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Particle energization through time-periodic helical magnetic fields

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We solve for the motion of charged particles in a helical time-periodic ABC (Arnold-Beltrami-Childress) magnetic field. The magnetic field lines of a stationary ABC field with coefficients $A = B = C = 1$ are chaotic, and we show that the motion of a charged particle in such a field is also chaotic at late times with positive Lyapunov exponent. We further show that in time-periodic ABC fields, the kinetic energy of a charged particle can increase indefinitely with time. At late times the mean kinetic energy grows as a power law in time with an exponent that approaches unity. For an initial distribution of particles, whose kinetic energy is uniformly distributed within some interval, the probability density function of kinetic energy is, at late times, close to a Gaussian but with steeper tails.

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I. INTRODUCTION

Production of charged particles with energies far exceeding the thermal energy is known to be a very common phenomenon in cosmic plasma. Such energetic particles, which include interplanetary, interstellar, and galactic cosmic rays, are believed to be produced in various astrophysical bodies, from magnetospheric to cosmic plasmas, including solar flares, coronal mass ejections (CMEs), and supernova remnants. In other words, acceleration of charged particles occurs ubiquitously in the plasma universe. Investigations of particle energization remains a major topic of astrophysics. The seminal paper in this field is by Fermi [1], who proposed that charged particles in cosmic rays can attain very high energies by being repeatedly reflected by two magnetic mirrors moving towards each other.

Fermi's idea of energization was tested in a simple setting in the now-famous Fermi-Ulam model [2] whose numerical simulations showed that, although the motion of a particle reflected by moving walls can be chaotic, on average no energy is gained by the particle if the motion of the moving wall is a smooth function of time. This result was elaborated upon in the early days of research in nonlinear dynamical systems [3–6], to show that energy can grow as a power law in time if the motion of the wall is not smooth in time, e.g., random or a sawtooth profile in time. The Fermi-Ulam problem in more than one dimension, sometimes called "billiard problems with breathing walls", allows energization of particles; see, e.g., [7] for a recent review. In some specially constructed cases, even exponential-in-time energy growth is possible [8]. Such problems, although of fundamental interest, are somewhat removed from the problem of energization of charged particles in time-dependent magnetic fields. In this paper, we show that the energy of charged particles can increase as a power law in time in a simple helical magnetic field, whose components are slowly varying sinusoidal functions of both space and time.

The motion of a charged particle in deceptively simple magnetic fields can be very complex, even chaotic. A recent paper [9] has shown that very simple current configurations, for example a circular current loop in the x-y plane plus a line current along the z axis passing through an off-center point, can give rise to a magnetic field whose magnetic lines of force are nonintegrable and chaotic [10]. The motion of a charged particle in such a chaotic magnetic field may or may not be chaotic. Recently, it was conjectured [11] that, if such a chaotic magnetic field changes with time, it may be able to impart significant energy to a charged particle. Our work, described below, demonstrates the feasibility of effective energization of charged particles by a time-varying chaotic magnetic field. We show that this process can lead to an indefinite energization of a charged particle to relativistic energies, given enough time.

II. MODEL

It is now generally accepted that astrophysical magnetic fields are generated by some dynamo mechanism, i.e., a mechanism by which the kinetic energy of the fluid is converted to magnetic energy [12]. If the characteristic length scale of the magnetic field, generated by the dynamo mechanism, is larger than the energy-containing scales of the fluid, the dynamo is called a large-scale dynamo. The most common examples of astrophysical magnetic fields, e.g., galactic magnetic fields, solar magnetic fields, and planetary magnetic fields, are generated by a large-scale dynamo. Almost all large-scale dynamo mechanisms demand that the fluid flow possess helicity, i.e., handedness. The helicity of the flow is often described by the well-known α effect which was proposed by Parker [13] with a detailed mathematical basis provided by Steenbeck, Krause, and Rädler [14]. The helical flow typically generates a helical field. Thus almost all large-scale astrophysical magnetic fields are helical in nature. This is also true of large-scale magnetic fields generated by numerical simulations [15]; see also [12] and references therein. Observations of solar flares also show the helical nature of magnetic field ejected from the Sun [16]. The helical nature of the magnetic field carried by the solar wind has also been observed [17]. One of the simple examples of a helical field, which is also a force-free field, is the Arnold-Beltrami-Childress (ABC) flow. The streamlines of the ABC flows are chaotic [18].

