Schwarzschild Black Hole in an Asymptotically Uniform Magnetic Field (*).

R. S. HANNI (**) and R. RUFFINI

Joseph Henry Laboratories, Princeton University - Princeton, N. J. 08540

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The discovery of binary X-ray sources (1) and the possible identification of some of them with black holes (2) in orbit around a normal star have made the accretion of material black holes of the greatest interest. The existence of magnetic fields in the main star and the tendency of plasma, falling into the collapsed object to align the magnetic lines of force along the direction of the infalling material (3) have made the analysis of magnetic fields in the neighborhood of a black hole essential to our understanding of the physics of these collapsed objects.

It has been known that a black hole in empty space can only be endowed with a magnetic field if the black hole itself is endowed with rotation and charge (4). The multipole structure of the field is uniquely determined and given asymptotically by a dipole field of strength

$$\mu = Qa\,,$$

where (5) Q is the charge of the black hole and $a = L/M$ is the angular momentum per unit mass. CHRISTODOULOU (6) has shown that any departure in the magnetic field from the configuration dictated by the Kerr-Newman (7) solution will be radiated away

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(**) Now at Stanford University, Stanford, Cal. 94305.

(1) See, *e.g.*, GIACCONI: *Binary* X-ray *sources*, invited talk delivered at the *Solvay Meeting, Bruxelles, 1973*, to appear in the *Proceedings*.

(2) See, *e.g.*, R. RUFFINI: *Neutron stars and black holes in our galaxy*, invited talk delivered at the *Solvay Meeting, Bruxelles, 1973*, to appear in the *Proceedings*.

(3) For a lucid and detailed treatment of plasma accreting in the magnetic field of a normal star see the classical work of M. N. MESTEL: *Roy. Amer. Soc.*, **119**, 223 (1959). Accretion of plasma into a black hole has recently been examined by R. RUFFINI and J. WILSON: to be published.

(4) For the analysis of the electromagnetic structure of the field of a black hole see D. CHRISTODOULOU and R. RUFFINI: *On the electrodynamics of collapsed objects*, in *Black Holes*, edited by B. DEWITT and C. DEWITT (New York, N. Y., 1973).

(5) Here and in the following we are using geometrized units $G = c = 1$.

(6) D. CHRISTODOULOU: private communication to one of us (R.R.).

(7) E. T. NEWMAN, E. COUCH, R. CHINNAPARED, A. EXTON, A. PRAKASH and R. TORRENCE: *Journ. Math. Phys.*, **6**, 918 (1965).

with the characteristic time

$$\tau = M$$

defined by the mass of the black hole, M.

A black hole embedded in a magnetic field rooted in interstellar plasma is radically different. HOYLE [8] has emphasized the importance of this distinction for the observation of black holes. As a first step in this investigation we find analytic formulae for an asymptotically uniform magnetic field in which we embed a Schwarzschild black hole. The more complex case in which plasma is present around the magnetic lines of force is currently being investigated [9].

We are going to study the structure of the magnetic field from three points of view. First we analyse locally the physical components of the Lorentz force acting on a magnetic monopole of unit strength at rest at a given point in the Schwarzschild background. Then we analyse the lines of force as seen from a far away observer to give a complete picture of the field. Finally we introduce the lines of constant flux which coincide with the lines of force and allow us to space them so that their density is proportional to the physical components of the Lorentz force.

We are dealing with weak magnetic fields whose energy-momentum tensor gives a negligible contribution to the background geometry. Consequently, it is sufficient to solve Maxwell's equations in the fixed background with metric

$$\mathrm{d}s^2 = -(1-2M/r)\,\mathrm{d}t^2 + (1-2M/r)^{-1}\,\mathrm{d}r^2 + r^2(\sin^2\theta\,\mathrm{d}\varphi^2 + \mathrm{d}\theta^2)\,. \tag{1}$$

