

Magnetic fields in the early Universe

K. Subramanian*

IUCAA, Post Bag 4, Pune University Campus, Ganeshkhind, Pune 411 007, India

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We give a pedagogical introduction to two aspects of magnetic fields in the early Universe. We first focus on how to formulate electrodynamics in curved space time, defining appropriate magnetic and electric fields and writing Maxwell equations in terms of these fields. We then specialize to the case of magnetohydrodynamics in the expanding Universe. We emphasize the usefulness of tetrads in this context. We then review the generation of magnetic fields during the inflationary era, deriving in detail the predicted magnetic and electric spectra for some models. We discuss potential problems arising from back reaction effects and from the large variation of the coupling constants required for such field generation.

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1 Introduction

The Universe is magnetized, from scales of planets to the largest collapsed objects like galaxy clusters. The origin and evolution of these magnetic fields is a subject of intense study. Much of the work on magnetic field origin centers on the idea that small seed magnetic fields are generated by purely astrophysical batteries and are subsequently amplified to the observed levels by a dynamo (cf. Brandenburg & Subramanian 2005 for a review). An interesting alternative is that the observed large-scale magnetic fields are partially a relic field from the early Universe, which have been further amplified by motions.

We provide here a pedagogical review of two aspects of magnetohydrodynamics (MHD) in the early Universe. In the first part of the review we focus on how to formulate electrodynamics in curved space time, especially how to define magnetic and electric fields and write Maxwell equations in terms of these fields. This issue may perhaps be well known to relativists but may not be so familiar to practitioners of MHD. The perspective provided is that of the author.

In the second part, we review, again in a pedagogical manner, generation of magnetic fields during the inflationary era. It is well known that scalar (density or potential) perturbations and gravitational waves (or tensor perturbations) can be generated during inflation. Could magnetic field perturbations also be generated? Indeed, a large number of mechanisms whereby magnetic fields are generated in the early Universe have been discussed in the literature (Turner & Widrow 1988; Ratra 1992; Widrow 2002; Giovannini 2008). These generically involve the breaking of the conformal invariance of the electromagnetic action, and the predicted amplitudes are rather model dependent. Nev-

ertheless, if a primordial magnetic field is generated, with a present-day strength of $B \sim 10^{-9}$ G and coherent on Mpc scales, it can strongly influence early galaxy formation (cf. Sethi & Subramanian 2005), or induce signals on the cosmic microwave background radiation (CMBR) (cf. Subramanian 2006 for a review) including CMBR non-Gaussianity (cf. Seshadri & Subramanian 2009; Caprini et al. 2009). An even weaker field, sheared and amplified due to flux freezing, during galaxy and cluster formation may kick-start the dynamo. It is then worth considering if one can generate such primordial fields. Our discussion of inflationary generation of magnetic fields follows the standard literature, but again the perspective offered is that of the author. We hope that our pedagogical discussion of these aspects of MHD in the early Universe will be useful to those entering the subject.

2 Electrodynamics in curved spacetime

We discuss to begin with Maxwell equations in a general curved spacetime and then focus on FRW models. Electrodynamics in curved spacetime is most conveniently formulated by giving the action for electromagnetic fields and their interaction with charged particles:

$$S = - \int \sqrt{-g} \, d^4x \left[\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right]. \quad (1)$$

Here $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic (EM) field tensor, with A_μ being the standard electromagnetic 4-potential and J^μ the 4-current density. Further, here and below, we use Greek indices μ, ν etc for spacetime co-ordinates and Roman indices i, j, k, \dots for purely spatial co-ordinates; repeated indices are summed over all the co-ordinates. We also adopt units where the speed of light $c = 1$ and a metric signature $(-, +, +, +)$. Demanding that

* Corresponding author: kandu@iucaa.ernet.in

the action is stationary under the variation of A_μ , gives one half of the Maxwell equations,

$$F^{\mu\nu}_{;\nu} = 4\pi J^\mu. \quad (2)$$

And from the definition of the electromagnetic field tensor we also get the other half of the Maxwell equations,

$$F_{[\mu\nu}; \gamma] = F_{[\mu\nu}, \gamma] = 0; \quad \text{or } {}^*F^{\mu\nu}_{;\nu} = 0. \quad (3)$$

The square brackets $[\mu\nu, \gamma]$ means adding terms with cyclic permutations of μ, ν, γ . In the latter half of Eq. (3), we have defined the dual electromagnetic field tensor

$${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}.$$

Here $\epsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor,

$$\epsilon^{\mu\nu\rho\lambda} = \frac{1}{\sqrt{-g}}\mathcal{A}^{\mu\nu\rho\lambda}, \quad \epsilon_{\mu\nu\rho\lambda} = \sqrt{-g}\mathcal{A}_{\mu\nu\rho\lambda},$$

and g is the determinant of the metric tensor. Further $\mathcal{A}^{\mu\nu\rho\lambda}$ is the totally antisymmetric symbol such that $\mathcal{A}^{0123} = 1$ and ± 1 for any even or odd permutations of $(0, 1, 2, 3)$ respectively. Note that $\mathcal{A}_{0123} = -1$.

We would like to cast these equations in terms of electric and magnetic fields (Ellis 1973; Tsagas 2005; Barrow, Maartens & Tsagas 2007). In flat spacetime the electric and magnetic fields are written in terms of different components of the EM tensor $F_{\mu\nu}$. This tensor is antisymmetric, thus its diagonal components are zero and it has 6 independent components, which can be thought of the 3 components of the electric field and the 3 components of the magnetic field. The electric field E^i is given by time-space components of the EM tensor, while the magnetic field B^i is given by the space-space components

$$F^{0i} = E^i, \quad F^{12} = B^3, \quad F^{23} = B^1, \quad F^{31} = B^2. \quad (4)$$

In a general spacetime, to define corresponding electric and magnetic fields from the EM tensor, one needs to isolate a time direction. This can be done by using a family of observers who measure the EM fields and whose four-velocity is described by the 4-vector

$$u^\mu = \frac{dx^\mu}{ds}, \quad u^\mu u_\mu = -1.$$

Given this 4-velocity field, one can also define the ‘projection tensor’

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu.$$

This projects all quantities into the 3-space orthogonal to u^μ and is also the effective spatial metric for these observers, i.e.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(u_\mu dx^\mu)^2 + h_{\mu\nu}dx^\mu dx^\nu. \quad (5)$$

Using the four-velocity of these observers, the EM fields can be expressed in a more compact form as a four-vector electric field E_μ and magnetic field B_μ as

$$E_\mu = F_{\mu\nu}u^\nu, \quad B_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}u^\nu F^{\rho\lambda} = {}^*F_{\mu\nu}u^\nu. \quad (6)$$

From the definition of E_μ and B_μ , we have

$$E_\mu u^\mu = 0, \quad B_\mu u^\mu = 0.$$

Thus the four-vectors B_μ and E_μ have purely spatial components and are effectively 3-vectors in the space orthogonal to u^μ . One can also invert Eq. (6) to write the EM tensor and its dual in terms of the electric and magnetic fields

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu + \epsilon_{\mu\nu\alpha\beta}B^\alpha u^\beta, \quad (7)$$

$$\begin{aligned} {}^*F^{\alpha\beta} &= \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu} \\ &= \epsilon^{\alpha\beta\mu\nu}u_\mu E_\nu + (u^\alpha B^\beta - B^\alpha u^\beta). \end{aligned} \quad (8)$$

We can now use the time-like vector u^μ and the spatial metric h^μ_ν to decompose the Maxwell equations into timelike and spacelike parts. Consider projection of Eq. (3) on u_α . We have

$$u_\alpha({}^*F^{\alpha\beta}_{;\beta}) = 0 = (u_\alpha {}^*F^{\alpha\beta})_{;\beta} - u_{\alpha;\beta} {}^*F^{\alpha\beta}. \quad (9)$$

Substituting Eq. (8) into Eq. (9) we get

$$\begin{aligned} [u_\alpha \epsilon^{\alpha\beta\mu\nu}u_\mu E_\nu + u_\alpha u^\alpha B^\beta - u^\beta u_\alpha B^\alpha]_{;\beta} \\ - u_{\alpha;\beta} [\epsilon^{\alpha\beta\mu\nu}u_\mu E_\nu + u^\alpha B^\beta - u^\beta B^\alpha] = 0. \end{aligned} \quad (10)$$

