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Mean electromotive force proportional to mean flow in MHD turbulence

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In mean-field magnetohydrodynamics the mean electromotive force due to velocity and magnetic-field fluctuations plays a crucial role. In general it consists of two parts, one independent of and another one proportional to the mean magnetic field. The first part may be nonzero only in the presence of mhd turbulence, maintained, e.g., by small-scale dynamo action. It corresponds to a battery, which lets a mean magnetic field grow from zero to a finite value. The second part, which covers, e.g., the α effect, is important for large-scale dynamos. Only a few examples of the aforementioned first part of the mean electromotive force have been discussed so far. It is shown that a mean electromotive force proportional to the mean fluid velocity, but independent of the mean magnetic field, may occur in an originally homogeneous isotropic mhd turbulence if there are nonzero correlations of velocity and electric current fluctuations or, what is equivalent, of vorticity and magnetic field fluctuations. This goes beyond the Yoshizawa effect, which consists in the occurrence of mean electromotive forces proportional to the mean vorticity or to the angular velocity defining the Coriolis force in a rotating frame and depends on the cross-helicity defined by the velocity and magnetic field fluctuations. Contributions to the mean electromotive force due to inhomogeneity of the turbulence are also considered. Possible consequences of the above findings for the generation of magnetic fields in cosmic bodies are discussed.

1 Introduction

Mean-field magnetohydrodynamics has proved to be a useful tool for studying the behavior of mean magnetic fields in turbulently moving electrically conducting fluids (see, e.g., Moffatt 1979; Krause & Rädler 1980; Brandenburg & Subramanian 2005). Within this framework both the magnetic field $\boldsymbol{B}$ and the fluid velocity $\boldsymbol{U}$ are split into mean parts, $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{U}}$, and fluctuating parts, $\boldsymbol{b}$ and $\boldsymbol{u}$. Starting from the induction equation governing $\boldsymbol{B}$ it is concluded that the mean magnetic field $\overline{\boldsymbol{B}}$ has to obey

$$\partial_t \overline{\boldsymbol{B}} = \eta \nabla^2 \overline{\boldsymbol{B}} + \nabla \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}}) , \quad \nabla \cdot \overline{\boldsymbol{B}} = 0 . \quad (1)$$

Here, η means the magnetic diffusivity of the fluid, for simplicity considered as independent of position, and $\boldsymbol{\mathcal{E}}$ the mean electromotive force caused by the velocity and magnetic fluctuations,

$$\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle . \quad (2)$$

Mean fields are defined by some kind of averaging satisfying the Reynolds rules. They are denoted either by overbars or synonymously by angle brackets.

The induction equation governing $\boldsymbol{B}$ also implies

$$\partial_t \boldsymbol{b} = \eta \nabla^2 \boldsymbol{b} + \nabla \times [\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \overline{\boldsymbol{B}} + (\boldsymbol{u} \times \boldsymbol{b})'], \quad \nabla \cdot \boldsymbol{b} = 0 , \quad (3)$$

where $(\boldsymbol{u} \times \boldsymbol{b})' = \boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$. With this in mind we may conclude that $\boldsymbol{\mathcal{E}}$ can be represented as a sum of two parts,

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(\overline{B})} , \quad (4)$$

where $\boldsymbol{\mathcal{E}}^{(0)}$ is a functional of $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$, and $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$, which is linear in $\overline{\boldsymbol{B}}$ but vanishes if $\overline{\boldsymbol{B}}$ is zero everywhere and at all past times (see, e.g., Rädler 1976, 2000; Rädler & Rheinhardt 2007). These statements apply irrespectively of whether or not $\boldsymbol{u}$ or $\overline{\boldsymbol{U}}$ depend on $\overline{\boldsymbol{B}}$. If they depend on $\overline{\boldsymbol{B}}$ and the total variation of $\boldsymbol{\mathcal{E}}$ with $\overline{\boldsymbol{B}}$ is considered, $\boldsymbol{\mathcal{E}}^{(0)}$ may well vary with $\overline{\boldsymbol{B}}$, and $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ need not be linear in $\overline{\boldsymbol{B}}$.

A non-zero $\boldsymbol{\mathcal{E}}^{(0)}$ corresponds to a battery. Assume for a moment that equation (1) for $\overline{\boldsymbol{B}}$ with $\boldsymbol{\mathcal{E}} = \boldsymbol{0}$ has no growing solutions. If then $\boldsymbol{\mathcal{E}}^{(0)}$ takes non-zero values, but $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ remains equal to zero, $\overline{\boldsymbol{B}}$ grows, even if initially equal to zero, to a finite magnitude determined by $\boldsymbol{\mathcal{E}}^{(0)}$. If, on the other hand, $\boldsymbol{\mathcal{E}}^{(0)}$ remains equal to zero a non-zero $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ may allow (if it has a suitable structure) a dynamo, that is, let an arbitrarily small seed magnetic field $\overline{\boldsymbol{B}}$ grow exponentially (in the absence of back-reaction on the fluid motion even endlessly). A small non-zero $\boldsymbol{\mathcal{E}}^{(0)}$ may deliver a seed field for such a dynamo. This possibility has been already discussed in the context of young galaxies (Brandenburg & Urpin 1998).

In most of the general representations and applications of mean-field magnetohydrodynamics the part $\boldsymbol{\mathcal{E}}^{(0)}$ of the

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electromotive force $\boldsymbol{\mathcal{E}}$ has been ignored. Indeed, if it occurs at all, it decays to zero in the course of time except in cases in which an independent magnetohydrodynamic turbulence exists, e.g., as a result of a small-scale dynamo.

The possibility of a non-zero $\boldsymbol{\mathcal{E}}^{(0)}$ due to local, that is, small-scale dynamos in the solar convection zone has been discussed by Rädler (1976). We express his statements here by

$$\boldsymbol{\mathcal{E}}^{(0)} = c_\gamma \boldsymbol{\gamma} + c_\Omega \boldsymbol{\Omega} + c_{\gamma\Omega} \boldsymbol{\gamma} \times \boldsymbol{\Omega} \,. \tag{5}$$

The vector $\boldsymbol{\gamma}$ was interpreted as a gradient, e.g., of the turbulence intensity. $\boldsymbol{\Omega}$ is the angular velocity responsible for the Coriolis force, and c_γ, c_Ω and $c_{\gamma\Omega}$ are some coefficients. More precisely, c_γ and $c_{\gamma\Omega}$ are scalars and c_Ω is a pseudoscalar.

