Chandrasekhar–Kendall functions in astrophysical dynamos

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Abstract. Some of the contributions of Chandrasekhar to the field of magnetohydrodynamics are highlighted. Particular emphasis is placed on the Chandrasekhar–Kendall functions that allow a decomposition of a vector field into right- and left-handed contributions. Magnetic energy spectra of both contributions are shown for a new set of helically forced simulations at resolutions higher than what has been available so far. For a forcing function with positive helicity, these simulations show a forward cascade of the right-handed contributions to the magnetic field and nonlocal inverse transfer for the left-handed contributions. The speed of inverse transfer is shown to decrease with increasing value of the magnetic Reynolds number.

1. Introduction

During the 1950s, Chandrasekhar was deeply immersed in studying problems in hydrodynamics, especially magnetohydrodynamics (MHD), which is the study of flow of electrically conducting media such as liquid metals and ionized gases or plasmas. Between 1950 and 1960, roughly 2/3 of his papers were related to hydrodynamic and magnetohydrodynamics stability, turbulence, and aspects of dynamo theory. Chandra's scientific activities during that period are reviewed in detail by Parker [1], who draws a close connection between Chandra's interests in Heisenberg's theory of turbulence [2] and the then discovered polarization of starlight in the galaxy [3,4]. This led to a collaboration between Chandrasekhar and Fermi [5], where they estimated the galactic field strength to be 6–7 μG, which is close to the commonly accepted value today [6]. Parker also suggests that observations of magnetic fields in the Sun [7] and other stars [8] may have contributed to Chandra's interest in magnetohydrodynamic stability and perhaps dynamo theory.

Chandrasekhar's name is not normally connected with dynamo theory. Indeed, at the time there was still great disbelief in dynamo theory, given that Cowling [9] and others, including Chandra himself [10], proved dynamos impossible under the assumption of axisymmetry of the magnetic field. After all, Chandra's work in that field was concentrated to the period just before Herzenberg [11] produced the first existence proof of dynamos.

Therefore, Chandrasekhar kept his mind open for alternative explanations. In 1956 this led him to the question [12] "Can the time of decay of 14,000 years, in the absence of internal motions, be prolonged to 500,000 years (say) by velocities of reasonable magnitude and patterns?", which he answered tentatively with "Yes". He discussed this finding in connection with possible solutions to the problem of the geomagnetic field and even the field observed in sunspots. Now we know that the physically correct and relevant understanding, at least for the Sun, came from Parker's work [13] of 1955. However, the special significance of Parker's early work was perhaps not yet fully appreciated when Chandrasekhar [12] in 1956 merely noted that "the possibility, nevertheless, of dynamo action has been explored, intensively, in recent years by Bullard and Gellman [14], Parker [13], and by Elsasser [15,16]."

Chandrasekhar's most cited publication is his book *Hydrodynamic and hydromagnetic stability* [17] which, in 2010, received about 140 citations on ADS. The citation rate has been increasing and was about 70 citations some 10 years earlier. He studied the stability of virtually any imaginable combination of systems in slab and spherical geometries, with and without rotation, with and without magnetic fields, as well as systems with self-gravity. In astrophysics, hydrodynamic and hydromagnetic instabilities are important sources of turbulence. Unlike atmospheric and wind tunnel turbulence, which is typically driven through boundary layer instabilities, turbulence in astrophysical settings tends to be driven by instabilities within the volume. Important examples include Rayleigh–Bénard convection as well as the magneto-rotational instability. Chandrasekhar has contributed to both types of instabilities. It is now well accepted that at large enough Reynolds numbers, the resulting flows tend to become turbulent and are invariably subject to yet another instability, the dynamo instability. Of particular significance is the case of large-scale dynamo action. All known mechanisms produce helical large-scale magnetic fields that tend to be nearly force-free. This leads to yet another of Chandra's favourite topics at the time, namely the study of such fields in spherical geometry in terms of what is now known as Chandrasekhar–Kendall functions [18]. In the present paper we discuss the significance of these functions for understanding the dynamics of magnetic fields exhibiting inverse transfer behaviour from small to large length scales. This mechanism is significant not only in stars [19,20], accretion discs [21], and galaxies [22], but possibly even on the scale of the Universe in connection with primordial magnetic fields [23,24].