Here we study the possibility of energization of a test particle in a magnetic field, which is a time-dependent ABC function:

$$\boldsymbol{B} = \mathcal{B}\sin\omega t, \tag{1}$$

where $\mathcal{B}$ is an ABC function with wave number k:

$$\begin{aligned}\mathcal{B}_x &= B_0(A\sin kz + C\cos ky),\\ \mathcal{B}_y &= B_0(B\sin kx + A\cos kz),\\ \mathcal{B}_z &= B_0(C\sin ky + B\cos kx).\end{aligned} \tag{2}$$

Here we choose $A = B = C = 1$ (which is the case where the ABC flow has chaotic streamlines), where the time-dependent part is a sine function with circular frequency ω and t is time. Although time variations with multiple time scales are expected to occur in a realistic astrophysical environment, in this work we have chosen, as an example, a simple sinusoidal time variation of the helical field.

Furthermore, the energization of test particles has been observed in complex (in space, but constant in time) turbulent electric fields generated by time variation of fluctuating magnetic field in direct numerical simulations of MHD [19,20]. It then behooves us to ask the question, what are the essential ingredients of the energization process? Can a simple magnetic field like an ABC field that is periodic in both space and time, but is nevertheless expected to give rise to chaotic motion of test particles, energize test particles indefinitely? Finally, the ABC field is an eigenfunction of the curl operator; hence it is easy to solve Maxwell's equations to obtain the electric field generated by the time-dependent part of the ABC field.

Through Maxwell's equations, the time-varying ABC field shall generate a fluctuating electric field, given by

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} = -\mathcal{B}\omega\cos\omega t. \tag{3}$$

Here, $\boldsymbol{E}$ is the electric field, which is given by

$$\boldsymbol{E} = \frac{\omega}{k}\mathcal{B}\cos\omega t. \tag{4}$$

In addition, we assume that $\omega/k \ll c$, where c is the speed of light. Hence we can safely ignore the displacement current in Maxwell's equations of electrodynamics. We further assume the particle to be nonrelativistic; hence its equations of motion are given by Newton's second law of motion

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \quad \dot{\boldsymbol{v}} = \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}). \tag{5}$$

Here and henceforth, the dot denotes a time derivative. In what follows, unless otherwise stated, length scales are normalized by $1/k$, time by ω_c where ω_c is defined to be the characteristic gyrofrequency,

$$\omega_c = \frac{qB_0}{m}. \tag{6}$$

We solve for $N_p = 409\,600$ copies of this six-dimensional dynamical system. Initially, the particle positions are uniformly distributed inside a cube of dimensions $2\pi\times 2\pi\times 2\pi$ with velocity along the x-axis uniformly distributed between -0.01 to 0.01 and $k = 1$. We use a fourth-order Runge-Kutta method with fixed step size [21] as our time-stepping algorithm. This algorithm is not energy-conserving by construction. However we have checked that, in practice, energy conservation is satisfied with a high degree of accuracy [22]. We have also checked for representative runs that a Runge-Kutta-Fehlberg [21] scheme with variable step size gives the same results. The computations are done with a PYTHON package to solve ordinary differential equations, and the figures are prepared using MATPLOTLIB [23].

