Handout 1: numerical approaches to magnetogenesis

By magnetogenesis, we usually mean the generation and amplification of *primordial magnetic fields*—as opposed to *contemporary magnetic fields* that are amplified by maintained by dynamos (lecture 4), which convert kinetic energy into magnetic energy. The time of primordial magnetic field generation can be anywhere between the epochs of inflation and recombination. Some time after that, gravitational collapse provides a source of kinetic energy.

1 Maxwell equations

Magnetic field evolution is described by the Maxwell equations, or some extensions of them. In the SI system, they are given by

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{B} - \mu_0\boldsymbol{J}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_\mathrm{e}/\epsilon_0, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla}\times\boldsymbol{E}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{2}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the current density, ρ_e is the charge density, $\boldsymbol{B}$ is the magnetic field, μ_0 is the vacuum permeability, ϵ_0 is the vacuum permittivity, and c is the vacuum speed of light with $c^2 = (\mu_0\epsilon_0)^{-1}$.

In the Heaviside–Lorentz system of units (`https://en.wikipedia.org/wiki/Heaviside%E2%80%93Lorentz_units`) with $c = 1$, we measure $\boldsymbol{E}$ and $\boldsymbol{B}$ in units such that

$$\mu_0 = \epsilon_0 = 1. \tag{3}$$

We will frequently make use of this. But if we want to measure the magnetic field in gauss, it is useful to convert to gaussian units. Loosely speaking, this means $\mu_0 = \epsilon_0 = 4\pi$, and factors of c have to be restored in the right places.

Equations (1) and (2) need to be closed by some prescription for $\boldsymbol{J}$. Such type of equations are sometimes called material equations. One approach is to solve the so-called kinetic equations for positively and negatively charged particles. In that case, $\boldsymbol{J}$ can be obtained by averaging the charge fluxes for positively and negatively charged particles over some small volume. Such an approach is called *Particle in Cell* (PiC) simulations. We will not do this here. Instead, we adopt Ohm's law, which states that $\boldsymbol{J}$ in a frame comoving with the gas is proportional to the comoving $\boldsymbol{E}$. In a fixed frame (sometimes also called lab frame), where the gas moves with velocity uu, we have to perform a Lorentz boost and obtain the current density as

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}); \tag{4}$$

see `https://en.wikipedia.org/wiki/Ohm%27s_law` for details.

There may well be additional contributions to $\boldsymbol{J}$. One of them is proportional to $\boldsymbol{B}$. This becomes relevant when we talk about the chiral magnetic effect.

Because of $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, it is convenient to introduce the magnetic vector potential such that $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. We can then uncurl Equation (2) to obtain

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \boldsymbol{\nabla}\phi, \tag{5}$$

where ϕ is the electrostatic potential. There is some gauge freedom. A commonly used gauge is the Coulomb gauge ($\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$), but one of the reasons to work with $\boldsymbol{A}$ is that it removes the problem of ensuring $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, so it wouldn't help if we then still have to worry about $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. For numerical purposes, we therefore always use $\phi = 0$, which is also called the Weyl gauge or the temporal gauge. Thus, from now on, we write

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E}. \tag{6}$$

Replacing in Equation (1) $\boldsymbol{E}$ by $-\partial \boldsymbol{A}/\partial t$, we have

$$-\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A} - \mu_0\sigma\left(-\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{u}\times\boldsymbol{B}\right). \tag{7}$$

From this, we see two interesting limits. One is $\sigma = 0$, in which we obtain purely electromagnetic waves. The other is $c \to \infty$, which leads to what is called the induction equation.

2 Decay experiments with $u = 0$

For $c \to \infty$, we obtain a diffusion equation of the form

$$\frac{\partial \boldsymbol{A}}{\partial t} = \eta\nabla^2\boldsymbol{A}, \tag{8}$$

where $\eta = 1/(\mu_0\sigma)$ is the magnetic diffusivity and we have used the Coulomb gauge. (Note that the symbol σ is often also used for conformal time!) Solutions are then proportional to $e^{-\eta k^2 t}$, where k is the wavenumber of a sinusoidal initial field. The decay rate (or negative growth rate) is then $-\gamma = \eta k^2$.