2. Chandrasekhar–Kendall functions and application to Cartesian domains

When it was realized that astrophysical magnetic fields might often be force-free [25], i.e. $(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0}$, Chandrasekhar began studying the mathematical properties of such fields both in cylindrical and spherical coordinates [26]. Such fields are eigenfunctions of the curl operator, i.e.,

$$\nabla \times \mathbf{h}_n = k_n \mathbf{h}_n, \tag{1}$$

where k_n are eigenvalues and $\mathbf{h}_n$ are eigenfunctions. These functions play important roles as basis functions for solenoidal vector fields as well as for decomposing them into right- and left-handed components. This led to an important joint paper with Kendall [18], whose supervisor (V C A Ferraro) remarked in a footnote of their paper that "The results in

this paper were derived independently by the two authors; and they agreed to write it as one". Their work gained importance since the 1970s, when force-free magnetic fields were observed in the solar corona [27] and produced in laboratory plasmas [28].

Chandrasekhar–Kendall or CK functions continue to be used in solar physics, for example in describing the dynamics of coronal loops [29]. These functions have subsequently been exploited also in other geometries, including cylindrical geometries [30], where they are also referred to as Lundquist fields [31]. In the plasma physics community, these functions are sometimes also referred to as Chandrasekhar–Kendall–Furth or CKF functions [32]. In the following, however, we shall focus on the Cartesian domains with periodic boundaries [33], where such functions are also known as Beltrami waves.

In periodic Cartesian domains it is convenient to adopt CK functions in Fourier space. For example, the magnetic vector potential can be written in Fourier space (indicated by subscript $\mathbf{k}$) as

$$\mathbf{A_k}(t) = \int \mathbf{A}(\mathbf{x}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{x}, \tag{2}$$

and it can be expanded in terms of eigenfunctions of the curl operator with positive and negative eigenvalues, as well as a longitudinal part that vanishes upon taking the curl to compute the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Thus, we write [24,33,34]

$$\mathbf{A_k}(t) = a_{\mathbf{k}}^{+}(t)\mathbf{h}_{\mathbf{k}}^{+} + a_{\mathbf{k}}^{-}(t)\mathbf{h}_{\mathbf{k}}^{-} + a_{\mathbf{k}}^{\parallel}(t)\mathbf{h}_{\mathbf{k}}^{\parallel}, \tag{3}$$

with

$$i\mathbf{k} \times \mathbf{h}_{\mathbf{k}}^{\pm} = \pm\, k\mathbf{h}_{\mathbf{k}}^{\pm}, \qquad k = |\mathbf{k}|, \tag{4}$$

where $\mathbf{h}_{\mathbf{k}}^{\pm}$ are the eigenfunctions with positive and negative eigenvalues, respectively. These functions are orthonormal, i.e.,

$$\left\langle \mathbf{h}_{\mathbf{k}}^{+*} \cdot \mathbf{h}_{\mathbf{k}}^{-} \right\rangle = \left\langle \mathbf{h}_{\mathbf{k}}^{+*} \cdot \mathbf{h}_{\mathbf{k}}^{\parallel} \right\rangle = \left\langle \mathbf{h}_{\mathbf{k}}^{-*} \cdot \mathbf{h}_{\mathbf{k}}^{\parallel} \right\rangle = 0 \tag{5}$$