III. RESULTS

Let us first study the properties of trajectories of particles for the case in which the magnetic field is constant in time. In this case we define the characteristic gyroradius of the particle,

$$r_c = \frac{v}{\omega_c}, \tag{7}$$

where v is the magnitude of the velocity of the particle, and ω_c, the characteristic gyrofrequency, is constant in time, because the energy of the particle is conserved. Let us first consider the case $r_c \ll 2\pi/k$ with k being the characteristic wave number of the magnetic field. In this case, for times $t \ll t_{trans} = 1/(kv)$, with $v = |\boldsymbol{v}|$, the particles move in a field that is almost a constant and hence its motion is not random. Here, t_{trans} is the time it takes a typical particle to go a distance equal to the wavelength of the magnetic field. The net displacement of the particle is due to the component of its velocity parallel to the local magnetic field. This component of velocity is also constant. Hence, the mean-square displacement of the particles obeys

$$\langle r^2\rangle \sim t^2 \quad \text{for} \quad t \ll t_{trans}. \tag{8}$$

In the other limit, $r_c \gg 2\pi/k$, the particles encounter significant change in the magnetic field even within one gyro-orbit. Hence, we expect nontrivial behavior in this regime. The mean-square displacement (normalized by k^2) as a function of time (normalized by ω_c), in log-log scale, is plotted in Fig. 1. For short times, Eq. (8) is indeed verified; for large times we

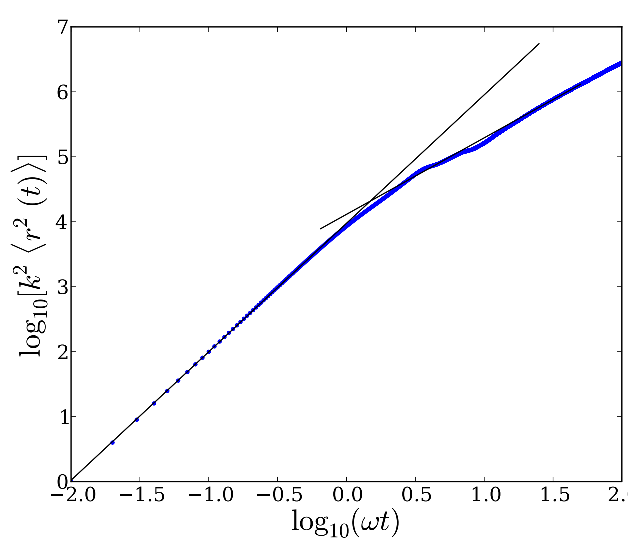

FIG. 1. (Color online) Average displacement $\langle r^2\rangle$ as a function of time for particles in a stationary ABC magnetic field ($\omega = 0$). The ordinate is normalized with k^2. The two straight lines to the left and right of the curve are fits with slopes 1.9 and 1.1, respectively.

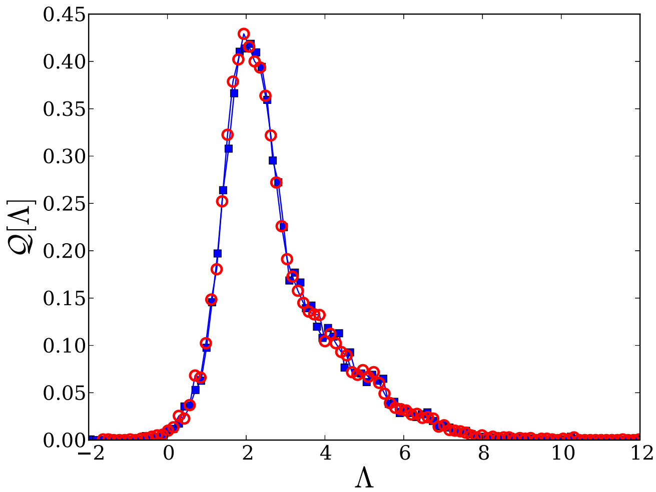

FIG. 2. (Color online) The PDF $\mathcal{Q}(\Lambda)$ of the Lyapunov exponents, defined in Eq. (10), for two time differences, $\omega_c \delta t = 130$ (blue filled squares) and 140 (red open circles).

find that the mean-square displacement grows approximately linearly with time (slope of the fit is 1.1); i.e., Brownian motion is observed.