- Check out the input files from `https://norlx65.nordita.org/~brandenb/teach/PencilCode/COSMOMAG2026/1_magnetogenesis/samples/decay-runs/`
- Look through the src/*.local files to see what you are doing. One parameter you might want to vary is the number of mesh points.
- Compile, start, run. Note that we work here with the diffusivity $\eta = 1/\sigma$ ($\mu_0 = 1$ by default).
- Think about the length of the time step. There are constraints from σ and c (here called c_light_set=1 in start.in). Check out what you limits are
- Look at snapshots—both after running the initial condition (pc_start) and after a small number of time steps.
- Measure the decay rate by plotting $\ln B_{\rm rms}$ versus t and study the dependence on η and c.
- Consider the possibility of artifacts, especially near limits of applicability.
- Compare single and double precision.
- Make a list of further interesting points that you studied!

The left panel of Figure 1 shows that when the conductivity becomes less than about twice the light travel time, the displacement current matters. The right panel of Figure 1 shows a related (and modified) plot from Brandenburg, et al. (2025) where an α effect was included.[1]

[1] Brandenburg, et al. (2025) considered the mean-field dynamo behavior with an α effect at a finite speed of light. Ohm's law is then of the form $\boldsymbol{J} = \sigma(\boldsymbol{E} + \alpha\boldsymbol{B})$ and $\boldsymbol{u} = 0$. Seeking solutions proportional to $\exp(\gamma t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$, the dispersion relation $\gamma(\boldsymbol{k})$ can be obtained by solving for the roots of

$$\left[\gamma + \eta_{\rm T}k^2\left(1 + \frac{\gamma^2}{c^2k^2}\right)\right]^2 - \alpha^2k^2 = 0. \tag{9}$$

In the limit $c \to \infty$, the dispersion relation agrees with the conventional one,

$$\gamma = \pm|\alpha|k - \eta_{\rm T}k^2. \tag{10}$$

Figure 1 shows $\gamma(\alpha)$ for a fixed value of k and three values of $c/\eta_{\rm T}k$. The dependence matches the conventional one for $c \to \infty$. We see that the effect of a finite speed of light is to lower the value of γ.

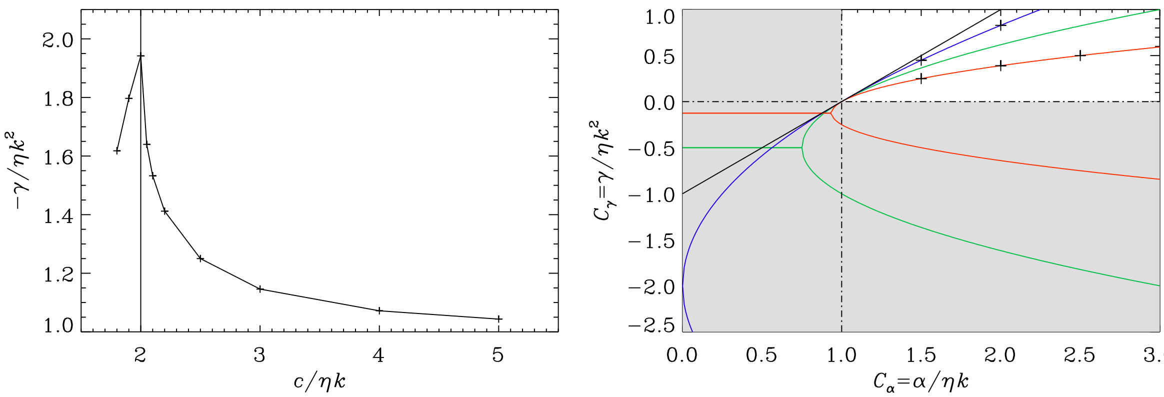

Figure 1: Left: decay rate versus conductivity, here expressed as $c/\eta k$. Right: comparison with Brandenburg, et al. (2025). Growth rate versus dynamo number $\alpha/\eta_{\rm T}k$ for a fixed value of k and $c/\eta_{\rm T}k = 0.5$ (red), 1 (green), and 2 (blue). The gray areas mark the regions of no growth. The conventional dependence for $c \to \infty$ corresponds to the black line. The black plus signs refer to data points obtained with the PENCIL CODE.