Since no electric fields are present, the only nonzero covariant components of the electromagnetic field tensor are

$$F_{r\varphi} = A_{\varphi,r}\,, \tag{2.1}$$

$$F_{\theta\varphi} = A_{\varphi,\theta}\,, \tag{2.2}$$

with contravariant components

$$F^{r\varphi} = (1-2m/r)r^{-2}\sin^{-2}\theta A_{\varphi,r}\,, \tag{3.1}$$

$$F^{\theta\varphi} = r^{-4}\sin^{-2}\theta A_{\varphi,\theta}\,. \tag{3.2}$$

The first set of Maxwell's equations, $*F^{\alpha\beta}{}_{;\beta} = 0$, is automatically satisfied. The only equation derived from the second set of equations $F^{\alpha\beta}{}_{;\beta} = 0$, which is not trivially satisfied, is

$$r^2\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)\frac{\partial A_\varphi}{\partial r}\right] + \sin\theta\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial A_\varphi}{\partial\theta}\right] = 0\,. \tag{4}$$

We separate the variables r and θ by assuming

$$A_\varphi = R(r)\,T(\theta)\,. \tag{5}$$

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[8] F. HOYLE: *Communication at the 1971 Meeting on Collapsed Objects in Honour of the 60th Birthday of* W. FOWLER, Cambridge, England.

[9] R. RUFFINI and J. R. WILSON: to be published.

We then have

$$r^2 \frac{\mathrm{d}}{\mathrm{d}r}(1-2M/r)\frac{\mathrm{d}R(r)}{\mathrm{d}r} = l(l+1)R(r)\,, \tag{6.1}$$

$$\sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\sin\theta}\frac{\mathrm{d}T(\theta)}{\mathrm{d}\theta} = -l(l+1)\,T(\theta)\,. \tag{6.2}$$

Setting $z = 2M/r$ we can transform eq. (6.1) to

$$\frac{\mathrm{d}}{\mathrm{d}z}(1-z)z^2\frac{\mathrm{d}R(z)}{\mathrm{d}z} = l(l+1)R(z)\,. \tag{7}$$

We then have two linearly independent solutions

$$\varphi_l(z) = z^l \sum_{n=0}^{\infty} a_n z^n \tag{8.1}$$

and

$$\psi_l(z) = z^{-(l+1)} \sum_{n=0}^{\infty} b_n z^n\,, \tag{8.2}$$

where a_n and b_n satisfy the following recurrence relations:

$$a_{n+1} = \frac{(n+l)(n-l+1)}{(n+l+1)(n+l+2)-l(l+1)}\,a_n\,, \tag{9.1}$$

$$b_{n+1} = \frac{(n-l-1)(n-l+1)}{(n-l)(n-l+1)-l(l+1)}\,b_n\,. \tag{9.2}$$

For a fixed value of l the solutions $\psi_l(z)$ are given by a finite series. The solutions φ_l can be obtained in close form from the functions ψ_l, we have then ([10])

$$\varphi_l(z) = \psi_l(z)\left[A + B\frac{\mathrm{d}z}{z^2(z-1)\,\psi_l^2(z)}\right]. \tag{10}$$

A and B being two constants. It is then clear from the explicit solution of eq. (10) that all the functions $\varphi_l(z)$ diverge at infinity as $z\to 1$.

The first solutions for $\psi_l(z)$ are

$$\psi_0(r) = 1\,, \tag{11.1}$$

$$\psi_1(r) = \frac{r^2}{4M^2}\,, \tag{11.2}$$

$$\psi_2(r) = \frac{r^3}{8M^3} - \frac{3r^2}{16M^2}\,, \tag{11.3}$$

and they all diverge at infinity with the exception of $\psi_0(r)$.

([10]) R. LEACH: Senior Thesis, presented at Princeton University, unpublished (1974).