Whether or not chaotic trajectories exist in this system (i.e., with stationary magnetic field) can be investigated by studying the tangent system,

$$\begin{aligned} \dot{\delta \boldsymbol{x}} &= \delta \boldsymbol{v}, \\ \dot{\delta \boldsymbol{v}} &= \frac{q}{m}[(\delta \boldsymbol{x} \cdot \nabla)\boldsymbol{E} + \boldsymbol{v} \times (\delta \boldsymbol{x} \cdot \nabla)\boldsymbol{B} + \delta \boldsymbol{v} \times \boldsymbol{B}]. \end{aligned} \tag{9}$$

This dynamical system clearly depends on the trajectory in phase space given by $\boldsymbol{x}(t), \boldsymbol{v}(t)$. It is possible to solve this system for each trajectory given by the solutions of Eq. (5). For each such trajectory one can calculate the quantity

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\ \delta x(t)\ |}{|\ \delta x(0)\ |}, \tag{10}$$

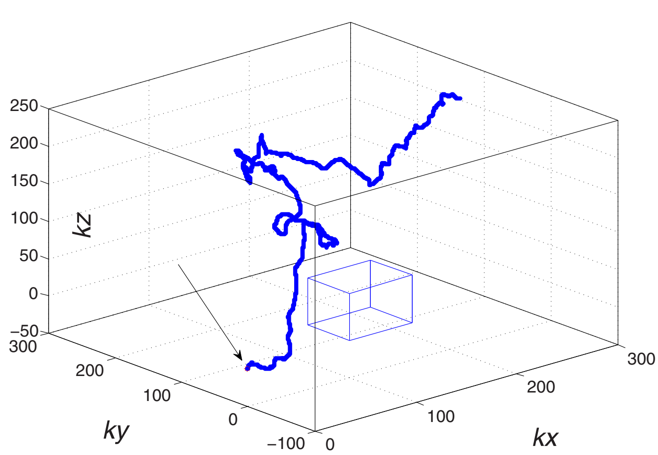

FIG. 3. (Color online) A typical track of a particle in the magnetic field. The initial position of the particle is shown by an arrow. The ABC field is periodic over a cube of unit length in units plotted in this figure. To give an idea of scales, a cube with each side equal to 10 is shown in the figure.

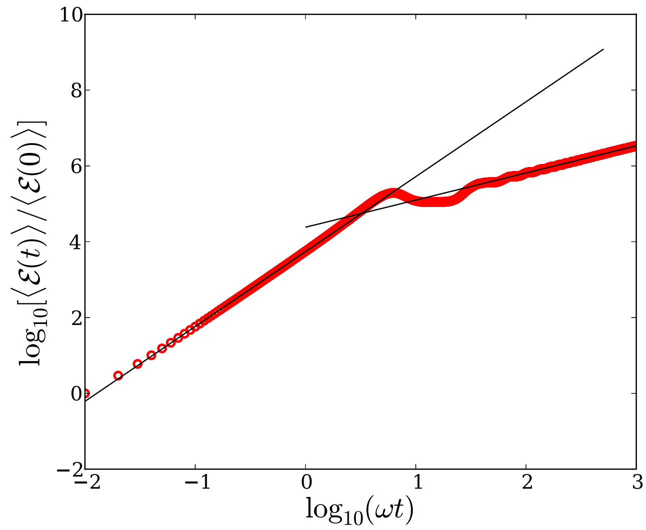

FIG. 4. (Color online) Mean energy of the particles as a function of time for run R2; see Table I. The two straight lines are fits (in the least-squares sense) with slopes 1.95 and 0.77 for small t and large t, respectively.

which is the largest Lyapunov exponent for that particular trajectory. Each trajectory is then labeled by the initial choice of position and velocity. For a set of random initial conditions, we have calculated the probability density function (PDF), $\mathcal{Q}(\Lambda)$. In Fig. 2 we plot $\mathcal{Q}(\Lambda)$ for two time differences. The two PDFs are quite close, and both of them are Gaussian with a positive mean. Hence, we conclude that at large times, $\mathcal{Q}(\Lambda)$ is Gaussian with a positive mean; i.e., the dynamical system (5) even with a time-independent magnetic field, which also implies zero electric field, has chaotic trajectories.

Now let us study the case with a time-periodic magnetic field. We limit ourselves to the case where ω/ω_c is very small and ω/k remains equal to unity. In other words, we consider a magnetic field that varies slowly in both space and time.