The 1st, 3rd and 5th terms are zero because respectively $u_\alpha \epsilon^{\alpha\beta\mu\nu}u_\mu = 0$, $u_\alpha B^\alpha = 0$ and $u_{\alpha;\beta}u^\alpha = 0$. The rest of the terms give

$$B^\beta_{;\beta} - B^\beta \dot{u}_\beta + u_{\alpha;\beta} \epsilon^{\alpha\beta\mu\nu}u_\mu E_\nu = 0, \quad (11)$$

where we have defined the acceleration 4-vector

$$\dot{u}_\beta = u^\alpha u_{\beta;\alpha}$$

as the directional derivative of u^β , along the timelike direction specified by u^α . The covariant velocity gradient tensor, $u_{\alpha;\beta}$, can be decomposed into, shear, expansion, vorticity and acceleration parts in the following manner: We first write

$$\begin{aligned} u_{\alpha;\beta} &= \delta^\mu_\alpha u_{\mu;\nu} [\delta^\nu_\beta + u^\nu u_\beta] - \delta^\mu_\alpha u_{\mu;\nu} u^\nu u_\beta \\ &= h^\mu_\alpha h^\nu_\beta u_{\mu;\nu} - \dot{u}_\alpha u_\beta \\ &= \Theta_{\alpha\beta} + \omega_{\alpha\beta} - \dot{u}_\alpha u_\beta. \end{aligned} \quad (12)$$

In the last line, we have decomposed the spatial part of the co-variant derivative $h^\mu_\alpha h^\nu_\beta u_{\mu;\nu}$, into symmetric expansion tensor $\Theta_{\alpha\beta}$ and an antisymmetric vorticity tensor $\omega_{\alpha\beta}$. Note that both the expansion and vorticity tensors are purely spatial, in the sense that $\Theta_{\alpha\beta}u^\beta = 0 = \omega_{\alpha\beta}u^\beta$. One can further split $\Theta_{\alpha\beta}$ into its trace and trace free part,

$$\Theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta},$$

where $\Theta = \Theta^\alpha_\alpha = u^\alpha_{;\alpha}$ is called the expansion scalar, and $\sigma_{\alpha\beta}$ is called the shear tensor. Note that the shear tensor satisfies, $\sigma^\alpha_\alpha = 0$, and $\sigma_{\alpha\beta}u^\beta = 0$, that is it is traceless and also purely spatial. Thus we have

$$u_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} + \omega_{\alpha\beta} - \dot{u}_\alpha u_\beta. \quad (13)$$

Only the antisymmetric part of $u_{\alpha;\beta}$ contributes when we substitute Eq. (13) into Eq. (11). Further simplification can be made by defining the vorticity vector

$$\omega^\nu = -\frac{1}{2}\omega_{\alpha\beta}\epsilon^{\alpha\beta\mu\nu}u_\mu$$

and the spatial projection of the covariant derivative

$$D_\beta B^\alpha = h^\mu_\beta h^\nu_\mu B^\alpha_{;\mu}.$$

Then we have

$$\begin{aligned} D_\beta B^\beta &= h^\mu_\beta h^\beta_\nu B^\nu_{;\mu} = h^\mu_\nu B^\nu_{;\mu} = (\delta^\mu_\nu + u^\mu u_\nu) B^\nu_{;\mu} \\ &= B^\mu_{;\mu} - u^\mu u_{\nu;\mu} B^\nu. \end{aligned} \quad (14)$$

Thus Eq. (11) becomes

$$D_\beta B^\beta = 2\omega^\beta E_\beta. \quad (15)$$

This equation generalizes the flat space equation $\nabla \cdot \mathbf{B} = 0$, to a general curved spacetime. We see that $2\omega^\beta E_\beta$ acts as an effective magnetic charge, driven by the vorticity of the relative motion of the observers measuring the electromagnetic field.

Now turn to the spatial projection of of Eq. (3) on h^κ_α . We have

$$\begin{aligned} h^\kappa_\alpha (*F^{\alpha\beta}_{;\beta}) &= h^\kappa_\alpha [\epsilon^{\alpha\beta\mu\nu} u_\mu E_\nu + u^\alpha B^\beta - u^\beta B^\alpha]_{;\beta} \\ &= 0, \end{aligned} \quad (16)$$

where we have again substituted Eq. (8) for the dual EM tensor. Expanding out the covariant derivative in Eq. (16) we get

$$\begin{aligned} h^\kappa_\alpha [\epsilon^{\alpha\beta\mu\nu} (u_{\mu;\beta} E_\nu + u_\mu E_{\nu;\beta}) + u^\alpha_{;\beta} B^\beta + u^\alpha B^\beta_{;\beta} \\ - h^\kappa_\alpha [u^\beta_{;\beta} B^\alpha + \dot{B}^\alpha] = 0. \end{aligned} \quad (17)$$

Using $h^\kappa_\alpha = (\delta^\kappa_\alpha + u^\kappa u_\alpha)$ above, only the contribution of δ^κ_α survives in the 2nd term (since $\epsilon^{\alpha\beta\mu\nu} u_\alpha u_\mu = 0$), 3rd term (since $u_\alpha u^\alpha_{;\beta} = 0$) and the 5th term (since $u_\alpha B^\alpha = 0$), while the 4th vanishes because $h^\kappa_\alpha u^\alpha = 0$. Thus we have from the remaining terms

$$\begin{aligned} h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} u_{\mu;\beta} E_\nu + \epsilon^{\kappa\beta\mu\nu} u_\mu E_{\nu;\beta} \\ + u^\kappa_{;\beta} B^\beta - u^\beta_{;\beta} B^\kappa - h^\kappa_\alpha \dot{B}^\alpha = 0. \end{aligned} \quad (18)$$

The first term can be simplified using Eq. (13). Due to the antisymmetry of the Levi-Civita tensor, the symmetric parts in Eq. (13) drop out and we have

$$\begin{aligned} h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} u_{\mu;\beta} E_\nu &= h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} E_\nu \omega_{\mu\beta} \\ &- h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} \dot{u}_\mu u_\beta E_\nu. \end{aligned} \quad (19)$$

The first term on the RHS of the above equation vanishes, i.e. $h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} E_\nu \omega_{\mu\beta} = 0$. To see this it is convenient to define a 3-d fully antisymmetric tensor

$$\bar{\epsilon}^{\kappa\beta\nu} = \epsilon^{\kappa\beta\nu\mu} u_\mu.$$

Note that this tensor is also purely spatial in the sense that $\bar{\epsilon}^{\kappa\beta\nu} u_\nu = \bar{\epsilon}^{\kappa\beta\nu} u_\beta = \bar{\epsilon}^{\kappa\beta\nu} u_\kappa = 0$, a result which follows from the anti-symmetry of $\epsilon^{\kappa\beta\nu\mu}$. We can then write the 4-d Levi-Civita tensor as

$$\epsilon^{\kappa\beta\nu\mu} = 2 \left(u^{[\kappa} \bar{\epsilon}^{\beta\nu]\mu} - \bar{\epsilon}^{\kappa\beta[\nu} u^{\mu]} \right), \quad (20)$$

where we use the notation, $A^{[\alpha\beta]} = (A^{\alpha\beta} - A^{\beta\alpha})/2$. The first term in Eq. (19) then becomes,

$$\begin{aligned} h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} E_\nu \omega_{\mu\beta} &= h^\kappa_\alpha E_\nu \omega_{\mu\beta} \times \\ [u^\alpha \bar{\epsilon}^{\beta\nu\mu} - u^\beta \bar{\epsilon}^{\alpha\nu\mu} - \bar{\epsilon}^{\alpha\beta\nu} u^\mu + \bar{\epsilon}^{\alpha\beta\mu} u^\nu] &= 0. \end{aligned} \quad (21)$$

The last equality follows from the fact that $h^\kappa_\alpha u^\alpha = 0$, $E_\nu u^\nu = 0$ and $\omega_{\mu\beta} u^\beta = \omega_{\mu\beta} u^\mu = 0$.¹ Therefore we have

$$\begin{aligned} h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} u_{\mu;\beta} E_\nu &= -h^\kappa_\alpha \epsilon^{\alpha\beta\mu\nu} \dot{u}_\mu u_\beta E_\nu \\ &= -\epsilon^{\kappa\beta\mu\nu} \dot{u}_\mu u_\beta E_\nu. \end{aligned} \quad (22)$$

The second term of Eq. (18) can be written more transparently by defining a ‘Curl’ operator

$$\text{Curl}(E^\kappa) = \bar{\epsilon}^{\kappa\beta\nu} E_{\nu;\beta}.$$

Thus $\epsilon^{\kappa\beta\mu\nu} u_\mu E_{\nu;\beta} = -\text{Curl}(E^\kappa)$. The third and fourth terms in Eq. (18) can also combined and rewritten using Eq. (13) to give

$$\begin{aligned} u^\kappa_{;\beta} B^\beta - u^\beta_{;\beta} B^\kappa &= [u^\kappa_{;\beta} - \Theta \delta^\kappa_\beta] B^\beta \\ &= \left[\sigma^\kappa_\beta + \omega^\kappa_\beta - \frac{2}{3} \Theta \delta^\kappa_\beta \right] B^\beta. \end{aligned} \quad (23)$$

