \frac{\mathcal{A}_C(\mathbf{q}, -\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{q}, \mathbf{q} - \mathbf{p}_2, \mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3)}{(\sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|}\,)^7 \, (\tau^C_{\mathrm{E}2,1} \, \tau^C_{\mathrm{E}2,2} \, \tau^C_{\mathrm{E}2,3} \, \tau^C_{\mathrm{E}2,4})^{2\pi\delta}} \\
+ & 2 \, \frac{p_2^{1/2} \, q^{1/2} \, |\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3|^{1/2} \, |\mathbf{p}_1 + \mathbf{p}_3 - \mathbf{q}|^{1/2} \, \mathcal{A}_E(\mathbf{p}_2, -\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3, \mathbf{q}, \mathbf{p}_1 + \mathbf{p}_3 - \mathbf{q})}{(\sqrt{p_2} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3|}\,)^{14} \, (\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_3 - \mathbf{q}|}\,)^{14} \, (\tau^E_{\mathrm{E}2,1} \, \tau^E_{\mathrm{E}2,2} \, \tau^E_{\mathrm{E}2,3} \, \tau^E_{\mathrm{E}2,4}))^{2\pi\delta}} \Bigg) ,
\end{aligned} \tag{C.7}$$

where we have neglected the k terms in the sums and with the time parameters given by

$$\begin{aligned}
&\tau^A_{\rm E2,1} = \tau^A_{\rm E2,2} = \tau^B_{\rm E2,1} = \tau^B_{\rm E2,2} = \tau^C_{\rm E2,1} \simeq -(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|}\,)^{-2}\,,\\
&\tau^A_{\rm E2,3} = \tau^A_{\rm E2,4} \simeq -(\sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{q}|}\,)^{-2}\,,\\
&\tau^B_{\rm E2,3} = \tau^B_{\rm E2,4} \simeq -(\sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|}\,)^{-2}\,,\\
&\tau^C_{\rm E2,2} \simeq -(\sqrt{q} + \sqrt{|\mathbf{q} - \mathbf{p}_2|}\,)^{-2}\,,\\
&\tau^C_{\rm E2,3} \simeq -(\sqrt{|\mathbf{q} - \mathbf{p}_2|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|}\,)^{-2}\,,\\
&\tau^C_{\rm E2,4} \simeq -(\sqrt{|\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{q}|} + \sqrt{|\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2 + \mathbf{p}_3|}\,)^{-2}\,,\\
&\tau^E_{\rm E2,1} = \tau^E_{\rm E2,2} \simeq -(\sqrt{p_2} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3|}\,)^{-2}\,,\\
&\tau^E_{\rm E2,3} = \tau^E_{\rm E2,4} \simeq -(\sqrt{q} + \sqrt{|\mathbf{p}_1 + \mathbf{p}_3 - \mathbf{q}|}\,)^{-2}\,.
\end{aligned}$$

We perform again one integration at the time considering all the possible orderings of the four momenta, i.e. $4! = 24$ permutations, but neglecting all the terms suppressed by 2^7 factors at the denominators, as done in Appendix B. For the first expression of (C.7) the only cases that survive are the four cases with $q < p_1$ or p_3 and $p_2 > p_1, p_3$ and q:

$p_2 > p_1 > p_3 > q$	$\dfrac{1.4 \times 10^4}{k^3_{BR}\,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}$
$p_2 > p_3 > p_1 > q$	$\dfrac{7.8 \times 10^3}{k^3_{BR}\,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}$
$p_2 > p_1 > q > p_3$	$\dfrac{1.4 \times 10^4\,\left(1.2 \times 10^3\pi^2\delta^2 - 4.7 \times 10^2\pi\delta - 22.\right)}{k^3_{BR}\,(8\pi\delta - 3)^2(8\pi\delta + 1)^2}$
$p_2 > p_3 > q > p_1$	$\dfrac{9.1 \times 10^3}{k^3_{BR}\,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}$

valid for $\delta > \frac{3}{8\pi}$. The second expression of (C.7) has the same dependence on momenta as the first one and so gives the same result. The third expression contains only suppressed combinations. Finally, for the fourth expression, we consider all combinations where p_1 or $p_3 > q$ and p_1 or $p_3 > p_2$. Although there are twelve such combinations, the symmetry between p_2 and q allow us to compute only six of them:

$p_1 > p_3 > q > p_2$ (or $p_1 > p_3 > p_2 > q$)	$\dfrac{1.3 \times 10^2}{k^3_{BR}\,(8\pi\delta - 3)}$
$p_1 > q > p_3 > p_2$ (or $p_1 > p_2 > p_3 > q$)	$\dfrac{2.6 \times 10^2}{k^3_{BR}\,(8\pi\delta - 3)}$
$p_1 > q > p_2 > p_3$ (or $p_1 > p_2 > q > p_3$)	$\dfrac{1.3 \times 10^2\,(1.4 \times 10^2\,\pi\delta - 6.1 \times 10))}{k^3_{BR}\,(8\pi\delta - 3)^2}$
$p_3 > p_1 > q > p_2$ (or $p_3 > p_1 > p_2 > q$)	$\dfrac{9. \times 10^1}{k^3_{BR}\,(8\pi\delta - 3)}$
$p_3 > q > p_1 > p_2$ (or $p_3 > p_2 > p_1 > q$)	$\dfrac{9.7 \times 10^1}{k^3_{BR}\,(8\pi\delta - 3)}$
$p_3 > q > p_2 > p_1$ (or $p_3 > p_2 > q > p_1$)	$\dfrac{1.1 \times 10^2}{k^3_{BR}\,(8\pi\delta - 3)}$

valid for $\delta > \frac{3}{8\pi}$. Summing all contributions, we obtain

$$I_{\mathrm{E},2} = \frac{-2.6 - 1.3\times 10^2\,\delta + 1.\times 10^3\,\delta^2 + 4.8\times 10^2\,\delta^3}{k_{BR}^3\,(4.7\times 10^{-3} + 8.\times 10^{-2}\,\delta - 1.\,\delta^2)^2}\,,\quad \text{for } \delta > \frac{3}{8\pi}\,. \tag{C.8}$$

In particular for $\delta = 0.2$ the integral takes the value

$$I_{\mathrm{E},2} \simeq \frac{4.7\times 10^4}{k^3}\left(\frac{k}{k_{BR}}\right)^3, \tag{C.9}$$

used in (4.31).

C.4 Term: $\mathcal{C}^{\mathrm{E}}_{\Omega\Omega,3}(\mathbf{k})'$

The non scale-invariant part of the third contribution to the extrinsic correlator of the gravitational wave energy densities in the regime of large momenta, eq. (4.32), is found by evaluating the integral

$$\begin{aligned}
I_{\mathrm{E},3} = {} & \frac{1}{k_{BR}^{8\pi\delta}} \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3\, d\mathbf{q}\, \frac{1}{p_3^3} \\
& \times \Bigg(\frac{q^{1/2}\,|\mathbf{p}_1+\mathbf{q}|^{1/2}\,|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|^{1/2}\,|\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}|^{1/2}}{(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|}\,)^7\,(\sqrt{q}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|}\,)^7\,(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}|}\,)^7} \\
& \times \frac{\mathcal{A}_A(\mathbf{q},-\mathbf{p}_1-\mathbf{q},\mathbf{p}_1-\mathbf{p}_3+\mathbf{q},\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})}{(\sqrt{|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}|}\,)^7\,(\tau^A_{\mathrm{E3},1}\,\tau^A_{\mathrm{E3},2}\,\tau^A_{\mathrm{E3},3}\,\tau^A_{\mathrm{E3},4})^{2\pi\delta}} \\
& + \frac{q^{1/2}\,|\mathbf{p}_1+\mathbf{q}|^{1/2}\,|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|^{1/2}\,|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|^{1/2}}{(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|}\,)^7\,(\sqrt{q}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|}\,)^7\,(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^7} \\
& \times \frac{\mathcal{A}_B(\mathbf{q},-\mathbf{p}_1-\mathbf{q},\mathbf{p}_1-\mathbf{p}_3+\mathbf{q},\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q})}{(\sqrt{|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^7\,(\tau^B_{\mathrm{E3},1}\,\tau^B_{\mathrm{E3},2}\,\tau^B_{\mathrm{E3},3}\,\tau^B_{\mathrm{E3},4})^{2\pi\delta}} \\
& + \frac{q^{1/2}\,|\mathbf{p}_1+\mathbf{q}|^{1/2}\,|\mathbf{q}-\mathbf{p}_2|^{1/2}\,|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|^{1/2}}{(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|}\,)^7\,(\sqrt{q}+\sqrt{|\mathbf{q}-\mathbf{p}_2|}\,)^7\,(\sqrt{|\mathbf{q}-\mathbf{p}_2|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^7} \\
& \times \frac{\mathcal{A}_C(\mathbf{q},-\mathbf{p}_1-\mathbf{q},\mathbf{q}-\mathbf{p}_2,\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q})}{(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^7\,(\tau^C_{\mathrm{E3},1}\,\tau^C_{\mathrm{E3},2}\,\tau^C_{\mathrm{E3},3}\,\tau^C_{\mathrm{E3},4})^{2\pi\delta}} \\
& + 2\,\frac{p_2^{1/2}\,q^{1/2}\,|\mathbf{p}_1+\mathbf{p}_2|^{1/2}\,|\mathbf{p}_1-\mathbf{p}_3-\mathbf{q}|^{1/2}\,\mathcal{A}_E(\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}_2,\mathbf{q},\mathbf{p}_1-\mathbf{p}_3-\mathbf{q})}{(\sqrt{p_2}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2|}\,)^{14}\,(\sqrt{q}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{q}|}\,)^{14}\,(\tau^E_{\mathrm{E3},1}\,\tau^E_{\mathrm{E3},2}\,\tau^E_{\mathrm{E3},3}\,\tau^E_{\mathrm{E3},4})^{2\pi\delta}}\Bigg)\,,
\end{aligned} \tag{C.10}$$