and

$$\left\langle \mathbf{h}_{\mathbf{k}}^{+*} \cdot \mathbf{h}_{\mathbf{k}}^{+} \right\rangle = \left\langle \mathbf{h}_{\mathbf{k}}^{-*} \cdot \mathbf{h}_{\mathbf{k}}^{-} \right\rangle = \left\langle \mathbf{h}_{\mathbf{k}}^{\parallel *} \cdot \mathbf{h}_{\mathbf{k}}^{\parallel} \right\rangle = 1, \tag{6}$$

where asterisks denote the complex conjugate and angular brackets denote volume averages. The longitudinal part $a_{\mathbf{k}}^{\parallel}\mathbf{h}_{\mathbf{k}}^{\parallel}$ is parallel to $\mathbf{k}$ and vanishes after taking the curl (or crossing with $i\mathbf{k}$) to calculate the magnetic field, i.e., we have

$$\mathbf{B_k} = k\left(a_{\mathbf{k}}^{+}\mathbf{h}_{\mathbf{k}}^{+} - a_{\mathbf{k}}^{-}\mathbf{h}_{\mathbf{k}}^{-}\right). \tag{7}$$

Magnetic helicity and magnetic energy spectra are then given by

$$\mathbf{A_k} \cdot \mathbf{B_k^*} = k\left[\left(a_{\mathbf{k}}^{+}\right)^2 - \left(a_{\mathbf{k}}^{-}\right)^2\right], \tag{8}$$

$$\mathbf{B_k} \cdot \mathbf{B_k^*} = k^2\left[\left(a_{\mathbf{k}}^{+}\right)^2 + \left(a_{\mathbf{k}}^{-}\right)^2\right]. \tag{9}$$

This implies evidently

$$\mathbf{B_k} \cdot \mathbf{B_k^*} \geq k\left|\mathbf{A_k} \cdot \mathbf{B_k^*}\right|, \tag{10}$$

which is also known as the realizability condition [35], that is often quoted in terms of shell-integrated spectra of magnetic energy and magnetic helicity, M_k and H_k, respectively, i.e.,

$$M_k \geq k|H_k|/2\mu_0. \tag{11}$$

Here, μ_0 is the vacuum permeability and the $2\mu_0$ factor comes from the normalization of the magnetic energy spectrum, $\int M_k \, dk = \langle \mathbf{B}^2/2\mu_0 \rangle$, while the magnetic helicity spectrum is normalized such that $\int H_k \, dk = \langle \mathbf{A} \cdot \mathbf{B} \rangle$.

To monitor the chiral dynamics of a system, it is convenient to plot energy spectra of the contributions from right- and left-handedly polarized parts. Fortunately, this can be done without ever performing the explicit decomposition (7) in terms of $a_{\mathbf{k}}^{\pm}$.

3. Application to the inverse cascade

Eigenfunctions of the curl operator are well suited as function basis for hydrodynamic turbulence [33]. More recently they have also been used in magnetohydrodynamics [24], and in dynamo theory [34], where flows allow a conversion of kinetic energy into magnetic energy.

Dynamos are possible under completely isotropic conditions without helicity. In that case we often speak about small-scale dynamos (see ref. [36] for a review). However, in the presence of helicity, there is an inverse cascade of magnetic helicity that leads to the gradual build-up of magnetic fields at the largest scale of the system [19,20]. This has been demonstrated in real space with turbulence driven by an ABC-flow forcing function [37] and, more clearly, with turbulence driven by a forcing function proportional to CK functions that are δ-correlated in time [38]. In the latter case, the inverse cascade process is clearly demonstrated in terms of CK-decomposed spectra, $M_k^{\pm}$, as was done in ref. [34]. They used a relatively big scale separation ratio, i.e., their forcing wavenumber k_f was large compared to the smallest wavenumber of the domain, k_1 (they used $k_f/k_1 = 27$). This implies, however, that the magnetic Reynolds number

$$R_m = u_{rms}/\eta k_f, \tag{12}$$

is smaller than if the scale separation was less. It is therefore useful to increase the numerical resolution as much as possible. In this paper we present new simulations of helically driven isotropic turbulence in a periodic domain at a resolution of 512^3 mesh points and scale separation ratios of $k_f/k_1 = 15$ and 30. The resulting magnetic Reynolds number reaches values up to $R_m \approx 57$ and the magnetic Prandtl number is unity. The simulations have been performed with the PENCIL CODE [38a], which is a modular high-order code (sixth order in space and third order in time, by default) for solving a large range of partial differential equations.

