Supplemental Material

to "Classes of hydrodynamic and magnetohydrodynamic turbulent decay" (arXiv:1607.01360)

by A. Brandenburg & T. Kahniashvili

December 14, 2016, Revision: 1.19

EFFECT OF PHASE ERRORS

By default, the PENCIL CODE uses sixth order accurate finite difference representations for the first and second derivatives. A low spatial order of the scheme implies that at high wavenumbers the magnitude of the numerical derivative is reduced, leading to lower advection speeds of the high wavenumber Fourier components. This is generally referred to as phase error. Thus, for an advected tophat function, the high wavenumber constituents will lag behind, creating the well-known Gibbs phenomenon which needs to be controlled by a certain amount of viscosity. Higher order schemes require less viscosity to control the Gibbs phenomenon [1]. On the other hand, any turbulence simulation requires a sufficient amount of viscosity to dissipate kinetic energy. It is therefore thought that for a sixth orders scheme the two limits on the viscosity are similar and that it is not advantageous to use higher order representations of the spatial derivatives.

To verify this in the present context, we have run a high Reynolds number case both with sixth and tenth order schemes. In the PENCIL CODE, the order of the scheme can easily be changed by setting `DERIV=deriv_10th`. In that case, first and second derivatives are represented as

$$\mathrm{d}^n f_i/\mathrm{d}x^n = \sum_{j=-N}^{N} (\mathrm{sgn}\, j)^n c_{|j|}^{(n)} f_{i+j}/\delta x^n, \qquad (1)$$

with coefficient $c_j^{(n)}$ given in Table I for schemes of order

TABLE I: Coefficients $c_j^{(n)} \equiv a_j^{(n)}/b^{(n)}$

N	n	$b^{(n)}$	$a_0^{(n)}$	$a_1^{(n)}$	$a_2^{(n)}$	$a_3^{(n)}$	$a_4^{(n)}$	$a_5^{(n)}$
10	1	2520	0	2100	−600	150	−25	2
8	1	840	0	672	−168	32	−3	
6	1	60	0	45	−9	1		
4	1	12	0	8	−1			
2	1	2	0	1				
10	2	25200	−73766	42000	−6000	1000	−125	8
8	2	5040	−14350	8064	−1008	128	−9	
6	2	180	−490	270	−27	2		
4	2	12	−30	16	−1			
2	2	1	−2	1				

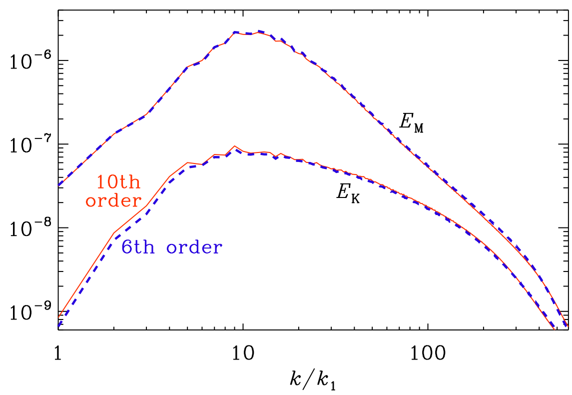

FIG. 1: Magnetic (upper curves) and kinetic (lower curves) energy spectra for at $t = 110$ for the sixth order (blue, dashed) and tenth order (red, solid) finite difference schemes.

N. The result of the comparison is shown in Figure 1. The differences between the two cases are negligible, except that with the more accurate tenth order scheme the inverse transfer of kinetic energy to larger scales is now slightly stronger. This is consistent with our earlier findings that the inverse transfer in nonhelical MHD becomes more pronounced at larger resolution.