Note that the acceleration term in Eq. (13) does not contribute above because $u_\beta B^\beta = 0$. Putting all the above results together, Eq. (18) gives the generalization of Faraday law to curved spacetime,

$$\begin{aligned} h^\kappa_\alpha \dot{B}^\alpha &= \left[\sigma^\kappa_\beta + \omega^\kappa_\beta - \frac{2}{3} \Theta \delta^\kappa_\beta \right] B^\beta \\ &- \bar{\epsilon}^{\kappa\mu\nu} \dot{u}_\mu E_\nu - \text{Curl}(E^\kappa). \end{aligned} \quad (24)$$

The other two Maxwell equations, involving source terms, can be derived in very similar manner. Note that if we map $\mathbf{E} \rightarrow -\mathbf{B}$, and $\mathbf{B} \rightarrow \mathbf{E}$, then the dual EM tensor is mapped to the EM tensor, that is $*F^{\mu\nu} \rightarrow F^{\mu\nu}$. We can use this symmetry between the EM tensor and its dual to read off the Maxwell equations involving the source terms from the source free equations, Eq. (15) and Eq. (24). We also note that in deriving Eq. (15) and Eq. (24), we changed the sign of all the terms appearing in Eq. (10) and Eq. (17). Thus mapping $\mathbf{E} \rightarrow -\mathbf{B}$, and $\mathbf{B} \rightarrow \mathbf{E}$ in Eq. (15) and Eq. (24), respectively, and also changing the sign of the source term $4\pi J^\mu \rightarrow -4\pi J^\mu$, the Maxwell equations $F^{\mu\nu}_{;\nu} = 4\pi J^\mu$, in terms of the E^μ and B^μ fields, become

$$D_\beta E^\beta = 4\pi \rho_q - 2\omega^\beta B_\beta, \quad (25)$$

$$\begin{aligned} h^\kappa_\alpha \dot{E}^\alpha &= \left[\sigma^\kappa_\beta + \omega^\kappa_\beta - \frac{2}{3} \Theta \delta^\kappa_\beta \right] E^\beta \\ &+ \bar{\epsilon}^{\kappa\mu\nu} \dot{u}_\mu B_\nu + \text{Curl}(B^\kappa) - 4\pi j^\kappa. \end{aligned} \quad (26)$$

Here we have defined the charge and 3-current densities as perceived by the observer with 4-velocity u^α by projecting the 4-current density J^μ , along u^α and orthogonal to u^α . That is

$$\rho_q = -J^\mu u_\mu, \quad j^\kappa = J^\mu h^\kappa_\mu.$$

Note that $j^\kappa u_\kappa = 0$. To do MHD in the expanding Universe, we also need the relativistic generalization to Ohm’s law. This is given by

$$\begin{aligned} h^\alpha_{(f)\beta} J^\beta &= \sigma F^{\alpha\beta} w_\beta, \quad \text{or} \\ J^\alpha &= \rho_{(f)q} w^\alpha + \sigma E^\alpha_{(f)}. \end{aligned} \quad (27)$$

¹ A more physical way of seeing that the first term in Eq. (19) vanishes is to note that E_ν and $\omega_{\mu\beta}$ are purely spatial tensors, orthogonal to the ‘time’ direction specified by u^α . Thus only the time like component of $P^\alpha = \epsilon^{\alpha\beta\mu\nu} E_\nu \omega_{\mu\beta}$ is non-zero. This implies however that $h^\kappa_\alpha P^\alpha = 0$, thus proving the required result.

Here the symbol (f) stands for a fluid variable, that is w^α is the mean 4-velocity of the fluid, $h_{(f)\beta}^\alpha = (\delta_\beta^\alpha + w^\alpha w_\beta)$ and $E_{(f)}^\alpha = F^{\alpha\beta} w_\beta$ is the electric field as measured in the fluid rest frame. Also $\rho_{(f)q}$ and σ are the fluid charge densities and conductivity as measured in its rest frame. Note that the fluid 4-velocity w^α , need not be the 4-velocity u^α of the family of fundamental observers used to define the EM fields in Maxwell equations; indeed the conducting fluid will in general have a peculiar velocity in the rest frame of the fundamental observers.

Further discussion on electrodynamics in curved space-time (using the 3+1 formalism) and how the different parts of the spacetime geometry affect the EM field can be found in Tsagas (2005). We now specialize to case of the expanding Universe.

2.1 Electrodynamics in the expanding Universe

Let us now consider Maxwell equations for the particular case of the expanding Universe, with the metric that of spatially flat Friedmann-Robertson-Walker (FRW) spacetime, $ds^2 = -dt^2 + a^2(t) [dx^2 + dy^2 + dz^2]$.

Here t is the proper time as measured by the fundamental observers of the FRW Universe, while x, y, z are co-moving spatial co-ordinates. The expansion of the Universe is determined by the scale factor $a(t)$, and $H(t) = \dot{a}/a$ is the Hubble expansion rate (we have also defined $\dot{a} = da/dt$). We choose u^α corresponding to the fundamental observers of the FRW spacetime, that is $u^\alpha \equiv (1, 0, 0, 0)$. For such a choice and in the FRW spacetime, we have

$$\dot{u}^\alpha = 0, \quad \omega_{\alpha\beta} = 0, \quad \sigma_{\alpha\beta} = 0, \quad \Theta = 3\frac{\dot{a}}{a}. \quad (29)$$

Further, we can simplify $h_\alpha^\kappa \dot{B}^\alpha$ as follows:

$$\begin{aligned} h_\alpha^\kappa \dot{B}^\alpha &= (\delta_\alpha^\kappa + u^\kappa u_\alpha) u^\gamma B_{;\gamma}^\kappa \\ &= u^\gamma B_{;\gamma}^\kappa + u^\kappa u^\gamma [(u_\alpha B^\alpha)_{;\gamma} - u_{\alpha;\gamma} B^\alpha] \\ &= u^\gamma B_{;\gamma}^\kappa. \end{aligned} \quad (30)$$

Thus the Maxwell equations reduce to

$$\begin{aligned} B_{;\beta}^\beta &= 0, \quad E_{;\beta}^\beta = 4\pi\rho_q, \\ u^\gamma B_{;\gamma}^\kappa + \frac{2}{3}\Theta B^\kappa &= -\text{Curl}(E^\kappa), \\ u^\gamma E_{;\gamma}^\kappa + \frac{2}{3}\Theta E^\kappa &= \text{Curl}(B^\kappa) - 4\pi j^\kappa. \end{aligned} \quad (31)$$

In the spatially flat FRW metric the connection co-efficients take the form

$$\begin{aligned} \Gamma_{00}^0 &= 0 = \Gamma_{0i}^0 = \Gamma_{ij}^0, \quad \Gamma_{ij}^0 = \delta_{ij} a \dot{a}, \\ \Gamma_{0j}^i &= \delta_{ij} \frac{\dot{a}}{a}. \end{aligned} \quad (32)$$

Using these Eqs. (31) can be further simplified as follows:

$$\begin{aligned} \frac{\partial B^i}{\partial x^i} &= 0, \quad \frac{\partial E^i}{\partial x^i} = 4\pi\rho_q, \\ \frac{1}{a^3} \frac{\partial}{\partial t} [a^3 B^i] &= -\frac{1}{a} \epsilon_{ilm}^* \frac{\partial E^l}{\partial x^m}, \\ \frac{1}{a^3} \frac{\partial}{\partial t} [a^3 E^i] &= \frac{1}{a} \epsilon_{ilm}^* \frac{\partial B^l}{\partial x^m} - 4\pi j^i. \end{aligned} \quad (33)$$

Here we have defined the 3-d fully antisymmetric symbol ϵ_{ijk}^* , with $\epsilon_{123}^* = 1$. These equations resemble the flat space-time Maxwell equations except for the presence of the scale factor $a(t)$.