where we have neglected the k terms in the sums and with the time parameters given by

$$\begin{aligned}
&\tau^A_{\mathrm{E3},1} = \tau^B_{\mathrm{E3},1} = \tau^C_{\mathrm{E3},1} \simeq -(\sqrt{q}+\sqrt{|\mathbf{p}_1+\mathbf{q}|}\,)^{-2}\,, && \tau^B_{\mathrm{E3},3} \simeq -(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^{-2}\,,\\
&\tau^A_{\mathrm{E3},2} = \tau^B_{\mathrm{E3},2} \simeq -(\sqrt{q}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|}\,)^{-2}\,, && \tau^C_{\mathrm{E3},2} \simeq -(\sqrt{q}+\sqrt{|\mathbf{q}-\mathbf{p}_2|}\,)^{-2}\,,\\
&\tau^A_{\mathrm{E3},3} \simeq -(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}|}\,)^{-2}\,, && \tau^C_{\mathrm{E3},3} \simeq -(\sqrt{|\mathbf{q}-\mathbf{p}_2|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^{-2}\,,\\
&\tau^A_{\mathrm{E3},4} \simeq -(\sqrt{|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}|}\,)^{-2}\,, && \tau^E_{\mathrm{E3},1} = \tau^E_{\mathrm{E3},2} \simeq -(\sqrt{p_2}+\sqrt{|\mathbf{p}_1+\mathbf{p}_2|}\,)^{-2}\,,\\
&\tau^B_{\mathrm{E3},4} \simeq -(\sqrt{|\mathbf{p}_1-\mathbf{p}_3+\mathbf{q}|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^{-2}\,, && \tau^E_{\mathrm{E3},3} = \tau^E_{\mathrm{E3},4} \simeq -(\sqrt{q}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{q}|}\,)^{-2}\,,\\
&\tau^C_{\mathrm{E3},4} \simeq -(\sqrt{|\mathbf{p}_1+\mathbf{q}|}+\sqrt{|\mathbf{p}_1-\mathbf{p}_3-\mathbf{p}_2+\mathbf{q}|}\,)^{-2}\,.
\end{aligned}$$

We perform again one integration at the time neglecting all the terms suppressed by 2^7 factors, as done in Appendix B. For the first expression of (C.10) the only cases that survive are the four cases with $p_2 > p_1 > q$:

$p_2 > p_1 > q > p_3$	$\dfrac{1.4 \times 10^4 \left(1.2 \times 10^3 \pi^2 \delta^2 - 4.7 \times 10^2 \pi\delta - 22.\right)}{k_{BR}^3 \,(8\pi\delta - 3)^2 (8\pi\delta + 1)^2}$
$p_2 > p_1 > p_3 > q$	$\dfrac{1.4 \times 10^4}{k_{BR}^3 \,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}$
$p_2 > p_3 > p_1 > q$	$\dfrac{1.7 \times 10^4}{k_{BR}^3 \,(2\pi\delta + 7/2)\,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}$
$p_3 > p_2 > p_1 > q$	$\dfrac{1.7 \times 10^4}{k_{BR}^3 \,(2\pi\delta + 7/2)\,(4\pi\delta + 7/2)\,(8\pi\delta - 3)}$

valid for $\delta > \frac{3}{8\pi}$. For the second expression, the cases that contribute are those where p_2 is the largest among all momenta, with $p_1 > q$. This condition leaves us with three possible cases:

$p_2 > p_3 > p_1 > q$	$\dfrac{1.7 \times 10^4}{k_{BR}^3 \,(2\pi\delta + 7/2)\,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}$
$p_2 > p_1 > p_3 > q$	$\dfrac{1.4 \times 10^4}{k_{BR}^3 \,(4\pi\delta + 1/2)\,(8\pi\delta - 3)}$
$p_2 > p_1 > q > p_3$	$\dfrac{1.4 \times 10^4 \left(1.2 \times 10^3 \pi^2 \delta^2 - 4.7 \times 10^2 \pi\delta - 22.\right)}{k_{BR}^3 \,(8\pi\delta - 3)^2 (8\pi\delta + 1)^2}$

valid for $\delta > \frac{3}{8\pi}$. In the third expression, only two cases survive, i.e. $p_3 > p_2 > p_1 > q$ and $p_3 > p_1 > p_2 > q$, which give rise to the same result:

$$\frac{6.3 \times 10^3}{k_{BR}^3 \,(2\pi\delta + 7/2)\,(4\pi\delta + 7/2)\,(8\pi\delta - 3)}\,, \quad \text{for } \delta > \frac{3}{8\pi}\,. \tag{C.11}$$

In the fourth expression, we have nine cases, satisfying $p_1 > p_2$ and $q < p_1$ or p_3, which reduce to five if we use the symmetry under the exchange $p_2 \leftrightarrow q$:

$p_1 > p_2 > q > p_3$ (or $p_1 > q > p_2 > p_3$)	$\dfrac{1.3 \times 10^2 \,(1.4 \times 10^2 \,\pi\delta - 6.1 \times 10)}{k_{BR}^3 \,(8\pi\delta - 3)^2}$
$p_1 > p_2 > p_3 > q$ (or $p_1 > q > p_3 > p_2$)	$\dfrac{2.6 \times 10^2}{k_{BR}^3 \,(8\pi\delta - 3)}$
$p_1 > p_3 > p_2 > q$ (or $p_1 > p_3 > q > p_2$)	$\dfrac{1.3 \times 10^2}{k_{BR}^3 \,(8\pi\delta - 3)}$
$p_3 > p_1 > p_2 > q$ (or $p_3 > p_1 > q > p_2$)	$\dfrac{3. \times 10^2}{k_{BR}^3 \,(4\pi\delta + 7/2)\,(8\pi\delta - 3)}$
$p_3 > q > p_1 > p_2$	$\dfrac{2.1 \times 10^3}{k_{BR}^3 \,(4\pi\delta)\,(4\pi\delta + 7/2)\,(8\pi\delta - 3)}$

valid for $\delta > \frac{3}{8\pi}$. Summing all contributions, we obtain

$$I_{\mathrm{E},3} = \frac{1}{k_{BR}^3 \,\delta\,(2.8 \times 10^{-1} + 1.\,\delta)\,(5.6 \times 10^{-1} + 1.\,\delta)\,(4.7 \times 10^{-3} + 8. \times 10^{-2}\,\delta - 1.\,\delta^2)^2}$$

$$\times \left(-5.6\times10^{-5} - 3.5\times10^{-1}\,\delta - 2.\times10^{1}\,\delta^2 + 5.1\times10^{1}\,\delta^3 + 7.4\times10^{2}\,\delta^4 + 1.1\times10^{3}\,\delta^5 + 2.2\times10^{2}\,\delta^6\right), \tag{C.12}$$

valid for $\delta > \frac{3}{8\pi}$. In particular for $\delta = 0.2$ the integral takes the value

$$I_{\mathrm{E},3} = \frac{4.\times10^4}{k^3}\left(\frac{k}{k_{BR}}\right)^3, \tag{C.13}$$

used in (4.32).