Another interesting result has been derived by Yoshizawa (1990, see also Yoshizawa 1993 or Yoshizawa, Itoh & Itoh 2003). Considering originally homogeneous isotropic magnetohydrodynamic turbulence under the influence of a mean flow or a rigid-body rotation, or both, he found

$$\boldsymbol{\mathcal{E}}^{(0)} = c_W \boldsymbol{W} + c_\Omega \boldsymbol{\Omega} \,, \tag{6}$$

where $\boldsymbol{W} = \boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$ is the mean vorticity, $\boldsymbol{\Omega}$ again the angular velocity responsible for the Coriolis force, and c_W and c_Ω are pseudoscalars coefficients which are, roughly speaking, proportional to the cross-helicity $\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$. This result has recently been used for an interpretation of the Archontis dynamo (Sur & Brandenburg 2009).

The main purpose of this paper is to demonstrate that a mean electromotive force $\boldsymbol{\mathcal{E}}^{(0)}$ proportional to the mean fluid velocity $\overline{\boldsymbol{U}}$ may occur in originally homogeneous isotropic magnetohydrodynamic turbulence. Of course, an electromotive force of that kind is only possible if the correlation properties of the turbulence in a given frame of reference differ from those in another frame moving uniformly relative to that, in other words, if no Galilean invariance of these properties exists. In this case this electromotive force occurs as soon as there is a non-zero correlation between the fluctuating parts of velocity and electric current, $\boldsymbol{u}$ and $\boldsymbol{j} = \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{b}$, or, what is equivalent, between the fluctuating parts of vorticity and magnetic field, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{b}$. Here, μ_0 is the magnetic permeability of free space. We express this condition roughly by saying that $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ or $\langle \boldsymbol{\omega} \cdot \boldsymbol{b} \rangle$ have to be unequal to zero. Unlike $\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$, which characterizes the linkage between vortex tubes and magnetic flux tubes, $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ quantifies the linkage between vortex tubes and current tubes.

In Sect. 2 we explain the basis of our calculations and provide general relations for the determination of the mean electromotive force $\boldsymbol{\mathcal{E}}^{(0)}$. In Sect. 3 we derive results for homogeneous isotropic turbulence, in particular the last-mentioned one, and we also reproduce that given by (6). Proceeding then in Sect. 4 to inhomogeneous turbulence we report on results related to those indicated in (5). The relevance of the results obtained in this paper and the need of further work are discussed in Sect. 5.

2 General concept

2.1 Basic equations

We consider a magnetic field $\boldsymbol{B}$ in a homogeneous incompressible electrically conducting turbulent fluid in a rotating frame. It is assumed that $\boldsymbol{B}$ and the fluid velocity $\boldsymbol{U}$ are governed by

$$\partial_t \boldsymbol{B} = \eta \boldsymbol{\nabla}^2 \boldsymbol{B} + \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B} + \boldsymbol{H}) \,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{7}$$

$$\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{U} = -\varrho^{-1} \boldsymbol{\nabla} P + \nu \boldsymbol{\nabla}^2 \boldsymbol{U} - 2\,\boldsymbol{\Omega} \times \boldsymbol{U} + \boldsymbol{F} \,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{U} = 0 \,, \tag{8}$$

where η is again the magnetic diffusivity, ν the kinematic viscosity and ϱ the mass density of the fluid. P is the sum of hydrostatic and centrifugal pressure, and $\boldsymbol{\Omega}$ the angular velocity defining the Coriolis force. The external electromotive force $\boldsymbol{H}$ and the external ponderomotive force $\boldsymbol{F}$ will allow us to mimic magnetohydrodynamic turbulence. For the sake of simplicity we have ignored the back-reaction of the magnetic field on the fluid motion.

Adopting the mean-field concept and taking averages of Eqs. (7) and (8) we arrive at equations for the mean fields $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{U}}$. The equations for $\overline{\boldsymbol{B}}$ differ from (1) only in so far as instead of $\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}}$ the sum $\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{H}}$ occurs. For the magnetic and velocity fluctuations $\boldsymbol{b}$ and $\boldsymbol{u}$ we further may derive

$$\partial_t \boldsymbol{b} = \eta \boldsymbol{\nabla}^2 \boldsymbol{b} + \boldsymbol{\nabla} \times [\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \overline{\boldsymbol{B}} + (\boldsymbol{u} \times \boldsymbol{b})' + \boldsymbol{h}] \,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0 \tag{9}$$

$$\partial_t \boldsymbol{u} = -\varrho^{-1} \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} - \overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{U}} - 2\,\boldsymbol{\Omega} \times \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})' + \boldsymbol{f} \,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{10}$$

where again $(\boldsymbol{u} \times \boldsymbol{b})'$ stands for $\boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$, analogously $(\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})'$ for $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \langle \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \rangle$, and $\boldsymbol{h}$, p and $\boldsymbol{f}$ are the fluctuating parts of $\boldsymbol{H}$, P and $\boldsymbol{F}$. The Eqs. (9) for $\boldsymbol{b}$ differ from (3) only by the additional electromotive force $\boldsymbol{h}$. In view of a later discussion we give Eqs. (10) also in the slightly different but equivalent form

$$\partial_t \boldsymbol{u} = -\varrho^{-1} \boldsymbol{\nabla} (p + \varrho (\overline{\boldsymbol{U}} \cdot \boldsymbol{u})) + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} - (2\,\boldsymbol{\Omega} + \boldsymbol{W}) \times \boldsymbol{u} + \overline{\boldsymbol{U}} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) + (\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}))' + \boldsymbol{f} \,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \tag{11}$$

where $\boldsymbol{W}$ stands again for $\boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$.

We strive to calculate the part $\boldsymbol{\mathcal{E}}^{(0)}$ of the mean electromotive force. So we put simply $\overline{\boldsymbol{B}} = \boldsymbol{0}$ in (9). This is not germane to the following considerations and could always be justified by choosing a suitable $\overline{\boldsymbol{H}}$. Basically, the so modified Eqs. (9) and (10) with a given $\overline{\boldsymbol{U}}$ imply the possibility of a small-scale dynamo, that is, of non-decaying $\boldsymbol{b}$, even if $\boldsymbol{h}$ is equal to zero. In what follows we introduce however some further simplifying assumptions which undermine this possibility, and we mimic a small-scale dynamo with a proper non-zero $\boldsymbol{h}$.