3 MHD approximation

Whenever there are some currents, the $\partial \boldsymbol{E} \partial t$ term does not play a role. We can therefore *replace* this equation by

$$0 = \boldsymbol{\nabla} \times \boldsymbol{B} - \mu_0 \boldsymbol{J} \qquad \text{when } \boldsymbol{J} \text{ is present.} \tag{11}$$

This does not mean that $\boldsymbol{E}$ is constant, but just that its variation isn't important for the dynamics.

4 Magnetogenesis at the end of inflation

Inflation is an episode of exponential expansion. The Universe became so dilute that the particle density became negligible and comparable to that of a vacuum. The conductivity was therefore vanishing.

An early calculation by Ratra (1992) made use of the idea that a scalar field (for example the inflaton) couples to the electromagnetic action. We refer here to the work by Subramanian (2010), starting with his Eq. (43). Instead of the standard electromagnetic action involving $F_{\mu\nu}F^{\mu\nu}$, we considered the term

$$f(a)^2\, F_{\mu\nu}F^{\mu\nu} \tag{12}$$

in the action, where $f(a)$ is a simple parameterization of the modifying term as a function of the scale factor a. (Ratra's model corresponds to $f \propto e^{\alpha\phi}$, where ϕ is the scalar field and α is a coefficient.)

We follow here the equations of Subramanian (2010), who used conformal time, $\tilde{t} = \int \mathrm{d}t/a(t)$. Derivatives with respect to conformal time are denoted by primes. Owing to the presence of the term f^2 in the action, the resulting evolution equation for $\boldsymbol{A}$ became

$$\boldsymbol{A}'' + 2\frac{f'}{f}\boldsymbol{A}' - a^2\nabla^2\boldsymbol{A} = 0. \tag{13}$$

To gain insight into the modification caused by $f(a)$, it is convenient to do to Fourier space, so ∇^2 turns into $-k^2$, and to work with a rescaled vector potential $\boldsymbol{\mathcal{A}} \equiv a(\tilde{t})f(\tilde{t})\boldsymbol{A}(\tilde{t},k)$, so Equation (13) takes the form

$$\boldsymbol{\mathcal{A}}'' + \left(\frac{f''}{f} - k^2\right)\boldsymbol{\mathcal{A}} = 0. \tag{14}$$

At the end of inflation, we have $f = 1$, so as to recover the usual Maxwell equations. However, when $f(a)$ varies, f''/f no longer vanishes and can be positive. In that case, the field becomes unstable for $k < f''/f$, i.e., on very large length scales.

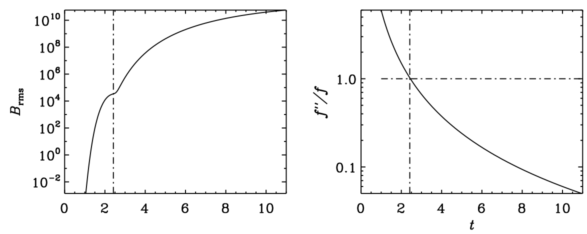

Figure 2: Time dependence of $B_{\rm rms}$ and f''/f for a run with $\beta = 1$.

A particularly interesting epoch is the end of inflation. The inflaton begins to describe oscillations, which renders the inflation dynamics essentially like that of the matter-dominated universe. As a function of conformal time, we then have $a(\tilde{t}) \propto \tilde{t}^2$. This leads us to the next exercise:

- Assume $f(a) \propto a^{-\beta}$, so f decreases to approach unity, which would mark the end of inflation and the beginning of the usual radiation-dominated era. Show that $f''/f = ...$
- Check out the input files from `https://norlx65.nordita.org/~brandenb/teach/PencilCode/COSMOMAG2026/1_magnetogenesis/samples/reheating/`
- Run the code with different values of β.
- For $\beta = 1$, you should something like what is shown in Figure 2 for $B_{\rm rms}$ and f''/f.
- Watch the evolution of the magnetic field structure.

5 Chiral magnetic effect

When a free neutron decays after about 10 minutes, it gets converted into a proton, and electron, and an electron antineutrino. The momentum of the electron, $\boldsymbol{p}$ points away from where it came from, but it also has a spin, $\boldsymbol{s}$. Since the discovery of the parity-breaking effect of the weak force, we know that the spin tends to point into the opposite direction of $\boldsymbol{p}$, so $\boldsymbol{p} \cdot \boldsymbol{s} < 0$. The quantity $\boldsymbol{p} \cdot \boldsymbol{s}$ is a pseudoscalar and changes sign when viewed in a mirror.