References

[1] B. P. Abbott *et al.* [LIGO Scientific and Virgo], Phys. Rev. Lett. **116**, no.6, 061102 (2016) doi:10.1103/PhysRevLett.116.061102 [arXiv:1602.03837 [gr-qc]].

[2] G. Agazie *et al.* [NANOGrav], Astrophys. J. Lett. **951**, no.1, L8 (2023) doi:10.3847/2041-8213/acdac6 [arXiv:2306.16213 [astro-ph.HE]].

[3] J. Antoniadis *et al.* [EPTA and InPTA:], "The second data release from the European Pulsar Timing Array - III. Search for gravitational wave signals," Astron. Astrophys. **678**, A50 (2023) doi:10.1051/0004-6361/202346844 [arXiv:2306.16214 [astro-ph.HE]].

[4] D. J. Reardon, A. Zic, R. M. Shannon, G. B. Hobbs, M. Bailes, V. Di Marco, A. Kapur, A. F. Rogers, E. Thrane and J. Askew, *et al.* "Search for an Isotropic Gravitational-wave Background with the Parkes Pulsar Timing Array," Astrophys. J. Lett. **951**, no.1, L6 (2023) doi:10.3847/2041-8213/acdd02 [arXiv:2306.16215 [astro-ph.HE]].

[5] B. Allen, [arXiv:gr-qc/9604033 [gr-qc]].

[6] C. Caprini and D. G. Figueroa, Class. Quant. Grav. **35**, no.16, 163001 (2018) doi:10.1088/1361-6382/aac608 [arXiv:1801.04268 [astro-ph.CO]].

[7] T. Regimbau, Res. Astron. Astrophys. **11**, 369-390 (2011) doi:10.1088/1674-4527/11/4/001 [arXiv:1101.2762 [astro-ph.CO]].

[8] E. Pajer and M. Peloso, "A review of Axion Inflation in the era of Planck," Class. Quant. Grav. **30**, 214002 (2013) doi:10.1088/0264-9381/30/21/214002 [arXiv:1305.3557 [hep-th]].

[9] K. Freese, J. A. Frieman and A. V. Olinto, "Natural inflation with pseudo - Nambu-Goldstone bosons," Phys. Rev. Lett. **65** (1990), 3233-3236 doi:10.1103/PhysRevLett.65.3233

[10] L. Sorbo, "Parity violation in the Cosmic Microwave Background from a pseudoscalar inflaton," JCAP **06**, 003 (2011) doi:10.1088/1475-7516/2011/06/003 [arXiv:1101.1525 [astro-ph.CO]].

[11] N. Barnaby and M. Peloso, "Large Nongaussianity in Axion Inflation," Phys. Rev. Lett. **106**, 181301 (2011) doi:10.1103/PhysRevLett.106.181301 [arXiv:1011.1500 [hep-ph]].

[12] R. Namba, M. Peloso, M. Shiraishi, L. Sorbo and C. Unal, "Scale-dependent gravitational waves from a rolling axion," JCAP **01**, 041 (2016) doi:10.1088/1475-7516/2016/01/041 [arXiv:1509.07521 [astro-ph.CO]].

[13] A. Linde, S. Mooij and E. Pajer, "Gauge field production in supergravity inflation: Local non-Gaussianity and primordial black holes," Phys. Rev. D **87** (2013) no.10, 103506 doi:10.1103/PhysRevD.87.103506 [arXiv:1212.1693 [hep-th]].

[14] M. M. Anber and E. Sabancilar, "Hypermagnetic Fields and Baryon Asymmetry from Pseudoscalar Inflation," Phys. Rev. D **92**, no.10, 101501 (2015) doi:10.1103/PhysRevD.92.101501 [arXiv:1507.00744 [hep-th]].

[15] W. D. Garretson, G. B. Field and S. M. Carroll, "Primordial magnetic fields from pseudoGoldstone bosons," Phys. Rev. D **46**, 5346-5351 (1992) doi:10.1103/PhysRevD.46.5346 [arXiv:hep-ph/9209238 [hep-ph]].

[16] M. M. Anber and L. Sorbo, JCAP **10**, 018 (2006) doi:10.1088/1475-7516/2006/10/018 [arXiv:astro-ph/0606534 [astro-ph]].