The electric and magnetic field 4-vectors we have used above are referred to a co-ordinate basis, where the space-time metric is of the FRW form. They have the following curious property. Consider for example the case when the plasma in the Universe has no peculiar velocity, that is $w^\alpha = u^\alpha$, and also highly conducting with $\sigma \rightarrow \infty$. Then from Eq. (27), we have $E_{(f)}^\alpha = 0 = E^\alpha$, and from Faraday's law in Eq. (33), $B^i \propto 1/a^3$. There is however a simple result derivable in flat space time that in a highly conducting fluid, the magnetic flux through a surface which co-moves with the fluid is constant. Since in the expanding Universe all proper surface areas increase as $a^2(t)$, one would expect the strength of a 'proper' magnetic field to go down with expansion as $1/a^2$. This naively seems to be at variance with the fact that $B^i \propto 1/a^3$ and $B_i = g_{i\mu} B^\mu \propto 1/a$. There are two comments to be made at this stage: First, if we define the magnetic field amplitude, say B , by looking at the norm of the four vector B^μ , that is let $B^2 = B^\mu B_\mu = B^i B_i \propto 1/a^4$, then we do get $B \propto 1/a^2$. This procedure however does not appear completely satisfactory as one would prefer to deal with the field components themselves. Another possibility is to refer all tensor quantities to a set of orthonormal basis vectors, referred to as tetrads.

Any observer can be thought to be carrying along her/his world line a set of four orthonormal vectors $e_{(a)}$, where $a = 0, 1, 2, 3$, which satisfy the relation

$$g_{\mu\nu} e_{(a)}^\mu e_{(b)}^\nu = \eta_{ab}, \quad \eta^{ab} e_{(a)}^\mu e_{(b)}^\nu = g^{\mu\nu}. \quad (34)$$

Here η_{ab} has the form of the flat space-time metric. We choose the observer's 4-velocity itself to be the tetrad with $a = 0$, i.e. $e_{(0)}^\mu = u^\mu$. The other three tetrads are orthogonal to the observer's 4-velocity. In the present case, we consider the observer to be the fundamental observer of the FRW space time, and the components of the tetrads, which satisfy Eq. (34) are given by

$$e_{(0)}^\mu = \delta_0^\mu, \quad e_{(i)}^\mu = \frac{1}{a} \delta_i^\mu, \quad i = 1, 2, 3.$$

The metric η_{ab} can also be used to raise and lower the index of the tetrad to define $e^{\mu(a)} = \eta^{ab} e_{(b)}^\mu$. Note that the fundamental observers move along geodesics, and as we noted earlier, do not have either relative acceleration or rotation. Such observers parallel transport their tetrad along their trajectory, i.e. $u^\mu e_{(a);\mu}^\alpha = 0$, as can be easily checked by direct calculation using the connection co-efficients given in Eq. (32). The magnetic and electric field components can now be represented as its projection along the four orthonormal tetrads using

$$\bar{B}^a = g_{\mu\nu} B^\mu e^{\nu(a)}, \quad \bar{E}^a = g_{\mu\nu} E^\mu e^{\nu(a)}, \quad (35)$$

which gives

$$\begin{aligned} \bar{B}^0 &= 0, \quad \bar{E}^0 = 0, \\ \bar{B}^a &= a(t) B^a, \quad \bar{E}^a = a(t) E^a, \quad \text{for } (a = 1, 2, 3). \end{aligned} \quad (36)$$

Note that \bar{B}^a, \bar{E}^a are co-ordinate scalars, but the set of four scalars \bar{B}^a is still a vector as far as local Lorentz transformation is concerned (which preserves the orthonormality conditions in Eq. (34)). If we define $\bar{B}_a = \eta_{ab}\bar{B}^b$, then numerically $\bar{B}^i = \bar{B}_i$ and $\bar{B}^0 = -\bar{B}_0 = 0$. Similar relations obtain for the electric field components. In the FRW Universe, as $B^i \propto 1/a^3$, we see that $\bar{B}^i = \bar{B}_i \propto 1/a^2$, as one naively expects from flux freezing of the magnetic field. Thus the magnetic field components projected onto the orthonormal tetrads seem to be the natural quantities to be used as the 'physical' components of the magnetic field. Note that this is similar to using the Cartesian components of a vector as the physical components in 3 dimensional vector analysis.

There is an additional feature of using tetrads which is of particular interest. Given the set of tetrads one can set up a local co-ordinate system around any event \mathcal{P} by using geodesics emanating from \mathcal{P} and whose tangent vectors at \mathcal{P} are the four tetrads $e_{(a)}$. This co-ordinate frame, is a locally inertial frame; that is the spacetime is locally flat with the metric in the form of η_{ab} and the connection co-efficients in these co-ordinates vanishing (see Sect. 13.6 of Misner, Thorne & Wheeler 1971 (MTW) for a proof). In fact such a co-ordinate system can be set up all along the world line of the fundamental observer, and are called then Fermi-Normal co-ordinates (Manasse & Misner 1963; MTW; Cooperstock, Faraoni & Vollich 1998). To leading order, the time direction in this inertial frame, is the proper time co-ordinate t of the FRW metric and the space co-ordinates become the proper space co-ordinates $r^i = a(t)x^i$. (There are second order deviations from these relations, cf. Cooperstock et al. 1998).

The Maxwell equations in such a locally inertial frame take a particularly transparent form. They can be derived from Eq. (31) by adopting the flat space metric and replacing the covariant derivative with an ordinary derivative. As the metric is locally flat, the curl operator also reduces to the ordinary curl with respect to r^i . Importantly Θ being a scalar is invariant under co-ordinate transformations, and so still $\Theta = 3\dot{a}/a$. The field components are the same as those defined above using the tetrads. This is because the tetrads become the co-ordinate basis vectors in the Fermi-Normal co-ordinates. The Maxwell equations then become

$$\begin{aligned} \frac{\partial \bar{B}^i}{\partial r^i} &= 0, & \frac{\partial(a^2 \bar{E}^i)}{\partial r^i} &= 4\pi\rho_q a^2, \\ \frac{\partial}{\partial t} [a^2 \bar{B}^i] &= -\epsilon_{ilm}^* \frac{\partial(a^2 \bar{E}^l)}{\partial r^m}, \\ \frac{\partial}{\partial t} [a^2 \bar{E}^i] &= \epsilon_{ilm}^* \frac{\partial(a^2 \bar{B}^l)}{\partial r^m} - 4\pi \bar{j}^i a^2. \end{aligned} \quad (37)$$

In the absence of charges and currents, Eq. (37) has electromagnetic wave solutions, with the amplitude of the electric and magnetic fields decaying with expansion as $1/a^2$. In the presence of a conducting medium, one has to again supplement these Maxwell equations with the Ohm's law. In the limit of non-relativistic fluid velocity v , this again reduces to

$$\bar{j}^i = \rho_q v^i + \sigma [\bar{E}^i + \epsilon^{*ilm} v_l \bar{B}_m]. \quad (38)$$

If we neglect the displacement current and charge density terms, as valid for a highly conducting medium (cf. Brandenburg & Subramanian 2005), the induction equation becomes

$$\frac{\partial}{\partial t} (B a^2) = \nabla_{\mathbf{r}} \times [\mathbf{v} \times (B a^2) - \eta \nabla \times (B a^2)], \quad (39)$$

where we have defined the vector $\mathbf{B} \equiv (\bar{B}^1, \bar{B}^2, \bar{B}^3)$. Thus we see that in the absence of resistivity ($\eta = 0$) or peculiar velocities ($\mathbf{v} = 0$), the magnetic field defined in the local inertial frame, decays with expansion factor as $\mathbf{B} \propto 1/a^2$. As pointed out above, this decays is as expected, when the magnetic flux is frozen to the plasma, since all proper areas in the FRW spacetime increase with expansion as a^2 . This completes our pedagogical discussion of doing magnetohydrodynamics in curved spacetime and in particular the expanding Universe. We turn to a pedagogical discussion of primordial magnetic field generation during the inflationary era.

3 Magnetic field generation during inflation

The early Universe is supposed to have gone through an epoch of accelerated expansion referred to as inflation. Inflation can provide a solution to several problems of standard big bang cosmology, one of them being to explain the origin of perturbations which eventually led to all the structures that we see. We refer the reader to several standard textbooks for a discussion of the inflationary paradigm (Liddle & Lyth 1999; Mukhanov 2005; Padmanabhan 2002) and recent reviews (Bassett, Tsujikawa & Wands 2006; Sramkumar 2009). Inflation provides several ideal conditions for the generation of primordial fields with large coherence scales (Turner & Widrow 1988). First the rapid expansion in the inflationary era provides the kinematical means to produce fields correlated on very large scales by just the exponential stretching of wave modes. Also vacuum fluctuations of the electromagnetic (or more correctly the hypermagnetic) field can be excited while a mode is within the Hubble radius and could be transformed to classical fluctuations as it transits outside the Hubble radius. Finally, during inflation any existing charged particle densities are diluted drastically by the expansion, so that the Universe is not a good conductor; thus magnetic flux conservation then does not exclude field generation from a zero field. Most of the models for magnetic field generation during inflation take the field to be described by the action of an Abelian gauge field, and have not considered the action obtained from for example the Electro-Weak or some Grand Unified theory. For simplicity we shall also adopt this approach below.