Let us assume that $\boldsymbol{u}$ and $\boldsymbol{b}$ depend only weakly on $\overline{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$ so that $\boldsymbol{\mathcal{E}}^{(0)}$ is linear in these quantities. We further assume that $\overline{\boldsymbol{U}}$ varies only weakly in space and time so that

$\boldsymbol{\mathcal{E}}^{(0)}$ in a given point depend only on $\overline{\boldsymbol{U}}$ and its first spatial derivatives in this point. Thus we have

$$\mathcal{E}_i^{(0)} = \mathcal{E}_i^{(00)} + a_{ip}\overline{U}_p + b_{ipq}\partial\overline{U}_p/\partial x_q + c_{ip}\Omega_p\,, \tag{12}$$

where $\mathcal{E}_i^{(00)}$ as well as the coefficients a_{ip}, b_{ipq} and c_{ip} are independent of $\overline{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$.

We now split $\boldsymbol{b}$ and $\boldsymbol{u}$ according to

$$\boldsymbol{b} = \boldsymbol{b}^{(0)} + \boldsymbol{b}^{(1)} + \cdots\,, \quad \boldsymbol{u} = \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(1)} + \cdots \tag{13}$$

into parts $\boldsymbol{b}^{(0)}$ and $\boldsymbol{u}^{(0)}$ independent of $\overline{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$, parts $\boldsymbol{b}^{(1)}$ and $\boldsymbol{u}^{(1)}$ of first order in $\overline{\boldsymbol{U}}$ or $\boldsymbol{\Omega}$, and higher-order contributions, which are however not considered in what follows. The assumption of the linearity of $\boldsymbol{\mathcal{E}}^{(0)}$ in $\overline{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$ implies

$$\boldsymbol{\mathcal{E}}^{(0)} = \langle \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(0)} \rangle + \langle \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(1)} \rangle + \langle \boldsymbol{u}^{(1)} \times \boldsymbol{b}^{(0)} \rangle\,. \tag{14}$$

Returning to Eqs. (9) and (10) and considering $\boldsymbol{h}$ and $\boldsymbol{f}$ as independent of $\overline{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$, we find for $\boldsymbol{b}^{(0)}$ and $\boldsymbol{u}^{(0)}$

$$\partial_t \boldsymbol{b}^{(0)} = \eta \boldsymbol{\nabla}^2 \boldsymbol{b}^{(0)} + \boldsymbol{\nabla} \times [(\boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(0)})' + \boldsymbol{h}]\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{b}^{(0)} = 0\,, \tag{15}$$

$$\partial_t \boldsymbol{u}^{(0)} = -\varrho^{-1}\boldsymbol{\nabla} p^{(0)} + \nu\boldsymbol{\nabla}^2\boldsymbol{u}^{(0)} - (\boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{(0)})' + \boldsymbol{f}\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u}^{(0)} = 0\,. \tag{16}$$

In the following we denote the turbulence defined by $\boldsymbol{b}^{(0)}$ and $\boldsymbol{u}^{(0)}$ as "background turbulence". In the equations resulting for $\boldsymbol{b}^{(1)}$ and $\boldsymbol{u}^{(1)}$ we introduce some generalized second-order correlation approximation, that is, neglect all terms originating from $(\boldsymbol{u}\times\boldsymbol{b})'$ and $(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u})'$. Hence we have

$$\partial_t \boldsymbol{b}^{(1)} = \eta\boldsymbol{\nabla}^2\boldsymbol{b}^{(1)} + \boldsymbol{\nabla}\times(\overline{\boldsymbol{U}}\times\boldsymbol{b}^{(0)})\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{b}^{(1)} = 0\,, \tag{17}$$

$$\partial_t \boldsymbol{u}^{(1)} = -\frac{1}{\varrho}\boldsymbol{\nabla} p^{(1)} + \nu\boldsymbol{\nabla}^2\boldsymbol{u}^{(1)} - 2\boldsymbol{\Omega}\times\boldsymbol{u}^{(0)} - \overline{\boldsymbol{U}}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{(0)} - \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\overline{\boldsymbol{U}}\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u}^{(1)} = 0\,. \tag{18}$$

2.2 Relation for $\boldsymbol{\mathcal{E}}^{(0)}$

In the following derivations we use a Fourier transformation of the form

$$F(\boldsymbol{x},t) = \iint \hat{F}(\boldsymbol{k},\omega)\,\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]\,\mathrm{d}^3k\,\mathrm{d}\omega, \tag{19}$$

with integrations over all $\boldsymbol{k}$ and ω.

In view of the determination of $\boldsymbol{\mathcal{E}}^{(0)}$ we first note

$$\langle \boldsymbol{u}(\boldsymbol{x},t)\times\boldsymbol{b}(\boldsymbol{x},t)\rangle_i = \epsilon_{ijk}\iint \hat{Q}_{jk}(\boldsymbol{x},t;\boldsymbol{k},\omega)\,\mathrm{d}^3k\,\mathrm{d}\omega\,, \tag{20}$$

where $\hat{Q}_{jk}(\boldsymbol{x},t;\boldsymbol{k},\omega)$ is the Fourier transform of

$$Q_{jk}(\boldsymbol{x},t;\boldsymbol{\xi},\tau) = \langle u_j(\boldsymbol{x}+\boldsymbol{\xi}/2,t+\tau/2)\,b_k(\boldsymbol{x}-\boldsymbol{\xi}/2,t-\tau/2)\rangle \tag{21}$$

with respect to $\boldsymbol{\xi}$ and τ. Adopting the formalism of Roberts & Soward (1975) we find that

$$\hat{Q}_{jk}(\boldsymbol{x},t;\boldsymbol{k},\omega) = \iint \langle \hat{u}_j(\boldsymbol{k}+\boldsymbol{k}'/2,\omega+\omega'/2)\,\hat{b}_k(-\boldsymbol{k}+\boldsymbol{k}'/2,-\omega+\omega'/2)\rangle \times\exp[\mathrm{i}(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega' t)]\,\mathrm{d}^3k'\,\mathrm{d}\omega'\,; \tag{22}$$

see Appendix A. As a consequence of $\boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{\nabla}\cdot\boldsymbol{b} = 0$ the conditions

$$(\nabla_j + 2\,\mathrm{i}k_j)\hat{Q}_{jk} = 0\,, \quad (\nabla_k - 2\,\mathrm{i}k_k)\hat{Q}_{jk} = 0 \tag{23}$$

have to be satisfied. Note that for the determination of $\boldsymbol{\mathcal{E}}^{(0)}$ only the antisymmetric part of $\hat{Q}_{jk}$ is needed.