[17] J. L. Cook and L. Sorbo, "Particle production during inflation and gravitational waves detectable by ground-based interferometers," Phys. Rev. D **85**, 023534 (2012) [erratum: Phys. Rev. D **86**, 069901 (2012)] doi:10.1103/PhysRevD.85.023534 [arXiv:1109.0022 [astro-ph.CO]].

[18] R. Abbott *et al.* [KAGRA, Virgo and LIGO Scientific], "Search for anisotropic gravitational-wave backgrounds using data from Advanced LIGO and Advanced Virgo's first three observing runs," Phys. Rev. D **104**, no.2, 022005 (2021) doi:10.1103/PhysRevD.104.022005 [arXiv:2103.08520 [gr-qc]].

[19] N. Bartolo *et al.* [LISA Cosmology Working Group], "Probing anisotropies of the Stochastic Gravitational Wave Background with LISA," JCAP **11**, 009 (2022) doi:10.1088/1475-7516/2022/11/009 [arXiv:2201.08782 [astro-ph.CO]].

[20] M. Geller, A. Hook, R. Sundrum and Y. Tsai, "Primordial Anisotropies in the Gravitational Wave Background from Cosmological Phase Transitions," Phys. Rev. Lett. **121** (2018) no.20, 201303 doi:10.1103/PhysRevLett.121.201303 [arXiv:1803.10780 [hep-ph]].

[21] A. Malhotra, E. Dimastrogiovanni, M. Fasiello and M. Shiraishi, JCAP **03**, 088 (2021) doi:10.1088/1475-7516/2021/03/088 [arXiv:2012.03498 [astro-ph.CO]].

[22] P. Adshead, N. Afshordi, E. Dimastrogiovanni, M. Fasiello, E. A. Lim and G. Tasinato, "Multimessenger cosmology: Correlating cosmic microwave background and stochastic gravitational wave background measurements," Phys. Rev. D **103**, no.2, 023532 (2021) doi:10.1103/PhysRevD.103.023532 [arXiv:2004.06619 [astro-ph.CO]].

[23] A. Ricciardone, L. V. Dall'Armi, N. Bartolo, D. Bertacca, M. Liguori and S. Matarrese, "Cross-Correlating Astrophysical and Cosmological Gravitational Wave Backgrounds with the Cosmic Microwave Background," Phys. Rev. Lett. **127** (2021) no.27, 271301 doi:10.1103/PhysRevLett.127.271301 [arXiv:2106.02591 [astro-ph.CO]].

[24] M. Braglia and S. Kuroyanagi, "Probing prerecombination physics by the cross-correlation of stochastic gravitational waves and CMB anisotropies," Phys. Rev. D **104**, no.12, 123547 (2021) doi:10.1103/PhysRevD.104.123547 [arXiv:2106.03786 [astro-ph.CO]].

[25] F. Schulze, L. Valbusa Dall'Armi, J. Lesgourgues, A. Ricciardone, N. Bartolo, D. Bertacca, C. Fidler and S. Matarrese, "GW_CLASS: Cosmological Gravitational Wave Background in the Cosmic Linear Anisotropy Solving System," [arXiv:2305.01602 [gr-qc]].

[26] S. P. Corbà and L. Sorbo, "Correlated scalar perturbations and gravitational waves from axion inflation," *Journal of Cosmology and Astroparticle Physics*, vol. 2024, no. 10, p. 024, Oct. 2024. doi:10.1088/1475-7516/2024/10/024 [arXiv:2403.03338 [astro-ph.CO]].

[27] G. Mentasti, C. R. Contaldi and M. Peloso, "Intrinsic Limits on the Detection of the Anisotropies of the Stochastic Gravitational Wave Background," Phys. Rev. Lett. **131**, no.22, 221403 (2023) doi:10.1103/PhysRevLett.131.221403 [arXiv:2301.08074 [gr-qc]].

[28] Y. Cui, S. Kumar, R. Sundrum and Y. Tsai, "Unraveling cosmological anisotropies within stochastic gravitational wave backgrounds," JCAP **10**, 064 (2023) doi:10.1088/1475-7516/2023/10/064 [arXiv:2307.10360 [astro-ph.CO]].