There is one major difficulty, which arises when one considers magnetic field generation during inflation. This is because the standard electromagnetic action is conformally invariant, and the Universe metric is conformally flat. Consider again the electromagnetic action

$$S = - \int \sqrt{-g} d^4x \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

$$= - \int \sqrt{-g} d^4x \frac{1}{16\pi} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}. \quad (40)$$

Suppose we make a conformal transformation of the metric given by

$$g_{\mu\nu}^* = \Omega^2 g_{\mu\nu}. \quad (41)$$

This implies $\sqrt{-g^*} = \Omega^4 \sqrt{-g}$ and $g^{*\mu\alpha} = \Omega^{-2} g^{\mu\alpha}$. Then taking

$$A_\mu^* = A_\mu \Rightarrow S^* = S. \quad (42)$$

Thus the action of the free electromagnetic field is invariant under conformal transformations. Note that the FRW models are conformally flat; that is the FRW metric can be written as $g_{\mu\nu}^{\text{FRW}} = \Omega^2 \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski, flat space metric. As we show below explicitly for a Universe with flat spatial sections, this implies that one can transform the electromagnetic wave equation into its flat space version. It turns out that one cannot then amplify electromagnetic wave fluctuations in such a FRW Universe and the field then always decreases² with expansion as $1/a^2(t)$.

Therefore mechanisms for magnetic field generation require the breaking of conformal invariance of the electromagnetic action, which changes the above behaviour to $B \sim 1/a^\epsilon$ with typically $\epsilon \ll 1$ for getting a strong field. A multitude of ways have been considered for breaking conformal invariance of the EM action during inflation. Some of them are illustrated in the action below:

$$S = \int \sqrt{-g} d^4x \left[-f^2(\phi, R) \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - bRA^2 + g\theta F_{\mu\nu} \tilde{F}^{\mu\nu} - D_\mu \psi (D^\mu \psi)^* \right]. \quad (43)$$

They include coupling of EM action to scalar fields (ϕ) like the inflaton and the dilation, coupling to curvature invariants (R), coupling to a pseudo-scalar field like the axion (θ), having charged scalar fields (ψ) and so on. If conformal invariance of the EM action can indeed be broken, the EM wave can be amplified from vacuum fluctuations, as its wavelength increases from sub-Hubble to super-Hubble scales. After inflation ends, the Universe reheats and leads to the production of charged particles leading to a dramatic increase in the plasma conductivity. Then the electric field \mathbf{E} would get shorted out while the magnetic field \mathbf{B} of the EM wave gets frozen in. This is the qualitative picture for the generation of primordial fields during the inflationary era.

There is however another potential difficulty; since $a(t)$ is almost exponentially increasing during slow roll inflation, the predicted field amplitude, which behaves say as $B \sim 1/a^\epsilon$ is exponentially sensitive to any changes of the parameters of the model which affects ϵ . Therefore models of magnetic field generation can lead to fields as large as $B \sim 10^{-9}$ G (as redshifted to the present epoch) down to fields which are much smaller than that required for even seeding the galactic dynamo. For example in model considered by Ratra (1992) with $f^2(\phi) \sim e^{\alpha\phi}$, with ϕ being

² Interestingly slower decay can occur for modes comparable to the curvature scale, in open FRW models due to coupling to the spatial curvature (Barrow & Tsagas 2008).

the inflaton, one gets $B \sim 10^{-9}$ to 10^{-65} G, for $\alpha \sim 20-0$. Note that the amplitude of scalar perturbations generated during inflation is also dependent on the parameters of the theory and has to be fixed by hand. But the sensitivity to parameters seems to be stronger for magnetic fields than for scalar perturbations due to the above reason. Nevertheless one may hope that there would arise a theory where the parameters are naturally such as to produce interesting primordial magnetic field strengths. We describe below one framework for magnetic field generation during inflation, keeping the discussion quite general without specifying any specific inflation model. A nice treatment of inflationary generation of magnetic fields, which we follow to some extent, is given by Martin & Yokoyama (2008), where some specific models are also discussed.

3.1 Quantizing the EM field

Let us assume that the scalar field ϕ in Eq. (43) is the field responsible for inflation and also assume that this is the sole term which breaks the conformal invariance of the electromagnetic action. The total action is given by

$$S = - \frac{1}{16\pi} \int d^4x \sqrt{-g} [g^{\alpha\beta} g^{\mu\nu} f^2(\phi) F_{\mu\alpha} F_{\nu\beta}] - \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right]. \quad (44)$$

Maxwell equation now become $[f^2 F^{\mu\nu}]_{;\nu} = 0$, or

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} [\sqrt{-g} g^{\alpha\beta} g^{\mu\nu} f^2(\phi) F_{\mu\alpha} F_{\nu\beta}] = 0. \quad (45)$$

The scalar field satisfies

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} [\sqrt{-g} g^{\mu\nu} \partial_\mu \phi] - \frac{dV}{d\phi} = \frac{f}{8\pi} \frac{df}{d\phi} F_{\mu\nu} F^{\mu\nu}. \quad (46)$$

We assume that the electromagnetic field is a ‘test’ field which does not perturb either the scalar field evolution or the evolution of the background FRW Universe. We take the metric to be spatially flat with

$$ds^2 = -dt^2 + a^2 [dx^2 + dy^2 + dz^2] = a^2(\eta) [-d\eta^2 + dx^2 + dy^2 + dz^2], \quad (47)$$

where $\eta = \int (dt/a)$ is the conformal time. Throughout the discussion in this section, we use conformal time (η) and co-moving space (x, y, z) as our four co-ordinates and all tensors when explicitly specified will be in this co-ordinate frame. It is convenient to adopt the Coulomb gauge

$$A_0(\eta, \mathbf{x}) = 0, \quad \partial_j A^j(\eta, \mathbf{x}) = 0. \quad (48)$$

In this case the time component of Eq. (45) becomes a trivial identity, while the space components give

$$A_i'' + 2 \frac{f'}{f} A_i' - a^2 \partial_j \partial^j A_i = 0, \quad (49)$$

where we have defined $\partial^j = g^{jk} \partial_k = a^{-2} \delta^{jk} \partial_k$, and a prime denotes derivative with respect to η . In fact $a^2 \partial_j \partial^j$ is the usual spatial ∇^2 operator with respect to the co-moving spatial co-ordinates.

We can also use Eq. (6) to write the electric and magnetic fields in terms of the vector potential. Note that the four velocity of the fundamental observers used to define these fields is now given by $u^\mu \equiv (1/a, 0, 0, 0)$. The time components of E_μ and B_μ are zero, while the spatial components are given by

$$E_i = -\frac{1}{a}A'_i, \quad B_i = \frac{1}{a}\epsilon_{ijk}^*\delta^{jl}\delta^{km}(\partial_l A_m). \quad (50)$$

Note that these spatial components are the same as when t is used instead of η as the time co-ordinate. This is because the transformation $t \rightarrow \eta$ is independent of \mathbf{x} , and we have not transformed the space co-ordinates. For a constant f Eq. (49) shows that A_i simply satisfies the usual wave equation in η and \mathbf{x} co-ordinates, whose solutions are plane waves with constant amplitude. Then the amplitude of B_i then scales as $1/a$, while that of B^i scales as $1/a^3$ and so the amplitude of \bar{B}^a scales as $1/a^2$ as before.