If we let $x = \cos\theta$, eq. (6.2) takes the form

$$(1-x^2)\frac{\mathrm{d}T_l(x)}{\mathrm{d}x^2} + l(l+1)\,T_l(x) = 0\,, \tag{12}$$

whose solution can be expressed in terms of Gegenbauer polynomials ([11]), the first four being

$$T_0 = \cos\theta\,, \tag{13.1}$$

$$T_1 = 1-\cos^2\theta\,, \tag{13.2}$$

$$T_2 = \cos\theta - \cos^3\theta\,, \tag{13.3}$$

$$T_3 = 1 - 6\cos^2\theta + 5\cos^5\theta\,. \tag{13.4}$$

For the solution to be regular at the horizon and asymptotically approach a uniform magnetic field it is required that $\varphi_l(r) = 0$ if $l \neq 1$. This determines the potential uniquely, we have

$$A_\varphi = \frac{r^2}{2}\sin^2\theta\,,$$

and for the covariant, contravariant components of the electromagnetic-field tensor we have

$$F_{\theta\varphi} = A_{\varphi,\theta} = r^2\cos\theta\sin\theta\,, \tag{14.1}$$

$$F_{r\varphi} = A_{\varphi,r} = r\sin^2\theta\,, \tag{14.2}$$

$$F^{\theta\varphi} = r^{-2}\operatorname{ctg}\theta\,, \tag{14.3}$$

$$F^{r\varphi} = r^{-2}(r-2m)\,. \tag{14.4}$$

The covariant components of the dual tensor are

$$*F_{rt} = \cos\theta\,, \qquad *F_{\theta t} = (r-2m)\sin\theta\,.$$

The Lorentz force on a magnetic monopole at rest in the Schwarzschild background is

$$L_\alpha = *F_{\alpha\beta}\mu^\beta\,, \qquad \mu^\beta = [0, 0, 0, (1-2m/r)^{-\frac{1}{2}}]\,.$$

In particular

$$L_r = \cos\theta/(1-2m/r)^{\frac{1}{2}} \qquad \text{and} \qquad L_\theta = r\left(1-\frac{2m}{r}\right)^{\frac{1}{2}}\sin\theta\,;$$

the physical components are

$$L_{\hat{r}} = \cos\theta\,, \tag{15.1}$$

$$L_{\hat{\theta}} = \left(1-\frac{2m}{r}\right)^{\frac{1}{2}}\sin\theta\,, \tag{15.2}$$

([11]) See, *e.g.*, M. Abramowitz and I. A. Stegun: *Handbook of Mathematical Functions* (1970).

which at large distances indeed correspond to a uniform magnetic field

$$L_{\hat{r}} = \cos\theta\,, \qquad L_{\hat{\theta}} = \sin\theta\,.$$

Let us now proceed to compute the lines of constant flux. The magnetic flux through a surface bounded by a surface of revolution at r and θ is

$$\Phi = \int_s F = \int_s F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 4\pi p\,, \tag{16}$$

where p is the total magnetic charge contained within the surface. We can then obtain from eq. (14) the following expressions for the flux if

$$\frac{\partial\varphi}{\partial r} = -4\pi r \sin^2\theta\,, \qquad \frac{\partial\varphi}{\partial\theta} = 4\pi r^2 \cos\theta \sin\theta\,.$$

The slope of the lines of constant flux is

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{\partial\Phi/\partial\theta}{\partial\Phi/\partial r} = r\cos\theta/\sin\theta\,. \tag{17}$$

The flux, through a surface bounded by a ring of revolution at r and θ, determined up to an arbitrary constant, is

$$\Phi = -2\pi r^2 \sin^2\theta\,.$$

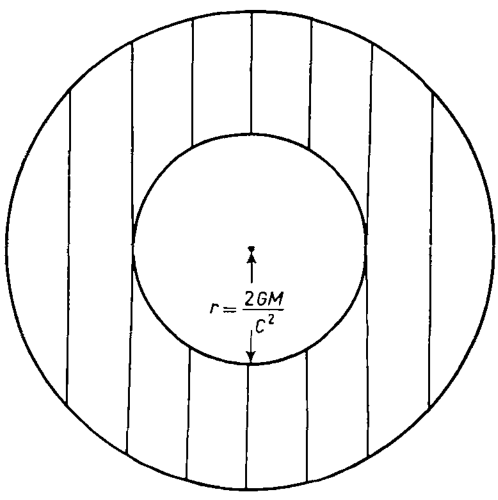

Fig. 1. – Magnetic lines of force of a uniform field at infinity plotted in Schwarzschild co-ordinates. The inner circle is the event horizon ($r = 2GM/c^2$) and the outer circle is an arbitrary cut-off ($r = 4GM/c^2$). This diagram seems inconsistent with the conclusion reached in eq. (15) that the lines of forces should cross the horizon orthogonally; however, the physical components have to be interpreted in an embedded diagram, see fig. 2 and 3.