In figure 1 we show the x component of the magnetic field at the periphery of the domain at two early times in the simulation. Note, that the magnetic field appears to organize itself gradually into larger scale patches. This is more clearly seen in energy spectra (see figure 2), where we show the spectral magnetic energy at different times. At early times the magnetic energy resides almost entirely at the dissipative scale. However, as time goes on, a magnetic field component at intermediate scale with wavenumber $k_m \approx k_f/2$ becomes discernible.
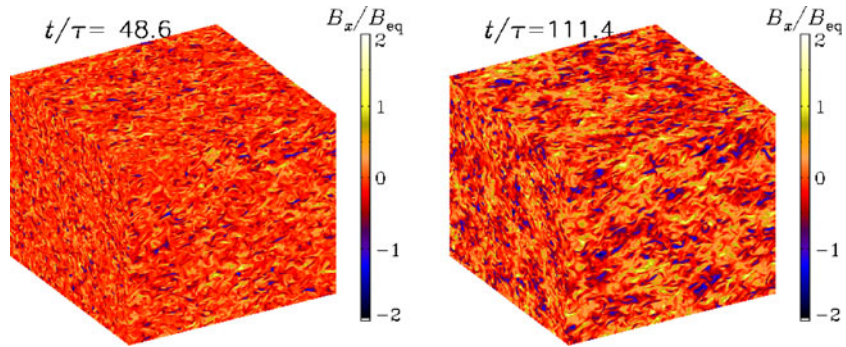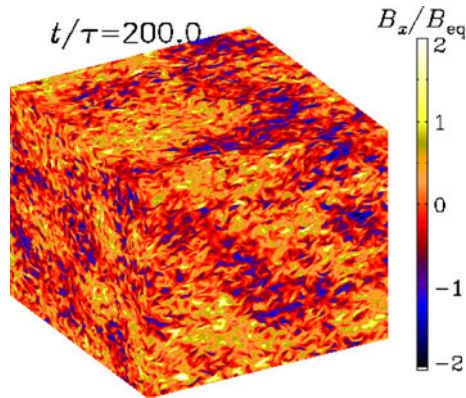
Figure 1. Visualizations of B_x/B_{eq} on the periphery of the domain at three times when the small-scale magnetic field was already saturated. Here, $B_{eq} = \sqrt{\mu_0 \rho_0}\, u_{rms}$ is the equipartition field strength where kinetic and magnetic energy densities are comparable, and ρ_0 is the mean density. Note that the maximum field strength is about twice B_{eq}.

This corresponds to the fastest growing eigenmode of a purely helical large-scale dynamo [34]. Note also that near the largest scale of the system, the magnetic field has a power spectrum comparable with k^4, which corresponds to white noise in the magnetic vector potential and obeys causality [39]. To the right of the injection wavenumber, k_f, there is a short range of a k^{-1} spectrum in M_k^-.