We would like quantize the electromagnetic field in the FRW background. For this we treat A_i as the co-ordinate, and find the conjugate momentum Π^i , by varying the EM Lagrangian density $\mathcal{L}_{\text{EM}} = -f^2 F_{\mu\nu} F^{\mu\nu} / (16\pi)$, with respect to A'_i . We get

$$\Pi^i = \frac{\delta \mathcal{L}_{\text{EM}}}{\delta A'_i} = \frac{1}{4\pi} f^2 a^2 g^{ij} A'_j, \quad \Pi_i = \frac{1}{4\pi} f^2 a^2 A'_i.$$

To quantize the electromagnetic field, we promote A^i and Π_i to operators and impose the canonical commutation relation between them,

$$[A^i(\mathbf{x}, \eta), \Pi_j(\mathbf{y}, \eta)] = i \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} P_j^i(\mathbf{k}) \\ = i \delta_{\perp j}^i(\mathbf{x} - \mathbf{y}). \quad (51)$$

Here the term $P_j^i(\mathbf{k}) = (\delta_j^i - \delta_{jm}(k^i k^m / k^2))$ is introduced to ensure that the Coulomb gauge condition is satisfied and $\delta_{\perp j}^i$ is the transverse delta function. This quantization condition is most simply incorporated in Fourier space. We expand the vector potential in terms of creation and annihilation operators, $b_\lambda^\dagger(\mathbf{k})$ and $b_\lambda(\mathbf{k})$, with \mathbf{k} the co-moving wave vector

$$A^i(\mathbf{x}, \eta) = \sqrt{4\pi} \int \frac{d^3 k}{(2\pi)^3} \sum_{\lambda=1}^2 e_\lambda^i(\mathbf{k}) \times \\ [b_\lambda(\mathbf{k}) A(k, \eta) e^{i\mathbf{k} \cdot \mathbf{x}} + b_\lambda^\dagger(\mathbf{k}) A^*(k, \eta) e^{-i\mathbf{k} \cdot \mathbf{x}}]. \quad (52)$$

Here the index $\lambda = 1, 2$ and $e_\lambda^i(\mathbf{k})$ are the polarization vectors, which form part of an orthonormal set of basis four-vectors,

$$e_0^\mu = \left(\frac{1}{a}, \mathbf{0}\right), \quad e_\lambda^\mu = \left(0, \frac{\bar{e}_\lambda^i}{a}\right), \quad e_3^\mu = \left(0, \frac{\hat{\mathbf{k}}}{a}\right). \quad (53)$$

The 3-vectors \bar{e}_λ^i are unit vectors, orthogonal to \mathbf{k} and to each other. The expansion in terms of the polarization vectors incorporates the Coulomb gauge condition in Fourier space. It also shows that the free electromagnetic field has two polarization degrees of freedom. If we substitute the

Fourier expansion given in Eq. (52) into Eq. (49), we find that the Fourier coefficients $\bar{A} = (aA(k, \eta))$ satisfy

$$\bar{A}'' + 2\frac{f'}{f}\bar{A}' + k^2\bar{A} = 0. \quad (54)$$

One can also define a new variable $\mathcal{A} = a(\eta)f(\eta)A(\eta, k)$ in order to eliminate the first derivative term, to get

$$\mathcal{A}''(\eta, k) + \left(k^2 - \frac{f''}{f}\right)\mathcal{A}(\eta, k) = 0. \quad (55)$$

We can see that the mode function \mathcal{A} satisfies the equation of a harmonic oscillator with a time dependent mass term. The case $f = 1$ corresponds to the standard EM action where \mathcal{A} oscillates harmonically with time. One needs $f'' > 0$ to have possible growth of magnetic fields.

The quantization condition given in Eq. (51) is implemented by imposing the following commutation relations between the creation and annihilation operators:

$$[b_\lambda(\mathbf{k}), b_{\lambda'}^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'}, \\ [b_\lambda(\mathbf{k}), b_{\lambda'}(\mathbf{k}')] = [b_\lambda^\dagger(\mathbf{k}), b_{\lambda'}^\dagger(\mathbf{k}')] = 0. \quad (56)$$

We also define the vacuum state $|0\rangle$ as one which is annihilated by $b_\lambda(\mathbf{k})$, that is $b_\lambda(\mathbf{k})|0\rangle = 0$. Note that the choice of the initial quantum state will be decided by the choice of the mode function \mathcal{A} as below. To check if Eq. (51) is indeed satisfied, we can substitute the Fourier expansion of A^i and Π_j into the commutator $[A^i, \Pi_j]$, and use the commutation relations Eq. (56). We get

$$[A^i(\mathbf{x}, \eta), \Pi_j(\mathbf{y}, \eta)] = \int \frac{d^3 k}{(2\pi)^3} \sum_{\lambda=1}^2 e_\lambda^i(\mathbf{k}) e_{j\lambda}(\mathbf{k}) \times \\ e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} W(k, \eta) f^2 a^2, \quad (57)$$

where we have defined the complex Wronskian,

$$W(k, \eta) = [AA'^* - A^*A'].$$

One can check that the polarization four-vectors satisfy the completeness relation

$$\sum_{\lambda=1}^2 e_\lambda^i(\mathbf{k}) e_{j\lambda}(\mathbf{k}) = P_j^i(\mathbf{k}). \quad (58)$$

Let us also define the Wronskian associated with \bar{A} , given by $\bar{W} = [\bar{A}\bar{A}'^* - \bar{A}'^*\bar{A}]$. We have $\bar{W} = a^2 W$, and since \bar{A} satisfies Eq. (54), we get $\bar{W}' = -(2f'/f)\bar{W}$. Integrating this equation we get $\bar{W} = a^2 W \propto (1/f^2)$. Substituting the expression for W into Eq. (57), and using Eq. (58), we can verify that the quantization condition Eq. (51) is indeed satisfied, provided we fix the constant of proportionality in W such that $W = (i/f^2 a^2)$.

Once we have set up the quantization of the EM field, it is of interest to ask how the energy density of the EM field evolves. The energy momentum tensor is given by varying the EM Lagrangian density with respect to the metric,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta[\sqrt{-g}\mathcal{L}_{\text{EM}}]}{\delta g^{\mu\nu}} \\ = \frac{f^2}{4\pi} \left[g^{\gamma\beta} F_{\mu\gamma} F_{\nu\beta} - g_{\mu\nu} \frac{F_{\alpha\beta} F^{\alpha\beta}}{4} \right]. \quad (59)$$

The energy density $T_{\mu\nu}u^\mu u^\nu$ can be written as the sum of a magnetic and electric contributions. The energy density due to the magnetic part of the EM field is given by

$$\begin{aligned} T_{\mu\nu}^B u^\mu u^\nu &= \frac{f^2}{16\pi} [A_{m,i} - A_{i,m}] [A_{l,j} - A_{j,l}] g^{ij} g^{ml} \\ &= f^2 \frac{B_i B^i}{8\pi}. \end{aligned} \quad (60)$$

Similarly, the energy density due to the electric field is

$$T_{\mu\nu}^E u^\mu u^\nu = \frac{f^2}{8\pi} [A'_i A'_j] g^{ij} = f^2 \frac{E_i E^i}{8\pi}. \quad (61)$$

We substitute the Fourier expansion of A_i into Eq. (60) and Eq. (61), and take the expectation value in the vacuum state $|0\rangle$. Let us define

$$\rho_B = \langle 0 | T_{\mu\nu}^B u^\mu u^\nu | 0 \rangle, \quad \rho_E = \langle 0 | T_{\mu\nu}^E u^\mu u^\nu | 0 \rangle.$$

Using the properties

$$\begin{aligned} b_\lambda(\mathbf{k})|0\rangle &= 0, \\ \langle 0 | b_\lambda(\mathbf{k}) b_{\lambda'}^\dagger(\mathbf{p}) | 0 \rangle &= (2\pi)^3 \delta(\mathbf{k} - \mathbf{p}) \delta_{\lambda\lambda'}, \end{aligned} \quad (62)$$

we get for the spectral energy densities in the magnetic and electric fields,

$$\frac{d\rho_B}{d\ln k} = \frac{1}{2\pi^2} \left(\frac{k}{a} \right)^4 k |\mathcal{A}(k, \eta)|^2, \quad (63)$$

$$\frac{d\rho_E}{d\ln k} = \frac{f^2}{2\pi^2} \frac{k^3}{a^4} \left| \left[\frac{\mathcal{A}(k, \eta)}{f} \right]' \right|^2. \quad (64)$$

Once we have calculated the evolution of the mode function $\mathcal{A}(k, \eta)$, the evolution of energy densities in the magnetic and electric parts of the EM field can be calculated.

3.2 Evolution of normal modes

Consider for example a case where the scale factor $a(\eta)$ evolves with conformal time as

$$a(\eta) = a_0 \left| \frac{\eta}{\eta_0} \right|^{1+\beta}. \quad (65)$$

The case when $\beta = -2$ corresponds to de Sitter space-time of exponential expansion in cosmic time, or $a(t) \propto \exp(Ht)$. On the other hand for an accelerated power law expansion with $a(t) = a_0(t/t_0)^p$ and $p > 1$, integrating $dt = ad\eta$, we have

$$\begin{aligned} \eta &= -\frac{t_0}{a_0(p-1)} \left(\frac{t}{t_0} \right)^{-1/(p-1)}, \\ a(\eta) &= a_0 \left[\frac{-a_0(p-1)\eta}{t_0} \right]^{-p/(p-1)}. \end{aligned} \quad (66)$$

Here we have assumed that $\eta \rightarrow 0_-$ as $t \rightarrow \infty$, such that during inflation, the conformal time lies in the range $-\infty < \eta < 0$. In the limit of $p \gg 1$, one goes over to an almost exponential expansion with $\beta \rightarrow -2 - 1/p$.