[29] L. Bethke, D. G. Figueroa and A. Rajantie, "Anisotropies in the Gravitational Wave Background from Preheating," Phys. Rev. Lett. **111**, no.1, 011301 (2013) doi:10.1103/PhysRevLett.111.011301 [arXiv:1304.2657 [astro-ph.CO]].

[30] Y. H. Yu and S. Wang, [arXiv:2504.07838 [astro-ph.CO]].

[31] N. Bartolo, D. Bertacca, S. Matarrese, M. Peloso, A. Ricciardone, A. Riotto and G. Tasinato, Phys. Rev. D **100**, no.12, 121501 (2019) doi:10.1103/PhysRevD.100.121501 [arXiv:1908.00527 [astro-ph.CO]].

[32] N. Bartolo, D. Bertacca, S. Matarrese, M. Peloso, A. Ricciardone, A. Riotto and G. Tasinato, "Characterizing the cosmological gravitational wave background: Anisotropies and non-Gaussianity," Phys. Rev. D **102**, no.2, 023527 (2020) doi:10.1103/PhysRevD.102.023527 [arXiv:1912.09433 [astro-ph.CO]].

[33] S. L. Cheng, W. Lee and K. W. Ng, "Numerical study of pseudoscalar inflation with an axion-gauge field coupling," Phys. Rev. D **93**, no. 6, 063510 (2016) [arXiv:1508.00251 [astro-ph.CO]].

[34] A. Notari and K. Tywoniuk,"Dissipative Axial Inflation," JCAP **1612**, 038 (2016) [arXiv:1608.06223 [hep-th]].

[35] O. O. Sobol, E. V. Gorbar and S. I. Vilchinskii, "Backreaction of electromagnetic fields and the Schwinger effect in pseudoscalar inflation magnetogenesis," Phys. Rev. D **100**, no.6, 063523 (2019) [arXiv:1907.10443 [astro-ph.CO]].

[36] G. Dall'Agata, S. González-Martín, A. Papageorgiou and M. Peloso, "Warm dark energy," JCAP **08**, 032 (2020) [arXiv:1912.09950 [hep-th]].

[37] V. Domcke, V. Guidetti, Y. Welling and A. Westphal, "Resonant backreaction in axion inflation," JCAP **09**, 009 (2020) [arXiv:2002.02952 [astro-ph.CO]].

[38] A. Caravano, E. Komatsu, K. D. Lozanov and J. Weller, "Lattice simulations of axion-U(1) inflation," Phys. Rev. D **108**, no.4, 043504 (2023) doi:10.1103/PhysRevD.108.043504 [arXiv:2204.12874 [astro-ph.CO]].

[39] M. Peloso and L. Sorbo, "Instability in axion inflation with strong backreaction from gauge modes," JCAP **01**, 038 (2023) doi:10.1088/1475-7516/2023/01/038 [arXiv:2209.08131 [astro-ph.CO]].

[40] D. G. Figueroa, J. Lizarraga, A. Urio and J. Urrestilla, "Strong Backreaction Regime in Axion Inflation," Phys. Rev. Lett. **131**, no.15, 151003 (2023) doi:10.1103/PhysRevLett.131.151003 [arXiv:2303.17436 [astro-ph.CO]].

[41] J. Garcia-Bellido, A. Papageorgiou, M. Peloso and L. Sorbo, "A flashing beacon in axion inflation: recurring bursts of gravitational waves in the strong backreaction regime," [arXiv:2303.13425 [astro-ph.CO]].

[42] R. von Eckardstein, M. Peloso, K. Schmitz, O. Sobol and L. Sorbo, "Axion inflation in the strong-backreaction regime: decay of the Anber-Sorbo solution," JHEP **11**, 183 (2023) doi:10.1007/JHEP11(2023)183 [arXiv:2309.04254 [hep-ph]].

[43] A. Caravano and M. Peloso, [arXiv:2407.13405 [astro-ph.CO]].

[44] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," JHEP **05**, 013 (2003) doi:10.1088/1126-6708/2003/05/013 [arXiv:astro-ph/0210603 [astro-ph]].

[45] N. Barnaby, R. Namba and M. Peloso, "Phenomenology of a Pseudo-Scalar Inflaton: Naturally Large Nongaussianity," JCAP **04**, 009 (2011) doi:10.1088/1475-7516/2011/04/009 [arXiv:1102.4333 [astro-ph.CO]].