Considering $\boldsymbol{\mathcal{E}}^{(0)}$ we restrict our attention for a moment to the point $\boldsymbol{x} = \boldsymbol{0}$. In that sense we put

$$\overline{U}_i = U_i + U_{ij}x_j \tag{24}$$

with two constant quantities U_i and U_{ij} satisfying $U_{ii} = 0$. We consider Eqs. (15) and (16) as solved, that is, $\boldsymbol{b}^{(0)}$ and $\boldsymbol{u}^{(0)}$ as known. Subjecting then Eqs. (17) and (18) for $\boldsymbol{b}^{(1)}$ and $\boldsymbol{u}^{(1)}$ with $\overline{\boldsymbol{U}}$ specified according to (24) to a Fourier transformation, eliminating the pressure term in the usual way and taking into account relation (B2) of Appendix B, we find

$$\hat{b}_i^{(1)} = -E\big[\mathrm{i}k_m U_m \hat{b}_i^{(0)} - U_{im}\hat{b}_m^{(0)} - k_m U_{mn}\partial\hat{b}_i^{(0)}/\partial k_n\big]\,, \qquad E = (\eta k^2 - \mathrm{i}\omega)^{-1}\,, \quad \hat{b}_i^{(1)}k_i = 0\,, \tag{25}$$

$$\hat{u}_i^{(1)} = -N\big[\mathrm{i}k_m U_m \hat{u}_i^{(0)} + U_{im}\hat{u}_m^{(0)} - k_m U_{mn}(2k_i\hat{u}_n^{(0)}/k^2 + \partial\hat{u}_i^{(0)}/\partial k_n) + 2\epsilon_{imn}k_m(\boldsymbol{k}\cdot\boldsymbol{\Omega})\hat{u}_n^{(0)}/k^2\big]\,, \qquad N = (\nu k^2 - \mathrm{i}\omega)^{-1}\,, \quad \hat{u}_i^{(1)}k_i = 0\,. \tag{26}$$

Calculating now $\hat{Q}_{jk}$ on the basis of (22), (25) and (26) we neglect again all contributions of higher than first order in $\overline{\boldsymbol{U}}$ and $\boldsymbol{\Omega}$. We further discard terms with more than one spatial derivative, in particular products of U_{ij} with any other spatial derivative. Since $\hat{Q}_{jk}$ should only weakly vary with $\boldsymbol{x}$ we expand $\langle\hat{u}_j\hat{b}_k\rangle$ in (22) for small $\boldsymbol{k}'$ and arrive so at

$$\begin{aligned}\hat{Q}_{jk} = {}& \hat{Q}_{jk}^{(0)} + \mathrm{i}(E^* - N)\,(\boldsymbol{k}\cdot\boldsymbol{U})\,\hat{Q}_{jk}^{(0)} \\ & + E^* U_{km}\,\hat{Q}_{jm}^{(0)} - N U_{jm}\,\hat{Q}_{mk}^{(0)} + 2N U_{mn}k_j k_m \hat{Q}_{nk}^{(0)}/k^2 \\ & + \tfrac{1}{2}(E^{*\prime} + N')U_{mn}k_m k_n \hat{Q}_{jk}^{(0)}/k^2 \\ & + \tfrac{1}{2}(E^* + N)U_{mn}k_m \partial\hat{Q}_{jk}^{(0)}/\partial k_n \\ & - 2N\,\epsilon_{jmn}k_m(\boldsymbol{k}\cdot\boldsymbol{\Omega})\hat{Q}_{nk}^{(0)}/k^2 \\ & - \tfrac{1}{2}(E^* + N)\,(\boldsymbol{U}\cdot\boldsymbol{\nabla})\hat{Q}_{jk}^{(0)} \\ & - \tfrac{1}{2}(E^{*\prime} + N')\,(\boldsymbol{k}\cdot\boldsymbol{U})\,(\boldsymbol{k}\cdot\boldsymbol{\nabla})\hat{Q}_{jk}^{(0)}/k^2 \\ & + \mathrm{i}\epsilon_{jmn}\big[N((\boldsymbol{k}\cdot\boldsymbol{\Omega})\nabla_m + k_m(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}))\hat{Q}_{nk}^{(0)}/k^2 \\ & \quad - (2N - N')k_m(\boldsymbol{k}\cdot\boldsymbol{\Omega})(\boldsymbol{k}\cdot\boldsymbol{\nabla})\hat{Q}_{nk}^{(0)}/k^4\big]\,.\end{aligned} \tag{27}$$

Here $\hat{Q}^{(0)}_{jk}$ means $\hat{Q}_{jk}$ for the background turbulence, that is, with $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{b}}$ replaced by $\hat{\boldsymbol{u}}^{(0)}$ and $\hat{\boldsymbol{b}}^{(0)}$. Relation (27), derived under the restriction $\boldsymbol{x} = \boldsymbol{0}$, applies for arbitrary $\boldsymbol{x}$ if only U_i and U_{ij} are interpreted as the values of $\overline{U}_i$ and $\partial\overline{U}_i/\partial x_j$ at the point $\boldsymbol{x}$. In that sense the arguments of $\hat{Q}_{jk}$ and $\hat{Q}^{(0)}_{jk}$, which were dropped for simplicity, are $(\boldsymbol{x}, t; \boldsymbol{k}, \omega)$. Those of E, N, etc. are (k, ω). The asterisk means complex conjugation and $F' = k\partial F/\partial k$.