At later times, magnetic energy is transported further to smaller wavenumbers (see figure 2b), where the peak of the spectrum at $k \equiv k_m$ travels to smaller wavenumbers. The effect of this is also evident in real space in that larger scale patches of magnetic field
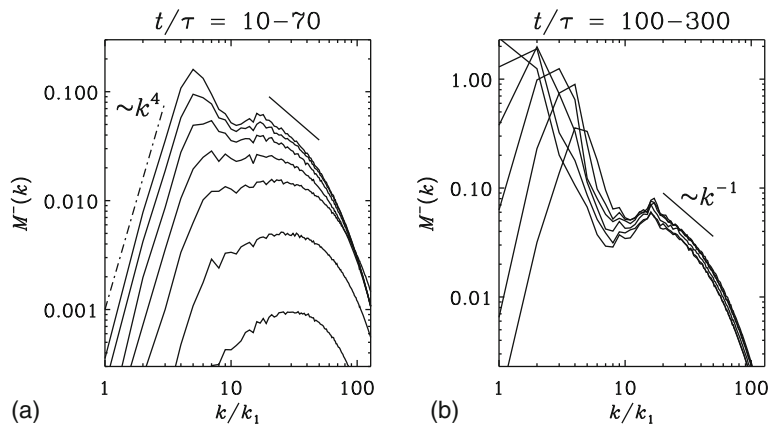
Figure 2. Spectra of magnetic energy of the negatively polarized contributions, M_k^-, at (**a**) earlier and (**b**) later times. The scale separation ratio $k_f/k_1 = 15$. The range of time t is given in units of the turn-over time, $\tau = 1/u_{rms}k_f$. At small wavenumbers, the M_k^- spectrum is proportional to k^4, while to the right of $k_f/k_1 = 15$ there is a short range with a k^{-1} spectrum.

Figure 3. Visualizations of B_x/B_{eq} on the periphery of the domain at a later time when a large-scale magnetic field is beginning to emerge. This time corresponds to the intermediate time shown in the second panel of figure 2 where the peak of the spectrum has reached the wavenumber $k_m/k_1 \approx 3$, corresponding to about three large-scale patches across the domain. The scale separation ratio $k_f/k_1 = 15$, and so there are about 15 eddies across the domain.

with the same orientation occur. In figure 3 we show the x component of the magnetic field at the time when a large-scale magnetic field has emerged, but it has still not reached the scale of the computational domain. Magnetic energy spectra of that time are shown in figure 4a. Note, however, that the magnetic energy does show a clearly pronounced secondary peak that is now at wavenumber $k_m/k_1 \approx 2$.

The process that produces magnetic fields at large length scales shown in the present paper can loosely be regarded as an inverse cascade of magnetic helicity and thereby magnetic field. However, it can also be treated in terms of mean-field theory, in which case it corresponds to the so-called α effect, whose physics was already identified by Parker [13]. This dynamo is quite different from the laminar dynamos that were studied at the time by Bullard and Gellman [14], although there remained substantial uncertainty about the convergence of the numerical solutions obtained back then.

One of the surprising results of the last 10 years is the fact that in a closed or periodic domain, the large-scale magnetic field saturates only on a resistive time-scale [38]. This is understood as a consequence of the magnetic helicity equation, which limits the time-scale for magnetic helicity evolution to the resistive time [40], and lowers the nonlinear α effect in an R_m-dependent fashion [41–44]. According to this picture, if we associate, as was done in ref. [38], the peak wavenumber k_m of the inversely cascading field with the α effect via $k_m = \alpha/2\eta_t$, where η_t is the turbulent magnetic diffusivity, we should expect the speed of inverse transfer to slow down as well. When this was first proposed [34], the results were not fully conclusive and of course in conflict with the usual understanding that the inverse cascade should operate on a dynamical time-scale [45]. Our present results allow us now a fresh look at this problem. As a continuous measure of k_m we define