Let us also consider a model potential where the gauge coupling function f evolves as a power law,

$$f(\eta) \propto a^\alpha. \quad (67)$$

This could obtain for example for exponential form of $f(\phi)$ and power law inflation. We then have

$$\frac{f''}{f} = \frac{\gamma(\gamma-1)}{\eta^2}, \quad \gamma = \alpha(1+\beta). \quad (68)$$

Then the evolution of the mode function \mathcal{A} is given by

$$\mathcal{A}''(k, \eta) + \left(k^2 - \frac{\gamma(\gamma-1)}{\eta^2} \right) \mathcal{A}(k, \eta) = 0, \quad (69)$$

whose solution can be written in terms of Bessel functions,

$$\begin{aligned} \mathcal{A} = (-k\eta)^{1/2} [C_1(k) J_{\gamma-1/2}(-k\eta) \\ + C_2(k) J_{-\gamma+1/2}(-k\eta)], \end{aligned} \quad (70)$$

where $C_1(k)$ and $C_2(k)$ are scale-dependent coefficients to be fixed by the initial conditions.

Let us define the Hubble radius at any epoch as the length scale $R_H = 1/H$ (in units where the speed of light $c = 1$). The initial conditions to determine the constants in Eq. (70) are specified for each mode (or wavenumber k), when it is deep within the Hubble radius. That is when the proper scale length associated with the mode $(k/a)^{-1}$ is much smaller than R_H , or $(k/aH) \gg 1$. For such small scale modes one assumes that the effects of space-time curvature are negligible and thus the mode function goes over to that relevant for the Minkowski space vacuum. Recall that the expansion rate is given by $H(t) = \dot{a}/a = a'/a^2$. (Here and below \dot{A} is derivative of A with respect to proper time.) For the expansion factor given in Eq. (66), we have $a'/a = -(p/(p-1))(1/\eta)$, and for $p \gg 1$, $aH \rightarrow -1/\eta$. Thus the ratio the Hubble radius to the proper scale of a perturbation is given by $(1/H)(a/k)^{-1} = k/(aH) = -k\eta$. A given mode is therefore within the Hubble radius for $-k\eta > 1$ and outside the Hubble radius when $-k\eta < 1$.

In the short wavelength limit, $(k/a)/H = (-k\eta) \rightarrow \infty$, the solutions of Eq. (69) are simply $\mathcal{A} \propto \exp(\pm ik\eta)$. The assumption that the gauge field for these modes is in the Minkowski space vacuum state leads us to pick the 'positive' frequency mode $\mathcal{A} = c_0 \exp(-ik\eta)$, and from the Wronskian condition that $W = i/f^2 a^2$, the constant c_0 is fixed to $c_0 = 1/\sqrt{2k}$. Thus we assume as initial condition that as $(-k\eta) \rightarrow \infty$,

$$\mathcal{A} \rightarrow \frac{1}{\sqrt{2k}} \exp^{-ik\eta}. \quad (71)$$

This fixes the constants in Eq. (70) to be

$$\begin{aligned} C_1(k) &= \sqrt{\frac{\pi}{4k}} \frac{\exp(-i\pi\gamma/2)}{\cos(\pi\gamma)}, \\ C_2(k) &= \sqrt{\frac{\pi}{4k}} \frac{\exp(i\pi(\gamma+1)/2)}{\cos(\pi\gamma)}, \end{aligned} \quad (72)$$

where we have used the asymptotic expansion, that for $x \rightarrow \infty$,

$$J_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos \left[x - \left(\nu + \frac{1}{2} \right) \frac{\pi}{2} \right].$$

In the opposite limit of modes well outside the Hubble radius, or at late times, with $(-k\eta) \rightarrow 0$, we get from Eq. (70)

$$\mathcal{A} \rightarrow k^{-1/2} [c_1(\gamma)(-k\eta)^\gamma + c_2(\gamma)(-k\eta)^{1-\gamma}], \quad (73)$$

where

$$\begin{aligned} c_1 &= \frac{\sqrt{\pi}}{2^{\gamma+1/2}} \frac{e^{-i\pi\gamma/2}}{\Gamma(\gamma + \frac{1}{2}) \cos(\pi\gamma)}, \\ c_2 &= \frac{\sqrt{\pi}}{2^{3/2-\gamma}} \frac{e^{i\pi(\gamma+1)/2}}{\Gamma(\frac{3}{2} - \gamma) \cos(\pi\gamma)}, \end{aligned} \quad (74)$$

Here we have used the property that

$$J_\nu(x) \rightarrow \frac{x^\nu}{2^\nu \Gamma(\nu+1)}, \quad \text{as } x \rightarrow 0.$$

From Eq. (73) one sees that the c_1 term dominates for $\gamma \leq 1/2$, while c_2 term dominates for $\gamma \geq 1/2$.

As an aside, we note that the late time (small k) solution can also be got for a more general f from directly integrating Eq. (55) in the limit $k \rightarrow 0$. We get

$$\mathcal{A} \rightarrow \bar{c}_1 f + \bar{c}_2 f \int \frac{d\eta}{f^2}. \quad (75)$$

If we substitute $f \propto a^\alpha$ and $a \propto |\eta|^{1+\beta}$ into Eq. (75), we will recover the long wavelength solution given in Eq. (73).

3.3 The generated magnetic and electric fields

We can now calculate the spectrum of ρ_B and ρ_E in the late time, super Hubble limit. Substituting Eq. (73) into Eq. (63), we get for the magnetic spectrum

$$\begin{aligned} \frac{d\rho_B}{d \ln k} &= \frac{\mathcal{F}(n)}{2\pi^2} H^4 \left(\frac{k}{aH} \right)^{4+2n} \\ &\approx \frac{\mathcal{F}(n)}{2\pi^2} H^4 (-k\eta)^{4+2n}, \end{aligned} \quad (76)$$

where $n = \gamma$ if $\gamma \leq 1/2$ and $n = 1 - \gamma$ for $\gamma \geq 1/2$, and

$$\mathcal{F}(n) = \frac{\pi}{2^{2n+1} \Gamma^2(n + \frac{1}{2}) \cos^2(\pi n)}. \quad (77)$$

During slow roll inflation, the Hubble parameter H is expected to vary very slowly, and thus most of the evolution of the magnetic spectrum is due to the $(-k\eta)^{4+2n}$ factor. One can see that the property of scale invariance of the spectrum (with $4 + 2n = 0$), and having $\rho_B \sim a^0$ go together, and they require either $\gamma = 3$ or $\gamma = -2$.

We can also calculate the electric field spectrum in a very similar manner. We first find $(\mathcal{A}/f)'$ from Eq. (70), using the identities

$$J'_\nu - \frac{\nu}{x} J_\nu = -J_{\nu+1}, \quad J'_\nu + \frac{\nu}{x} J_\nu = -J_{\nu-1},$$

and then take the limit $(-k\eta) \rightarrow 0$. Substituting the result into Eq. (64) gives

$$\begin{aligned} \frac{d\rho_E}{d \ln k} &= \frac{\mathcal{G}(m)}{2\pi^2} H^4 \left(\frac{k}{aH} \right)^{4+2m} \\ &\approx \frac{\mathcal{G}(m)}{2\pi^2} H^4 (-k\eta)^{4+2m}, \end{aligned} \quad (78)$$

where now $m = \gamma + 1$ if $\gamma \leq -1/2$ and $m = -\gamma$ for $\gamma \geq -1/2$, and

$$\mathcal{G}(m) = \frac{\pi}{2^{2m+3} \Gamma^2(m + \frac{3}{2}) \cos^2(\pi m)}. \quad (79)$$

Thus having a scale invariant magnetic spectrum implies that the electric spectrum is not scale invariant, and in addition can vary strongly with time. For example if $\gamma = 3$, then $(4 + 2m) = -2$, although $(4 + 2n) = 0$. In this case, at late times as $(-k\eta) \rightarrow 0$, the electric field increases rapidly, with $\rho_E \propto (-k\eta)^{-2} \rightarrow \infty$. There is then the danger of its energy density exceeding the energy density in the Universe during inflation itself, unless the scale of inflation (or the value of H^4 is sufficiently small. Such values of γ are strongly constrained by the back reaction on the background expansion they imply (Martin & Yokoyama 2008).