A typical path is shown in Fig. 3. Note that, although the magnetic field is periodic with wave number $k = 1$, the particle trajectories themselves are not periodic. To illustrate this in Fig. 3 we have also plotted a cube each side of which is 10 times the length scale over which the magnetic field is periodic. In Fig. 4 we show the growth of kinetic energy (per unit mass) averaged over the total number of particles,

$$\mathcal{E}(t) \equiv \tfrac{1}{2}\langle v^2 \rangle. \tag{11}$$

At short times, $\mathcal{E}(t)$ behaves as t^2, but at later time goes like t^ξ with an exponent ξ that is not universal but depends on ω;

TABLE I. The table shows how the exponent ξ depends on the frequency of the magnetic field ω. We study the limit where the magnetic field changes very slowly. For all the runs, $q/m = 1$, $\omega_c = 1$, and $\omega/k = 1$. As $\omega/\omega_c \to 0$, ξ approaches unity. The values of ξ for smaller ω have larger error as these runs have not run as long as the first two runs.

	R1	R2	R3	R4
ω	1/10	1/16	1/32	1/64
ξ	0.45	0.77	0.8	0.9

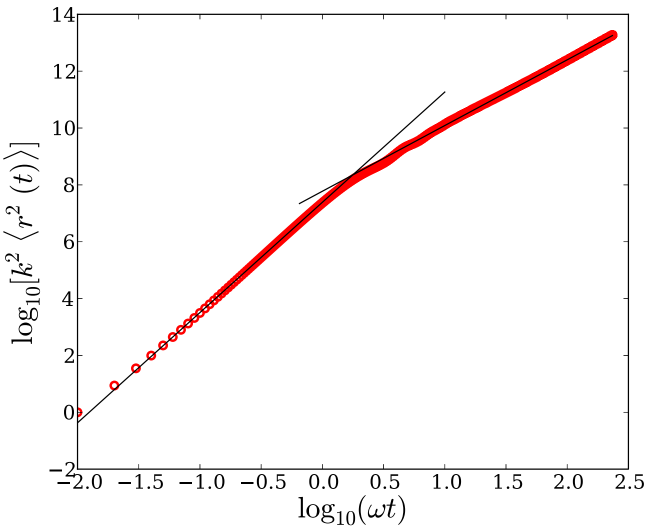

FIG. 5. (Color online) Plot of the mean-square displacement of the particles as a function of time, for run R2. The two straight lines are the fits (in the least-squares sense) with slopes 3.9 and 2.1 for small t and large t, respectively.

see Fig. 4 and Table I. The gyroradius of the charged particle grows with time as energy grows. The first stage of the growth, over which energy grows as t^2, continues until the gyroradius becomes of the same order as the characteristic scale of the ABC field. Figure 4 shows that the first stage of the growth, over which energy grows as t^2, continues until the gyroradius becomes of the same order as the characteristic scale of the ABC field. The growth at these short times can be understood by reminding ourselves that for $kr_c \ll 1$, a particle does not encounter significant spatial change in the magnetic field. As ω/ω_c is small, the particle effectively moves under a constant force (the electric field); hence its energy grows quadratically with time. At later times, the growth slows down to t^ξ where the exponent ξ for different values of ω is given in Table I. We find that, as $\omega/\omega_c \to 0$, ξ approaches unity. This shows that the process we observe can be interpreted as a subdiffusive process in momentum space. In real space, we simultaneously find that the mean-square displacement is proportional to t^4 for small times and goes as t^2 for large times. A representative plot of the mean-square displacement as a function of time is given in Fig. 5.

We further study the PDF of energies of particles with different random initial conditions; see inset of Fig. 6. The PDF of v_x is also plotted in log-linear scale in Fig. 6. For small values of its argument this PDF is well approximated by a Gaussian; i.e., the PDF of energy would be a Maxwellian, but for large values of its argument the PDF is sub-Gaussian.