Returning now to the representation (12) of $\boldsymbol{\mathcal{E}}^{(0)}$ we find

$$\mathcal{E}^{(00)}_i = \epsilon_{ijk} \iint \hat{Q}^{(0)}_{jk}\, \mathrm{d}^3k\mathrm{d}\omega\,, \tag{28}$$

$$a_{ip} = \epsilon_{ijk} \iint \big[\mathrm{i}(E^* - N)k_p - \tfrac{1}{2}(E^* + N)\nabla_p - \tfrac{1}{2}(E^{*\prime} + N')(k_p/k^2)(\boldsymbol{k}\cdot\boldsymbol{\nabla})\big]\hat{Q}^{(0)}_{jk}\mathrm{d}^3k\,\mathrm{d}\omega\,, \tag{29}$$

$$b_{ipq} = \epsilon_{ijk} \iint \big[E^*\delta_{kp}\hat{Q}^{(0)}_{jq} - N(\delta_{jp} - 2k_jk_p/k^2)\hat{Q}^{(0)}_{qk} + \tfrac{1}{2}(E^* + N)k_p\partial\hat{Q}_{jk}/\partial k_q + \tfrac{1}{2}(E^{*\prime} + N')k_pk_q\hat{Q}_{jk}/k^2\big]\,\mathrm{d}^3k\,\mathrm{d}\omega\,, \tag{30}$$

$$c_{ip} = \iint \big[(N/k^2)\big(2k_ik_p - \mathrm{i}(k_i\nabla_p + k_p\nabla_i)\big)\hat{Q}^{(0)}_{ll} + \mathrm{i}(N/k^2)(2k_p\nabla_k + k_k\nabla_p)\hat{Q}^{(0)}_{ik} - \mathrm{i}(2N - N')(k_p/k^4)(\boldsymbol{k}\cdot\boldsymbol{\nabla})\big(k_k\hat{Q}^{(0)}_{ik} - k_i\hat{Q}^{(0)}_{ll}\big)\big]\mathrm{d}^3k\,\mathrm{d}\omega\,, \tag{31}$$

where again the above remarks on arguments apply.

3 Homogeneous isotropic turbulence

3.1 General results

Consider now the simple case in which the background turbulence is homogeneous and isotropic and return first to (12). Since there is no isotropic vector we have $\mathcal{E}^{(00)}_i = 0$. As a consequence of isotropy we further put $a_{ip} = c_U\delta_{ip}$, $b_{ipq} = -c_W\epsilon_{ipq}$ and $c_{ip} = c_\Omega\delta_{ip}$. Hence we obtain

$$\boldsymbol{\mathcal{E}}^{(0)} = c_U\,\overline{\boldsymbol{U}} + c_W\,\boldsymbol{\nabla}\times\overline{\boldsymbol{U}} + c_\Omega\,\boldsymbol{\Omega} \tag{32}$$

with a scalar c_U and pseudoscalars c_W and c_Ω.

Employing homogeneity and isotropy of the turbulence with the conditions (23) we have

$$\hat{Q}^{(0)}_{jk}(\boldsymbol{k},\omega) = \frac{1}{2}\big(\delta_{jk} - \frac{k_jk_k}{k^2}\big)\hat{\Phi}^{(0)}(k,\omega) - \frac{\mathrm{i}}{2k^2}\epsilon_{jkl}k_l\hat{\Psi}^{(0)}(k,\omega)\,. \tag{33}$$

$\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ turn out to be the Fourier transforms of

$$\Phi^{(0)}(\boldsymbol{\xi},\tau) = \langle\boldsymbol{u}^{(0)}(\boldsymbol{x}+\boldsymbol{\xi}/2, t+\tau/2)\cdot\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}/2, t-\tau/2)\rangle\,, \tag{34}$$

$$\Psi^{(0)}(\boldsymbol{\xi},\tau) = \mu_0\langle\boldsymbol{u}^{(0)}(\boldsymbol{x}+\boldsymbol{\xi}/2, t+\tau/2)\cdot\boldsymbol{j}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}/2, t-\tau/2)\rangle \tag{35}$$

with respect to $\boldsymbol{\xi}$ and τ, where $\mu_0\boldsymbol{j}^{(0)}$ stands for $\boldsymbol{\nabla}\times\boldsymbol{b}^{(0)}$. Clearly $\Phi^{(0)}$ and $\hat{\Phi}^{(0)}$ are pseudoscalars but $\Psi^{(0)}$ and $\hat{\Psi}^{(0)}$ scalars. Of course, $\Phi^{(0)}$ and $\Psi^{(0)}$ as well as $\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ are independent of $\boldsymbol{x}$, and consequently (35) is equivalent to

$$\Psi^{(0)}(\boldsymbol{\xi},\tau) = \langle\boldsymbol{\omega}^{(0)}(\boldsymbol{x}+\boldsymbol{\xi}/2, t+\tau/2)\cdot\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}/2, t-\tau/2)\rangle\,, \tag{36}$$

where $\boldsymbol{\omega}^{(0)}$ is the vorticity of the velocity field $\boldsymbol{u}^{(0)}$, that is $\boldsymbol{\omega}^{(0)} = \boldsymbol{\nabla}\times\boldsymbol{u}^{(0)}$. Furthermore, both $\Phi^{(0)}$ and $\Psi^{(0)}$ are even in $\boldsymbol{\xi}$, and $\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ even in $\boldsymbol{k}$. In general $\hat{Q}_{jk}$, $\Phi^{(0)}$, $\Psi^{(0)}$ as well as $\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ may depend on t. If we however assume that the turbulence shows, in addition to its homogeneity, also statistical steadiness this dependence vanishes. In addition the arguments $(\boldsymbol{x}+\boldsymbol{\xi}/2, t+\tau/2)$ and $(\boldsymbol{x}-\boldsymbol{\xi}/2, t-\tau/2)$ in (34)–(36) may then be replaced, e.g., by $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)$ or by $(\boldsymbol{x}+\boldsymbol{\xi}, t+\tau)$ and $(\boldsymbol{x}, t)$, respectively.

With Eqs. (29)–(31) and (33) we find

$$c_U = \frac{1}{3}\iint (E^* - N)\,\hat{\Psi}^{(0)}\,k\,\mathrm{d}^3k\,\mathrm{d}\omega\,, \tag{37}$$

$$c_W = \frac{1}{3}\iint E^*\,\hat{\Phi}^{(0)}\,\mathrm{d}^3k\,\mathrm{d}\omega\,, \tag{38}$$

$$c_\Omega = \frac{2}{3}\iint N\,\hat{\Phi}^{(0)}\,\mathrm{d}^3k\,\mathrm{d}\omega\,. \tag{39}$$

We point out that $E = (2\pi)^4\hat{G^{(\eta)}}$ and $N = (2\pi)^4\hat{G^{(\nu)}}$ where the $G^{(\gamma)}$ are the well-known Green's functions defined by

$$G^{(\gamma)}(\xi,\tau) = (4\pi\gamma\tau)^{-3/2}\exp(-\xi^2/4\gamma\tau)\ \text{ for } \tau > 0\,,$$
$$G^{(\gamma)}(\xi,\tau) = 0\ \text{ for } \tau \le 0\,. \tag{40}$$