ISOTHERMAL VERSUS POLYTROPIC EQUATION OF STATE

An isothermal equation of state is often used in subsonic compressible turbulence to approximate the conditions of nearly incompressible flows. Using instead a polytropic equation of state means that in the momentum equation the pressure gradient term for an isothermal gas is amended by a factor $\propto \rho^{\gamma-1}$, i.e.,

$$c_\mathrm{s}^2 \boldsymbol{\nabla} \ln\rho \to c_{\mathrm{s}0}^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \boldsymbol{\nabla} \ln\rho, \qquad (2)$$

where $\gamma = 5/3$ is the polytropic index for a monatomic gas instead of $\gamma \to 1$ for an isothermal gas. Using $\gamma = 5/3$ implies a slightly stiffer equation of state, so one has to drive stronger to achieve the same compression; see Sect. 9.3.6 of [2]. In the present context of subsonic decaying turbulence, this leads to slightly smaller vorticity fluctuations, as is shown in Figure 2. It is seen that the difference between $\gamma = 5/3$ and 1 is negligible for all practical purposes.

TIME-DEPENDENT $\nu(t)$ AND $\eta(t)$

As pointed out by Olesen [4], the hydrodynamic and MHD equations are invariant under rescaling $x \to \tilde{x}\ell$ and

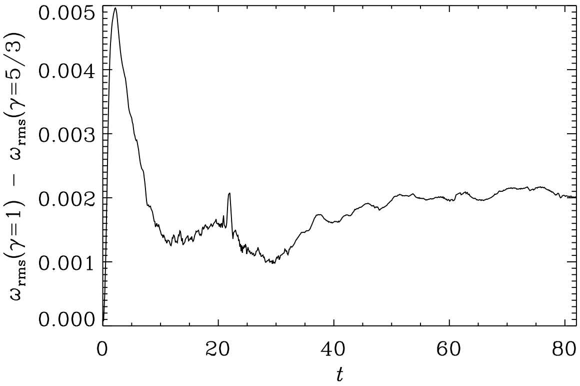

FIG. 2: Difference in rms vorticity, $\omega_{\rm rms}$, between the isothermal and polytropic solutions.

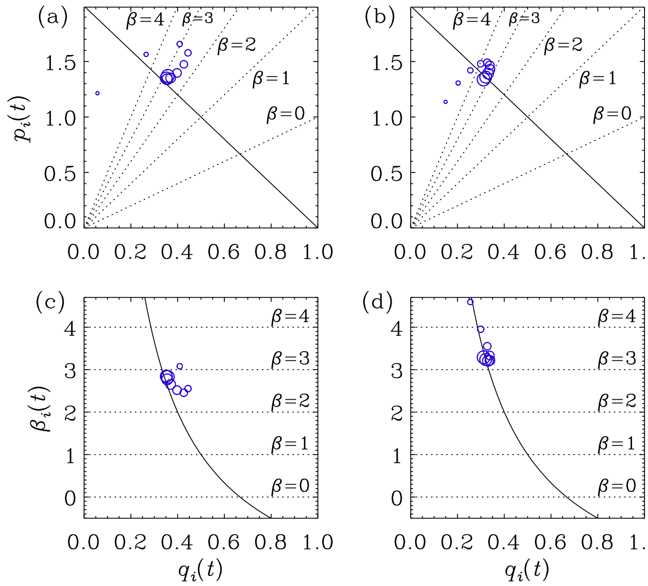

FIG. 3: pq diagrams for hydrodynamic turbulence with $\nu=$ const (a) and time-dependent $\nu(t)\propto t^r$ (b) with $r=-0.43$ and $\alpha=4$. Panels (c) and (d) show the corresponding βq diagrams. Open (closed) symbols corresponds to $i=$ K (M) and their sizes increase with time.

$t\to\tilde{t}\ell^{1/q}$ provided also ν and η are being dynamically rescaled such that

$$\nu(t)=\nu_0\,\max(t/t_0,1)^r,\quad \eta(t)=\eta_0\,\max(t/t_0,1)^r, \quad (3)$$

TABLE II: Exponents r for different α.