In the presence of a magnetic field, the spin aligns with the magnetic field. Since $\boldsymbol{p} \cdot \boldsymbol{s} < 0$, also the momentum aligns with the magnetic field. And since electrons carry a negative charge, this leads to a current along the magnetic field,

$$\boldsymbol{J} = ... + \frac{e^2}{2\pi^2\hbar^2 c}\mu_5 \boldsymbol{B}. \tag{15}$$

Here, e is the charge of the electron, $\hbar$ is the reduced Planck constant, and μ_5 is the chiral chemical potential. (The number 5 arises from using the γ_5 matrix in relativistic quantum mechanics to differentiate chirality, linking μ_5 to axial-vector currents and chiral symmetry.) It is convenient to normalize the chiral chemical potential such that it has the units of a wavenumber, i.e., $\tilde{\mu}_5 = (e^2/2\pi^2\hbar^2 c)\mu_5$. The electric field on the right-hand side of the Faraday equation is then $\boldsymbol{E} = \eta(\boldsymbol{\nabla} \times \boldsymbol{B} - \tilde{\mu}_5 \boldsymbol{B}$. It leads to a generation of magnetic field at the wavenumber $\tilde{\mu}_5/2$.

As we will see later, the quantity $\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$ changes in well conducting media (small magnetic diffusivity) only on a resistive timescale. Its evolution equation has $-2\langle \boldsymbol{E} \cdot \boldsymbol{B} \rangle$ on the right-hand side. The chiral magnetic effect (CME) has the property of conserving the total chirality in the system, and since the CME produces a helical magnetic field, the chiral chemical potential must diminish such that the combination

$$\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle + 2\tilde{\mu}_5/\lambda = \mathrm{const} \tag{16}$$

is conserved. This means that the CME can only produce as much (helical) magnetic field as there was chiral chemical potential initially. Estimates yield

$$\langle \boldsymbol{B}^2 \rangle \xi_\mathrm{M} \lesssim (0.5\,\mathrm{G})^2\,\mathrm{Mpc} \tag{17}$$

for our universe (Brandenburg et al., 2017).

The Pencil Code also comes with its own samples. The chiral magnetic effect is included in them; see `https://github.com/pencil-code/pencil-code/tree/master/samples/2d-tests/chiral_dynamo` underneath the full `https://github.com/pencil-code/pencil-code/` directory on github.

6 Seed magnetic fields

The magnetogenesis mechanisms discussed about required an initial seed magnetic field. This question continues to be raised all the time: what if you don't have one? Within the MHD approximation, there would never emerge a field on its own. But when the $\partial \boldsymbol{E} \partial t$ term is included, then $\boldsymbol{E}$ must obey $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_\mathrm{e}/\epsilon_0$, and ρ_e will always possess fluctuations. We can address this via some exercises:

- Check out the input files from `https://norlx65.nordita.org/~brandenb/teach/PencilCode/COSMOMAG2026/1_magnetogenesis/samples/seed_fields/`
- Run the code to produce different realizations of ρ_e. We do this here by computing it from $\rho_\mathrm{e} = \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}$.
- Study different spatial distributions of ρ_e.

References

Brandenburg, A., Schober, J., Rogachevskii, I., Kahniashvili, T., Boyarsky, A., Fröhlich, J., Ruchayskiy, O., & Kleeorin, N., "The turbulent chiral magnetic cascade in the early universe," *Astrophys. J. Lett.* **845**, L21 (2017).

Brandenburg, A., Ghosh, O., Vazza, F., & Neronov, A., "Magnetic field spreading from stellar and galactic dynamos into the exterior," *Astrophys. J.*, submitted, arXiv:2512.17770 (2025).

E. N. Parker, "Instability of thermal fields," *Astrophys. J.* **117**, 431–436 (1953).

Ratra, B., "Cosmological "seed" magnetic field from inflation," *Astrophys. J. Lett.* **391**, L1–L4 (1992).

Subramanian, K., "Magnetic fields in the early Universe," *Astron. Nachr.* **331**, 110–120 (2010).

$Header: /var/cvs/brandenb/tex/teach/COSMOMAG26/1_magnetogenesis/notes.tex,v 1.18 2026/01/13 05:56:55 brandenb Exp $