$$k_m^{-n} = \int k^{-n} M_k \, dk \bigg/ \int M_k \, dk, \qquad (13)$$
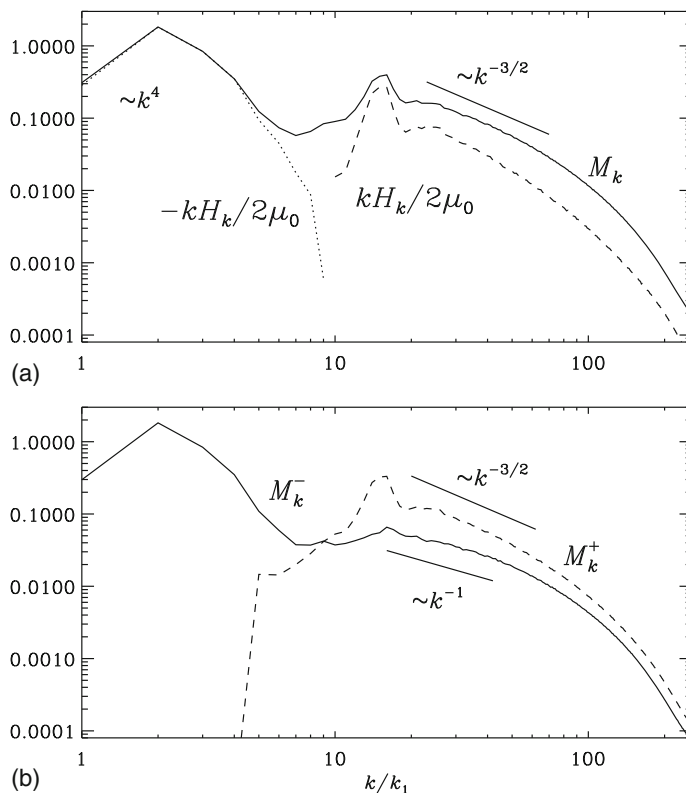
Figure 4. (a) Spectra of magnetic energy, M_k, and rescaled magnetic helicity, $\pm k H_k / 2\mu_0$. (b) Spectra of magnetic energy of positively and negatively polarized parts, $M_k^{\pm}$.

where we choose $n = 3$, which was found to give the best approximation to the actual location of the secondary peak in M_k. This is probably related to the sharp rise of M_k proportional to k^4 for $k_1 < k < k_{\mathrm{m}}$. Following the earlier work [34], the evolution of $k_{\mathrm{m}}(t)$ is well described by the fit formula

$$k_{\mathrm{m}}^{-1} = \alpha_{\mathrm{trav}}(t - t_{\mathrm{sat}}), \tag{14}$$

where the parameter α_{trav} characterizes the speed at which k_{m} travels toward smaller wavenumbers. This parameter is not directly related to the α effect, although α_{trav} also has the dimension of velocity. The result is shown in figure 5, where we plot k_{m}^{-1} vs. time for different values of R_{m} and k_{f}/k_1, and we also show the dependence of the slope α_{trav} on R_{m}. In agreement with earlier work [34], α_{trav} is clearly R_{m}-dependent and decreases with increasing R_{m} roughly proportional to $R_{\mathrm{m}}^{-1/2}$. The reason for this seems clear: the bump in the spectrum is a manifestation of an α effect causing nonlocal spectral transfer to smaller k, but it is not an inverse cascade in the usual sense. An inverse cascade may still occur at intermediate wavenumbers $k_{\mathrm{m}} \ll k \ll k_{\mathrm{f}}$, where evidence for constant (k-independent) spectral magnetic helicity transfer to larger scale has been found [46].
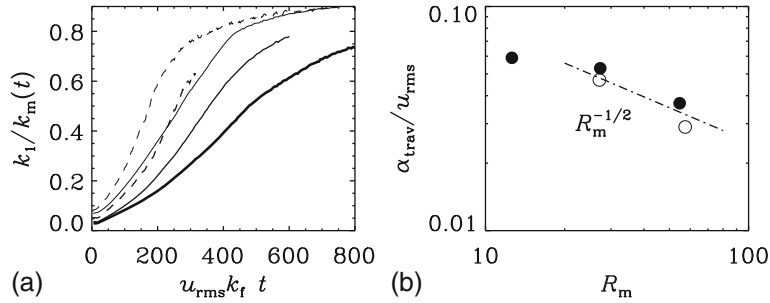
Figure 5. (**a**) Time dependence of the peak wavenumber for scale separation ratios of 15 (dashed) and 30 (solid lines) at $R_m = 12$, 27, and around 57 (increasing line thickness). (**b**) R_m dependence of the cascade speed for scale separation ratios of 15 (open circles) and 30 (filled circles). The straight lines give the $R_m^{-1/2}$ (dotted) and R_m^{-1} (dashed) dependences.