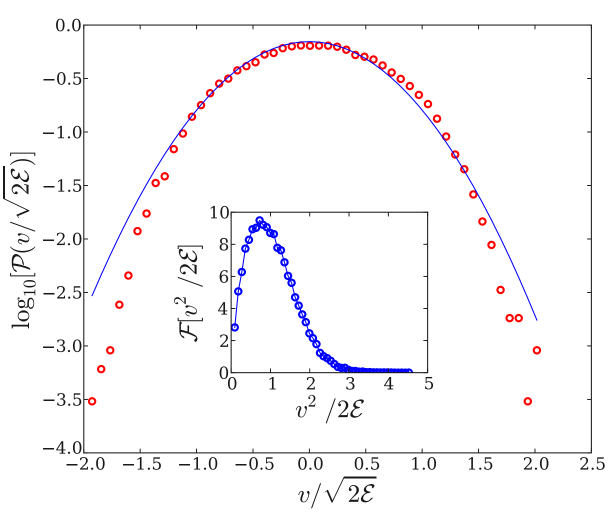

FIG. 6. (Color online) Plot of the log of PDF $\mathcal{P}$ of the x component of velocity of the particles at $\omega t = 100$. The abscissa is normalized by $\sqrt{2\mathcal{E}}$ at the same time. Figure 4 shows that by this time $\mathcal{E}$ has grown by more than five orders of magnitude compared to its initial value. The continuous line is the function (parabola), $f(x) = C - x^2/\sigma^2$ with $C = -0.155$ and $\sigma = 1.25$. The inset shows the PDF $\mathcal{F}$ of kinetic energy (v^2) at the same instant of time.

IV. CONCLUSION

In this paper we have shown that a helical magnetic field with sinusoidal spatiotemporal dependence can energize particles. The energization behavior is a power law in time and, given enough time, can energize the particles to very high energies where the relativistic effects start becoming important [24]. In particular, in our simulations we observe the mean energy of the particles to grow by six orders of magnitude. The chaotic nature of particle trajectories plays a crucial role in our model. If we change the ABC field such that $A = B = 0$ and $C = 1$ then we know that the magnetic lines of force are integrable and nonchaotic. In such a field, we do observe energization for short times, but at large times no systematic gain in energy is observed.

The Fermi model of acceleration of charged particle, often referred to as diffusive shock acceleration, is thought to be one of the primary mechanisms for energization of charged particles in the cosmos; see, e.g., Refs. [25–27] for a review. Fermi's theory also reproduces the experimental observation that the PDF of energies of cosmic rays has an inverse power-law tail. But diffusive shock acceleration of electrons can occur only if the initial energy of the electrons is at least of the order of a few MeV which is significantly higher than the thermal energies; this is the well-known *injection problem*.

The mechanism we propose is akin to second-order Fermi acceleration where a charged particle is energized due to collisions with random scatter centers moving with random velocities. The input "randomness" is a crucial ingredient of this process and one typically obtains diffusive properties in both real and momentum space [28]. The Fermi second-order process also produces a PDF of energies with power-law tail. By contrast, in our case the diffusive behavior is generated by deterministic chaos. We obtain subdiffusive behavior ($\xi < 1$) in momentum space, which becomes close to diffusive behavior as $\omega \to 0$. Furthermore, the PDF of energies at large times becomes Gaussian with steeper tails; i.e., the PDF can be characterized as Gaussian at low speed, but falls off more rapidly than a Gaussian at speeds in excess of the mean. On the positive side, our model is a possible mechanism that can generate a population of electrons with superthermal energies. They can now act as a resolution to the injection problem. What is truly remarkable in our model is that a deceptively simple magnetic field with a dynamics that is smooth in time is able to energize particles to indefinitely high energies.