Considering this and applying the convolution theorem to Eqs. (37)–(39) we obtain

$$c_U = \frac{\mu_0}{3}\iint \big(G^{(\eta)}(\xi,\tau) - G^{(\nu)}(\xi,\tau)\big)\langle\boldsymbol{u}^{(0)}(\boldsymbol{x},t)\cdot\boldsymbol{j}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\rangle\,\mathrm{d}^3\xi\,\mathrm{d}\tau\,, \tag{41}$$

$$c_W = \frac{1}{3}\iint G^{(\eta)}(\xi,\tau)\langle\boldsymbol{u}^{(0)}(\boldsymbol{x},t)\cdot\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\rangle\,\mathrm{d}^3\xi\,\mathrm{d}\tau\,, \tag{42}$$

$$c_\Omega = \frac{2}{3}\iint G^{(\nu)}(\xi,\tau)\langle\boldsymbol{u}^{(0)}(\boldsymbol{x},t)\cdot\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\rangle\,\mathrm{d}^3\xi\,\mathrm{d}\tau\,. \tag{43}$$

Here the integrations are over all $\boldsymbol{\xi}$ and primarily also over all τ. However, since the $G^{(\eta)} = G^{(\nu)} = 0$ for $\tau \le 0$, they involve in fact only positive τ.

The most remarkable result of our derivations is that a contribution to $\boldsymbol{\mathcal{E}}^{(0)}$ proportional to $\overline{\boldsymbol{U}}$, that is, a term $c_U\overline{\boldsymbol{U}}$ in (32), may occur. This possibility has not previously been considered in the literature. According to (41), as long as $\langle\boldsymbol{u}^{(0)}(\boldsymbol{x},t)\cdot\boldsymbol{j}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\rangle$ does not vanish and $\eta \ne \nu$, the coefficient c_U may well be different from zero. In the special case $\eta = \nu$, however, it is equal to zero. In what follows the occurrence of that contribution to $\boldsymbol{\mathcal{E}}^{(0)}$ proportional to $\overline{\boldsymbol{U}}$ is labeled as "$\langle\boldsymbol{u}\cdot\boldsymbol{j}\rangle$ effect".

For non-vanishing $\langle\boldsymbol{u}^{(0)}(\boldsymbol{x},t)\cdot\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\rangle$ the pseudoscalars c_W and c_Ω will in general be different from zero. Then, as already found by Yoshizawa (1990), contributions to $\boldsymbol{\mathcal{E}}^{(0)}$ proportional to the mean vorticity $\nabla \times \overline{\boldsymbol{U}}$ and to the angular velocity $\boldsymbol{\Omega}$ will occur. We refer to them as "$\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ effects" or "Yoshizawa effects".

We stress that the $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect is well possible under circumstances in which $\langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{b}^{(0)}(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \rangle$ is equal to zero, that is, in which there are no $\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ effects.

The fact that Eqs. (11) for $\boldsymbol{u}^{(1)}$ contain $\boldsymbol{\Omega}$ and $\boldsymbol{W}$ only in the combination $2\boldsymbol{\Omega} + \boldsymbol{W}$ might suggest that c_Ω is equal to $2c_W$. Of course, the term $(2\boldsymbol{\Omega} + \boldsymbol{W}) \times \boldsymbol{u}$ in (11) contributes to both c_Ω and c_W. If there were no other influences on these coefficients this equality was indeed to be expected. However, also $\nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b})$ in (9) as well as $\overline{\boldsymbol{U}} \times (\nabla \times \boldsymbol{u})$ in (11) may contribute to c_W without influencing c_Ω. This disturbs the equality of c_Ω and $2c_W$. Remarkably, c_Ω and c_W differ even in their dependence on ν and η. As (38) and (39) and also (42) and (43) show, c_Ω depends on ν but not on η, and c_W on η but not on ν.

3.2 Special cases

Let us first consider $\boldsymbol{\mathcal{E}}^{(0)}$ in some limiting cases with respect to η and ν. We use the fact that

$$G^{(\gamma)}(\xi, \tau) \to \delta^3(\boldsymbol{\xi}) \quad \text{as} \quad \gamma \to 0\,. \tag{44}$$

In the limit defined by $\eta \to 0$ and $\nu \to \infty$ we obtain

$$c_U = \tfrac{1}{3} A\,, \quad c_W = \tfrac{1}{3} C\,, \quad c_\Omega = 0\,,$$

in the limit $\eta \to \infty$ and $\nu \to 0$

$$c_U = -\tfrac{1}{3} A\,, \quad c_W = 0\,, \quad c_\Omega = \tfrac{2}{3} C\,, \tag{45}$$

and in the limit η, $\nu \to 0$

$$c_U = 0\,, \quad c_W = \tfrac{1}{3} C\,, \quad c_\Omega = \tfrac{2}{3} C\,, \tag{46}$$

where

$$A = \mu_0 \int_0^\infty \langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{j}^{(0)}(\boldsymbol{x}, t - \tau) \rangle \, \mathrm{d}\tau\,,$$
$$C = \int_0^\infty \langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{b}^{(0)}(\boldsymbol{x}, t - \tau)) \rangle \, \mathrm{d}\tau\,. \tag{47}$$

Instead of the last relations we may also write

$$A = \mu_0 \langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{j}^{(0)}(\boldsymbol{x}, t) \rangle \, \tau_A\,,$$
$$C = \langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{b}^{(0)}(\boldsymbol{x}, t)) \rangle \, \tau_C \tag{48}$$

with correlation times τ_A and τ_C defined by equating the respective right-hand sides of (47) and (48).

In view of a numerical test we also consider the case in which $\langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{j}^{(0)}(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \rangle$ and $\langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{b}^{(0)}(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \rangle$, as far as they enter into the integrals in (41)–(43), do not markedly vary with τ. Ignoring the dependence on τ completely and using

$$\int_0^\infty G^{(\gamma)}(\xi, \tau) \, \mathrm{d}\tau = \frac{1}{4\pi\gamma\xi} \tag{49}$$

we find

$$c_U = \frac{1}{3} \left(\frac{1}{\eta} - \frac{1}{\nu} \right) A^\dagger\,,$$
$$c_W = \frac{1}{3\eta} C^\dagger\,, \quad c_\Omega = \frac{2}{3\nu} C^\dagger\,, \tag{50}$$

with

$$A^\dagger = \frac{1}{4\pi} \int_\infty \langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot (\nabla \times \boldsymbol{b}^{(0)}(\boldsymbol{x} - \boldsymbol{\xi}, t)) \rangle \, \frac{\mathrm{d}^3\xi}{\xi}\,,$$
$$C^\dagger = \frac{1}{4\pi} \int_\infty \langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{b}^{(0)}(\boldsymbol{x} - \boldsymbol{\xi}, t) \rangle \, \frac{\mathrm{d}^3\xi}{\xi}\,. \tag{51}$$