α	0	1	2	3	4
r	0.33	0	−0.20	−0.33	−0.43

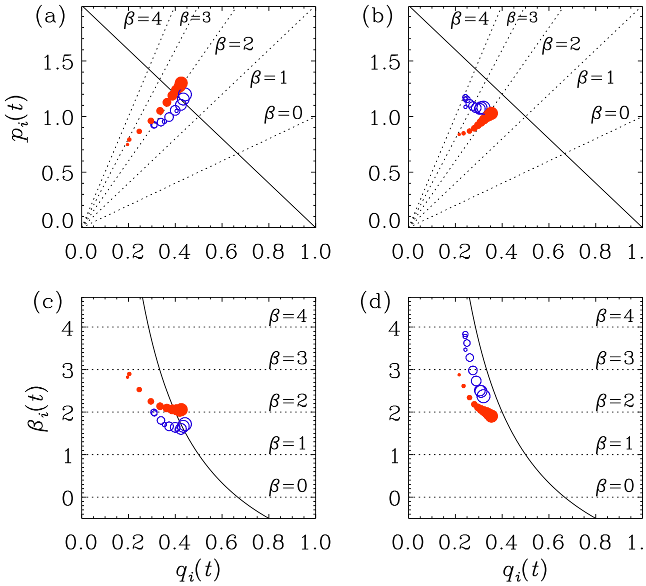

FIG. 4: Similar to Figure 3, but for nonhelical MHD turbulence with $\nu=\eta=$ const (a) and time-dependent $\nu(t)=\eta(t)\propto t^r$ (b) with $r=-0.43$ and $\alpha=4$.

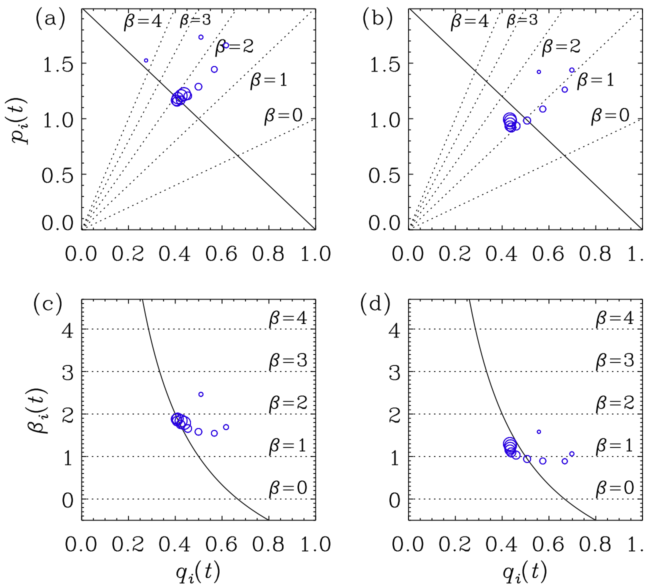

FIG. 5: Similar to Figure 3, but for $\alpha=2$ (a) and $\alpha=1$ (b) with $\nu=$ const.

with

$$r=2q-1=(1-\alpha)/(3+\alpha); \quad (4)$$

see Table II. The use of the max function in Equation (3) limits the values of $\nu\leq\nu_0$ and $\eta\leq\eta_0$ for $t\leq t_0$ when $r<0$. At large Reynolds numbers, the time-dependence is not expected to be important. To verify this, we compare in Figure 3 hydrodynamic runs with constant and time-dependent ν using $\alpha=4$. Both cases are similar and the case with time-dependent ν still has $\beta = 3 \neq \alpha$. Similar behavior is found in MHD; see Figure 4, where we compare runs with constant and time-dependent ν and η using again $\alpha = 4$. In both cases, we find $\beta = 2 \neq \alpha$.

In agreement with earlier work we find that in hydrodynamic cases with $\alpha = 2$ and $\alpha = 1$, we have $\beta = \alpha$ [5]. This is demonstrated in Figure 5, where we show the pq and βq diagrams for these two case.

[1] A. Brandenburg and W. Dobler, Comp. Phys. Comm. **147**, 471 (2002).

[2] A. Brandenburg, (ed. A. Ferriz-Mas & M. Núñez), pp. 269. Advances in nonlinear dynamos (The Fluid Mechanics of Astrophysics and Geophysics, Vol. **9**) (2003). Taylor & Francis, London and New York

[3] A. Brandenburg, T. Kahniashvili, and A. G. Tevzadze, Phys. Rev. Lett. **114**, 075001 (2015).

[4] P. Olesen, Phys. Lett. B **398**, 321 (1997).

[5] Yousef, T. A., Haugen, N. E. L., & Brandenburg, A., Phys. Rev. E **69**, 056303 (2004).