The lines of force must be at every point tangent to the Lorentz force. We find then from eq. (15)

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = *F^{\varphi}{}_{t}/*F^{\theta}{}_{t} = r\cos\theta/\sin\theta\,. \tag{18}$$

This expression is identical to that given by eq. (17). The lines of force and the lines of constant flux therefore coincide. In fig. 1, 2 and 3 the lines of force are reproduced as given in an embedded diagram (physical components) or projected in the co-ordinate space. In particular it is clarified in what sense we can say that the lines of force cross the horizon orthogonally.

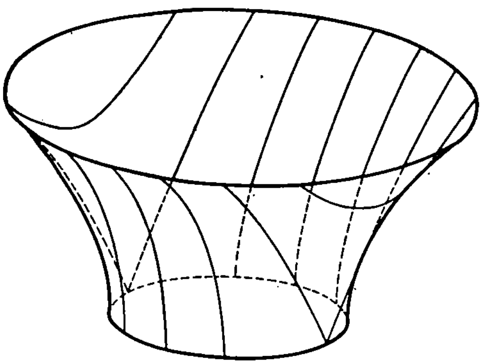

Fig. 2.

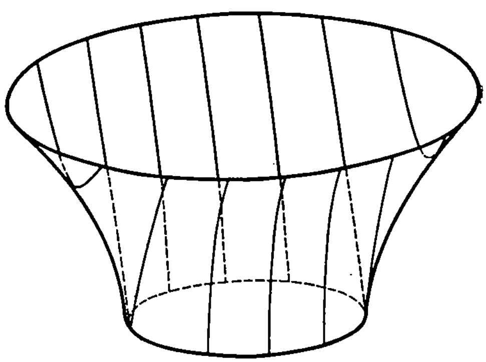

Fig. 3.

Fig. 2. – Magnetic lines of force of a uniform field at infinity plotted on a curved background. Exploiting the static nature and the axial symmetry of the Schwarzschild solution we suppress the temporal and azimuthal dependence of the metric. The exterior Schwarzschild solution can be visualized as a two-dimensional hyperboloid embedded in an Euclidean three space. We then have for a radial trajectory $\mathrm{d}\sigma^2 = \mathrm{d}r^2 + \mathrm{d}z^2 = (1 - 2M/r)^{-1}\mathrm{d}r^2$. Optical distortion of a five-degree field of view was assumed in projecting this surface. The smaller ellipse is the event horizon and the larger is the arbitrary cut-off of fig. 1. All dashed lines are hidden by the surface. The light continuous lines are magnetic lines of force seen from the outside of the hyperboloid; the medium continuous lines are seen from the inside. In this curved space the lines of force are strictly radial at the event horizon, as are the physical components of the Lorentz force. The two lines with cusps at the event horizon (tangent to the event horizon in Schwarzschild co-ordinates fig. 1) do not intersect the event horizon orthogonally, but all physical components of the magnetic field vanish at the points of intersection.

Fig. 3. – Same diagram as in fig. 2 from a different viewpoint.

The example treated here is of the greatest interest since it clarifies that lines of force can indeed cross the horizon of a Schwarzschild black hole and closed analytic formulae are given for analyzing the strength of the magnetic fields and the lines of force near the horizon. It fails to be realistic in one important respect: In a binary-system plasma will be largely affecting the magnetic lines of force. Material falling toward the black hole will greatly magnify the strength of the magnetic field ([9]). The treatment here presented, however, is still of relevance for the understanding of the boundary conditions to be adopted on the horizon in this more general treatment.