4. Closing the loop: Turbulence from the magnetic field itself

Large-scale dynamos show mean magnetic fields that can either be statistically steady, or can oscillate and propagate in space. In the language of mean-field dynamos, the former are typically the so-called α^2 dynamos while the latter are the $\alpha\Omega$ dynamos, where the Ω effect corresponds to differential rotation or shear amplifying the toroidal magnetic field from a poloidal one. A prominent example of such a dynamo is the one believed to exist in accretion discs, where shear plays a strong role [21]. This brings us to our final example where Chandrasekhar's contributions are still being mentioned in modern astrophysics, in connection with the magneto-rotational instability.

Already in 1953, Chandrasekhar studied the hydromagnetic stability of Couette flows, e.g. the flow of a liquid metal between two rotating cylinders with angular velocities Ω_1 and Ω_2 at radii R_1 and $R_2 > R_1$. In view of astrophysical applications to the stability of accretion discs, particularly interesting is the case where angular velocity decreases with radius, i.e. $\Omega_1 > \Omega_2$, but angular momentum increases, i.e. $R_1^2\Omega_1 < R_2^2\Omega_2$. Under the assumption of a small magnetic Prandtl number, where the viscosity is much smaller than the magnetic diffusivity, and the gap width is small, Chandrasekhar [47] found that the flow is stable both with and without an axial magnetic field. However, relaxing the assumption of a small gap width, Velikhov [48] found that for an ideal fluid (zero viscosity and zero magnetic diffusivity) the flow is unstable when $\Omega_1 > \Omega_2$. This was also confirmed by Chandrasekhar [49], but the astrophysical significance was hardly appreciated until Balbus and Hawley [50] rediscovered this instability, which is now usually called the magneto-rotational instability (MRI).

The point is that both Velikhov and Chandrasekhar were very much ahead of their time, because accretion discs entered the astrophysical literature only in the late 1960s and early 1970s. Nevertheless, it is important to realize that in the meantime the MRI was not completely forgotten. In his book, Safronov [51] discussed several mechanisms for causing turbulence in protostellar discs. Among the various mechanisms, he did discuss the MRI, but discarded it on the grounds that protostellar discs are cold and poorly ionized, and so magnetic fields would not be tied to the gas. His work was before the famous paper of

Shakura and Sunyaev [52], where the physics of accretion discs was applied to the much hotter discs around stellar-mass and supermassive black holes. Curiously enough, until then the focus has rather been on stability, as is evidenced by a quote from S H Hong [53] who wrote "A special form of this problem was first investigated by Chandrasekhar [47], and later by Velikhov in [48]. On the basis of the small gap approximation, they found that the effect of a sufficiently strong magnetic field is to inhibit the onset of instability."

The MRI is now generally believed to be the main agent that drives turbulence in accretion discs, and this turbulence can reinforce the magnetic field by dynamo action via a doubly-positive feedback [21,54–56]. These discoveries would not have been possible without the help of supercomputers. However, coming back to Chandrasekhar, it is fair to say that he always had an eye towards numerical solutions. For example, his approach to the dynamo problem in 1956, where he found a significant slow-down of the decay [12], was also entirely based on numerical solutions. It is therefore clear that his combined approach using numerical and analytical tools was not only a modern one, but much of his work attracts interest still today, as is evidenced by the increasing citation rate of his book titled *Hydrodynamic and hydromagnetic stability* [17].

Acknowledgements

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