We may introduce vector potentials $\boldsymbol{\psi}^{(0)}$ and $\boldsymbol{a}^{(0)}$ such that

$$\nabla \times \boldsymbol{\psi}^{(0)} = \boldsymbol{u}^{(0)}\,, \quad \nabla \cdot \boldsymbol{\psi}^{(0)} = 0\,,$$
$$\nabla \times \boldsymbol{a}^{(0)} = \boldsymbol{b}^{(0)}\,, \quad \nabla \cdot \boldsymbol{a}^{(0)} = 0\,, \tag{52}$$

and therefore

$$\boldsymbol{\psi}^{(0)}(\boldsymbol{x}) = \frac{1}{4\pi} \int_\infty \nabla \times \boldsymbol{u}^{(0)}(\boldsymbol{x} - \boldsymbol{\xi}) \, \frac{\mathrm{d}^3\xi}{\xi}\,,$$
$$\boldsymbol{a}^{(0)}(\boldsymbol{x}) = \frac{1}{4\pi} \int_\infty \nabla \times \boldsymbol{b}^{(0)}(\boldsymbol{x} - \boldsymbol{\xi}) \, \frac{\mathrm{d}^3\xi}{\xi}\,. \tag{53}$$

For simplicity the argument t is omitted everywhere. With (51) and (53) we obtain

$$A^\dagger = \langle \boldsymbol{u}^{(0)} \cdot \boldsymbol{a}^{(0)} \rangle = \langle \boldsymbol{\psi}^{(0)} \cdot \boldsymbol{b}^{(0)} \rangle\,,$$
$$C^\dagger = \langle \boldsymbol{\psi}^{(0)} \cdot \boldsymbol{a}^{(0)} \rangle\,. \tag{54}$$

The arguments of the quantities in the angle brackets are, of course, always $(\boldsymbol{x}, t)$.

3.3 A numerical test

As a check of the above derivations, the electromotive force $\boldsymbol{\mathcal{E}}^{(0)}$ and so the coefficient c_U have been determined with numerical solutions of Eqs. (9) and (10). For these calculations $\overline{\boldsymbol{B}}$ has been put equal to zero. $\overline{\boldsymbol{U}}$ was specified via the initial condition to be constant in space, and it turned out to remain nearly constant in time, too. The forcing fields $\boldsymbol{h}$ and $\boldsymbol{f}$ were taken as periodic in the space coordinates x, y and z, and steady. More precisely, $\boldsymbol{h}$ and $\boldsymbol{f}$ differed only by constant factors from the vector field $\boldsymbol{e}(k\boldsymbol{x}) \equiv (\sin kz, \sin kx, \sin ky)$, with a constant k. The flow which would result from $\boldsymbol{f}$ is the no-cosine ABC flow of Archontis (2000, see also Dorch & Archontis 2004 and Cameron & Galloway 2006). Flows of this type are nonhelical. There are good reasons to expect an $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect but no $\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ effects. Corresponding to the steadiness of $\boldsymbol{h}$ and $\boldsymbol{f}$ only steady $\boldsymbol{b}$ and $\boldsymbol{u}$ were considered. The average which defines mean fields was taken over all x, y and z or, equivalent to this, over a periodic box. No approximation such as, e.g., the second-order correlation approximation was used.

Let us specify the result for c_U given by (50) and (54), which has been derived in the second-order correlation approximation, to the described situation. Relying on (15) and (16) we assume that $\boldsymbol{u}^{(0)}$ and $\boldsymbol{b}^{(0)}$ are dominated by contributions proportional to $\boldsymbol{e}(k\boldsymbol{x})$. So we find

$$c_U = c_0 R_\mathrm{m} \left(1 - \frac{1}{P_\mathrm{m}} \right)\,, \quad c_0 = \frac{\mu_0 \langle \boldsymbol{u}^{(0)} \cdot \boldsymbol{j}^{(0)} \rangle}{3 u_\mathrm{rms}^{(0)} k}\,, \tag{55}$$

Table 1 Numerically calculated values of c_U/c_0 for several $R_{\rm m}$ and $P_{\rm m}$, to be compared with the values derived in the second-order correlation approximation, $R_{\rm m}(1-1/P_{\rm m})$.

$R_{\rm m}$	$P_{\rm m}$	c_U/c_0	$R_{\rm m}(1-1/P_{\rm m})$
0.2	4	0.15	0.15
	2	0.10	0.10
	1	2.6×10^{-6}	0
	0.2	−0.80	−0.80
	0.04	−4.80	−4.80
	0.01	−15.5	−19.8
1	5	0.80	0.80
	1	1.2×10^{-6}	0
	0.2	−4.0	−4.0
	0.05	−17.7	−19.0
10	50	9.8	9.8
	10	9.0	9.0
	2	4.8	5.0
	0.5	−9.05	−10.0

with the magnetic Reynolds number $R_{\rm m}$ and the magnetic Prandtl number $P_{\rm m}$ defined by

$$R_{\rm m} = u^{(0)}_{\rm rms}/\eta k\,, \quad P_{\rm m} = \nu/\eta\,, \tag{56}$$

and $u^{(0)}_{\rm rms} = \langle \boldsymbol{u}^{(0)\,2}\rangle^{1/2}$.

In Table 1 and Fig. 1 the numerically determined values of c_U/c_0 are given in dependence of $R_{\rm m}$ and $P_{\rm m}$. In agreement with what we have found in our analytical calculations in the second-order correlation approximation their signs change with growing $P_{\rm m}$ at $P_{\rm m}=1$. Moreover, in most cases the numerically determined values completely agree with those obtained in this approximation, that is, with $R_{\rm m}(1-1/P_{\rm m})$. Deviations occur only if the fluid Reynolds number $R_{\rm e} = u^{(0)}_{\rm rms}/\nu k$, that is $R_{\rm e} = R_{\rm m}/P_{\rm m}$, exceeds a value of about 5. We should emphasize that all our numerical solutions are laminar and perfectly regular, just as in Fig. 2 (upper row) of Sur & Brandenburg (2009).