On the other hand consider the case near $\gamma = -2$. In this case the magnetic spectrum is scale invariant, and at the same time $(4 + 2m) = 2$, and so the electric energy density goes as $(-k\eta)^2 \rightarrow 0$ as $(-k\eta) \rightarrow 0$. Thus these values of γ are acceptable for magnetic field generation without severe back reaction effects.

We now discuss the evolution of the field after inflation. Post inflationary reheating is expected to convert the energy in the inflaton field to radiation (which will include various species of relativistic charged particles). For simplicity let us assume this reheating to be instantaneous. After the Universe becomes radiation dominated its conductivity (σ) becomes important. Indeed Turner & Widrow (1988) showed that the ratio $\sigma/H \gg 1$. In order to take into account this conductivity, one has to reinstate the interaction term in the EM action, given in Eq. (1). Further, as the inflaton has decayed, we can take f to have become constant with time and settled to some value f_0 . Varying the action with respect to A_μ now gives

$$F^{\mu\nu}_{;\nu} = \frac{4\pi J^\mu}{f_0^2}.$$

The value of f_0 thus goes to renormalize the value of electric charge e to be $e_N = e/f_0^2$. This aspect raises an additional potential problem for values of $\gamma \approx -2$, which has been recently emphasized by Demozzi, Mukhanov & Rubinstein (2009) (DMR).

Suppose the inflationary expansion is almost exponential with $\beta = -2$, then for $\gamma \approx -2$, we have $\alpha = \gamma/(1 + \beta) \approx 2$. This implies that the function $f = f_i(a/a_i)^2$ increases greatly during inflation, from its initial value of f_i at $a = a_i$. Thus if we want $f_0 \sim 1$ at the end of inflation, then at early times $f_i \ll f_0$ and the renormalized charge at these early times $e_N = e/f_i^2 \gg e$. DMR argue that one is then in a strongly coupled regime at the beginning of inflation where such a theory is not trustable. There is however the following naive caveat to the above argument: Suppose one started with a weakly coupled theory where $f_i \sim 1$. Then at the end of inflation $f_0 \gg f_i$, and so the renormalized charge $e_N \ll e$. Such a situation does not seem to have the problem of strong coupling raised by DMR; however it does leave the gauge field extremely weakly coupled to the charges at the end of inflation. This also means that even if ρ_B is large, the magnetic field strength itself as deduced from Eq. (60) is $B^i B_i = 8\pi \rho_B / f_0^2 \ll 8\pi \rho_B$. Ideas to sort out this difficulty need to be explored, whether for example one can relax a

large f_0 back to f_i , without now re-generating strong electric fields.

Let us proceed by assuming that we have absorbed f_0^2 into e . In the conducting plasma which obtains after reheating, the current density will be given by the Ohm's law of Eq. (27). The fluid velocity at this stage is expected to be that of the fundamental observers, i.e. $w^\mu = u^\mu$. Thus the spatial components $J^i = \sigma E^i = -g^{ij} \dot{A}_j$. Let us assume that the net charge density is negligible and thus neglect gradients in the scalar potential A_0 . Then the evolution of the spatial components of the vector potential is given by

$$\ddot{A}_i + (H + 4\pi\sigma)\dot{A}_i - \partial_j \partial^j A_i = 0. \quad (80)$$

We see that any time dependence in A_i is damped out on the inverse conductivity time-scale. To see this explicitly, consider modes which have been amplified during inflation and hence have super Hubble scales $k/(aH) \ll 1$. Also let us look at the high conductivity limit of $\sigma/H \gg 1$. Then Eq. (80) reduces to

$$\ddot{A}_i + 4\pi\sigma \dot{A}_i = 0$$

whose solution is given by

$$A_i = \frac{D_1(\mathbf{x})}{4\pi\sigma} e^{-4\pi\sigma t} + D_2(\mathbf{x}). \quad (81)$$

We see that the D_1 term decays exponentially on a time-scale of $(4\pi\sigma)^{-1} \ll (1/H)$. This leaves behind a constant (in time) $A_i = D_2(\mathbf{x})$. Thus the electric field $E_i = 0$, and the high conductivity of the plasma has led to the shorting out of the electric field. Note that the time scale in which the electric field decays does not depend on the scale of the perturbation, that is the σ dependent damping term in Eq. (80) has no dependence on spatial derivatives. As far as the magnetic field is concerned, Eq. (50) shows that $B_i \sim 1/a$ when $A_i = D_2(\mathbf{x})$. Therefore $\dot{B}_i \sim 1/a^2$, as expected when the magnetic field is frozen into the highly conducting plasma.

Let us now make a numerical estimate of the strength of the magnetic fields generated in the scale invariant case. For both $\gamma = -2$ and $\gamma = 3$, we have from Eq. (76) and Eq. (77)

$$\frac{d\rho_B}{d \ln k} \approx \frac{9}{4\pi^2} H^4. \quad (82)$$

Cosmic Microwave Background limits on the amplitude of scalar perturbations generated during inflation, give an upper limit on $H/M_{\text{pl}} \sim 10^{-5}$ (cf. Bassett et al. 2006). Here $M_{\text{pl}} = 1/\sqrt{G}$ is the Planck mass. The magnetic energy density decreases with expansion as $1/a^4$, and so its present day value $\rho_B(0) = \rho_B(a_f/a_0)^4$, where a_f is the scale factor at end of inflation, while a_0 is its present day value. Let us assume that the Universe transitioned to radiation domination immediately after inflation and use entropy conservation, that is the constancy of $gT^3 a^3$ during its evolution, where g is the effective relativistic degrees of freedom and T the temperature of the relativistic fluid. We get

$$\frac{a_0}{a_f} \sim \frac{g_f^{1/12} H^{1/2} M_{\text{pl}}^{1/2}}{g_0^{1/3} T_0} \left(\frac{90}{8\pi^3} \right)^{1/4}.$$

Taking $g_f \sim 100$, gives $(a_0/a_f) \sim 10^{29} (H/10^{-5} M_{\text{pl}})^{1/2}$. This leads to an estimate the present day value of the magnetic field strength, B_0 at any scale,

$$B_0 \sim 5 \times 10^{-10} \text{ G} \left(\frac{H}{10^{-5} M_{\text{pl}}} \right). \quad (83)$$

Thus interesting field strengths can in principle be created if the parameters of the coupling function f are set appropriately and the problems highlighted by DMR can be circumvented. Note that the strength of the generated field is sensitive to even slight departures from scale invariance. Suppose $\gamma = -2 + \epsilon$, with $\epsilon \ll 1$, then

$$\frac{d\rho_B}{d \ln k} \approx \frac{9}{4\pi^2} H^4 \left(\frac{k}{aH} \right)^{2\epsilon}, \quad (84)$$

valid for $(k/aH) \ll 1$. Assuming a radiation dominated Universe immediately after inflation, and matter domination from a redshift $z_{\text{eq}} \sim 3300$, we estimate

$$(k/aH) \sim 3 \times 10^{-24} \left(\frac{k}{1 \text{ h Mpc}^{-1}} \right) \left(\frac{H}{10^{-5} M_{\text{pl}}} \right)^{1/2}.$$

Thus at galactic scales of $k = 1 \text{ h Mpc}^{-1}$, B_0 will be smaller or larger by a factor of $\sim 10^{-5}$, if one takes $\epsilon = \pm 0.2$ respectively. This shows the sensitivity of the magnetic field amplitude to small changes in the parameters of any generation model, as mentioned earlier.

4 Discussion

We end with a few comments. We have emphasized the use of tetrads in defining properly behaved magnetic and electric fields. Such a procedure is also followed when using with the $3+1$ formalism in the context of black-hole electrodynamics (cf. Macdonald et al. 1986). Thus it seems to us somewhat surprising why this does not usually get mentioned when dealing with MHD in the context of cosmology. Regarding magnetic field generation during inflation, we have a paradigm, but no compelling model as yet. Clearly more work is required to find such a model, which at the same time avoids the problems of back reaction, the strong coupling problem mentioned by DMR, or the strong decay of coupling constants.

Another possibility is generation of primordial fields is during a later phase transition, like the Electro-Weak or the quark-hadron transition. Here the main problem is that the generated field typically has a tiny correlation scale, less than the Hubble radius H^{-1} , at the epoch of the phase transition, unless magnetic helicity is also generated. Interestingly there are several ideas for such helicity generation during the Electro-weak phase transition (cf. Vachaspati 2001; Diaz-Gil et al. 2008; Copi et al. 2008). Magnetic energy decay conserving helicity can then lead to an inverse cascade and larger coherence scales, as first emphasized in the early Universe context by Brandenburg, Enquist & Olesen (1996) (see also Christensson, Hindmarsh & Brandenburg 2001; Banerjee & Jedamjick 2005), but that is a story for another review.

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