The question of energization of a test (charged) particle in a turbulent plasma has been numerically studied in recent times; see, e.g., Refs. [19,20]. These studies consider energization in an electric field that is frozen in time but obtained from one snapshot of a direct numerical simulation of magnetohydrodynamic turbulence. In such a setup, test particles show diffusion in real space. Energization is also observed, and at large times energy seems to grow linearly with time. In addition, the PDF of energies obtained in Refs. [19,20] has power-law tails. However, it is not clear what the exponent of this power law is and how that emerges. Our simple model is able to capture the first two aspects, viz., the random walk and the energization but not the power-law tail. One of the contributions to the electric field in a turbulent plasma comes from the current. The square of the current is the resistive contribution to energy dissipation rates. The energy dissipation rates calculated from the solar wind data [29] show intermittent behavior. Such intermittency is absent in our simple model. This could be one of the reasons why we do not observe the power-law tail of the PDF of energy of the test particles. This brings us to the question, what are the minimal ingredients necessary to add to our model to obtain a power-law tail in the PDF of energy? This will be the subject of future investigations.

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[1] E. Fermi, Phys. Rev. **75**, 1169 (1949).
[2] S. Ulam, in *4th Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, Berkeley, 1961).
[3] G. M. Zaskavskii and B. V. Chirikov, Sov. Phys. Dokl. **9**, 989 (1965).
[4] H. Brahic, Astron. Astrophys. **12**, 98 (1971).
[5] M. A. Lieberman and A. J. Lichtenberg, Phys. Rev. A **5**, 1852 (1972).
[6] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).
[7] V. Gelfreich, V. Rom-Kedar, and D. Turaev, Chaos **22**, 033116 (2012).
[8] V. Gelfreich, V. Rom-Kedar, K. Shah, and D. Turaev, Phys. Rev. Lett. **106**, 074101 (2011).
[9] A. K. Ram and B. Dasgupta, Phys. Plasmas **17**, 122104 (2010).
[10] Of course, charged particles in a magnetic field do not move along magnetic field lines, and their motion may be chaotic even in magnetic fields where the lines of force are integrable.
[11] B. Dasgupta, G. Li, X. Li, and A. Ram, AIP Conf. Proc. **1500**, 56 (2012).
[12] A. Brandenburg and K. Subramanian, Phys. Rep. **417**, 1 (2005).
[13] E. Parker, Astrophys. J. **1122**, 293 (1955).
[14] F. Krause, K.-H. Rädler, and M. Steenbeck, Technical Report No. NCAR-TN/IA-60, National Center for Atmospheric Research, Boulder, Colorado (unpublished).
[15] A. Brandenburg, Astrophys. J. **550**, 824 (2001).
[16] P. Démoulin *et al.*, Astron. Astrophys. **382**, 650 (2002).
[17] A. Brandenburg, K. Subramanian, A. Balogh, and M. L. Goldstein, Astrophys. J. **734**, 9 (2011).
[18] T. Dombre *et al.*, J. Fluid Mech. **167**, 353 (1986).
[19] M. Onofri, H. Isliker, and L. Vlahos, Phys. Rev. Lett. **96**, 151102 (2006).
[20] P. Dmitruk, W. Matthaeus, and N. Seenu, Astrophys. J. **617**, 667 (2004).
[21] W. Press, B. Flannery, S. Teukolsky, and W. Vetterling, *Numerical Recipes in Fortran* (Cambridge University Press, Cambridge, 1992).
[22] For the case of stationary magnetic field, in which case energy should be conserved, the fractional change in energy over the whole period of integration is negligible, less than 10^{-9}.
[23] J. D. Hunter, Comput. Sci. Eng. **9**, 90 (2007).
[24] A relativistic generalization of our model is possible in order to deal with ultrarelativistic particles. This is beyond the scope of this paper.
[25] L. O. Drury, Rep. Prog. Phys. **46**, 973 (1983).
[26] J. Jokipii, Astrophys. Space Sci. **277**, 15 (2001).
[27] A. Balogh *et al.*, Space Sci. Rev. **173**, 1 (2012).
[28] F. Bouchet, F. Cecconi, and A. Vulpiani, Phys. Rev. Lett. **92**, 040601 (2004).
[29] So far, the energy dissipation rate has not been directly measured from observations of solar wind turbulence, but proxies, e.g., average of the cube of the velocity difference across a time scale, have been used; see, e.g., Sec. 7 of Ref. [30] for a review.
[30] E. Marsch and C.-Y. Tu, Nonlinear Processes in Geophysics **4**, 101 (1997).