4 Inhomogeneous turbulence

Let us add some results for the case in which the turbulence is no longer homogeneous and so also no longer isotropic. We admit now that correlation functions as they occur on the right-hand sides of (34)–(36) may depend on $\boldsymbol{x}$ so that their gradients with respect to $\boldsymbol{x}$ do not generally vanish. Then in addition to the contributions to $\boldsymbol{\mathcal{E}}^{(0)}$ given in (32) other contributions are possible. Symmetry considerations suggest

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{(0)} \;=\;& c_U\,\overline{\boldsymbol{U}} + c_W\,\boldsymbol{\nabla}\times\overline{\boldsymbol{U}} + c_\Omega\,\boldsymbol{\Omega}\\ &+ \boldsymbol{g} + \boldsymbol{g}_U\times\overline{\boldsymbol{U}} + \boldsymbol{g}_\Omega\times\boldsymbol{\Omega}\,.\end{aligned} \tag{57}$$

For the sake of simplicity only terms up to first order in the spatial derivatives are included. The coefficients c_U, c_W and c_Ω may now vary in space. Further $\boldsymbol{g}$ and $\boldsymbol{g}_\Omega$ are vectors and $\boldsymbol{g}_U$ is a pseudovector, all determined by the anisotropy of the turbulence.

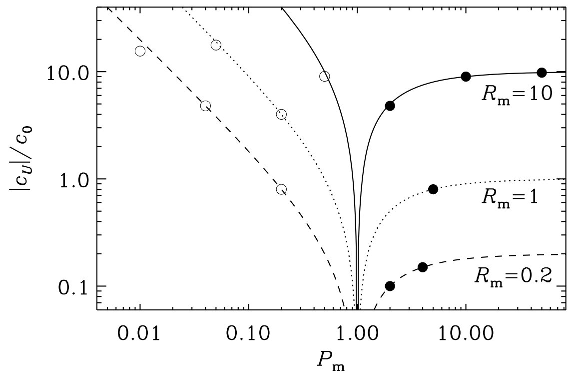

Fig. 1 Numerically obtained values of $|c_U|/c_0$, indicated by ∘ for $P_{\rm m}<1$ (where $c_U<0$) but by • for $P_{\rm m}>1$ (where $c_U>0$), and curves representing $R_{\rm m}|1-1/P_{\rm m}|$ for $R_m=10$ (solid), $R_m=1$ (dotted), and $R_m=0.2$ (dashed).

We may determine the contributions to $\boldsymbol{\mathcal{E}}^{(0)}$ mentioned in (57) again on the basis of (20) and (27). However, relation (33) for $\hat{Q}^{(0)}_{jk}$, which applies to homogeneous isotropic turbulence only, has to be modified. Taking into account the aforementioned gradients of correlation functions, but considering them as small, we may generalize (33) by adding, on its right-hand side, terms that are linear in these gradients and applying again the conditions (23). Without claiming to full generality, in order to arrive at a typical result, we use in what follows

$$\begin{aligned}&\hat{Q}^{(0)}_{jk}(\boldsymbol{x};\boldsymbol{k},\omega) =\\ &\quad\frac{1}{2}\Big[\big(\delta_{jk}-\frac{k_j k_k}{k^2}\big) + \frac{\mathrm{i}}{2k^2}\big(k_j\nabla_k - k_k\nabla_j\big)\Big]\,\hat{\Phi}^{(0)}(\boldsymbol{x};k,\omega)\\ &\quad-\frac{1}{2k^2}\Big[\mathrm{i}\epsilon_{jkl}k_l\\ &\qquad-\frac{1}{2k^2}(k_j\epsilon_{klm}+k_k\epsilon_{jlm})k_l\nabla_m\Big]\,\hat{\Psi}^{(0)}(\boldsymbol{x};k,\omega)\,.\end{aligned} \tag{58}$$

Here, $\hat{\Phi}^{(0)}$ is again the Fourier transform of

$$\begin{aligned}&\Phi^{(0)}(\boldsymbol{x};\boldsymbol{\xi},\tau) =\\ &\quad\langle\boldsymbol{u}^{(0)}(\boldsymbol{x}+\boldsymbol{\xi}/2,t+\tau/2)\cdot\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}/2,t-\tau/2)\rangle\end{aligned} \tag{59}$$

with respect to $\boldsymbol{\xi}$ and τ, and $\hat{\Psi}^{(0)}$ that of

$$\begin{aligned}&\Psi^{(0)}(\boldsymbol{x};\boldsymbol{\xi},\tau) =\\ &\quad\tfrac{1}{2}\langle(\boldsymbol{\nabla}\times\boldsymbol{u}^{(0)}(\boldsymbol{x}+\boldsymbol{\xi}/2,t+\tau/2))\\ &\qquad\qquad\cdot\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}/2,t-\tau/2)\rangle\\ &\quad+\tfrac{1}{2}\langle\boldsymbol{u}^{(0)}(\boldsymbol{x}+\boldsymbol{\xi}/2,t+\tau/2)\\ &\qquad\qquad\cdot(\boldsymbol{\nabla}\times\boldsymbol{b}^{(0)}(\boldsymbol{x}-\boldsymbol{\xi}/2,t-\tau/2))\rangle\,.\end{aligned} \tag{60}$$

In contrast to the case of homogeneous turbulence, $\Phi^{(0)}$ and $\Psi^{(0)}$ as well as $\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ may now depend on $\boldsymbol{x}$. Furthermore, $\Phi^{(0)}$ and $\Psi^{(0)}$ are no longer necessarily even in $\boldsymbol{\xi}$, and $\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ no longer even in $\boldsymbol{k}$. Again, $\hat{Q}_{jk}$, $\Phi^{(0)}$, $\Psi^{(0)}$, $\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ may depend on t, but this is of minor importance in this context and therefore not explicitly indicated.

A straightforward calculation confirms then (57). As expected, we find again the relations (37)–(39) for c_U, c_W and c_Ω, now with $\hat{\Phi}^{(0)}$ and $\hat{\Psi}^{(0)}$ according to (59) and (60). While the expression (41) for c_U has to be modified, (42) and (43) for c_W and c_Ω retain their validity. As for $\boldsymbol{g}$, $\boldsymbol{g}_U$ and $\boldsymbol{g}_\Omega$, the calculation yields

$$\boldsymbol{g} = -\mathrm{i} \iint (\boldsymbol{k}/k^2)\, \hat{\Psi}^{(0)}\, \mathrm{d}^3k\, \mathrm{d}\omega\,, \tag{61}$$

$$\boldsymbol{g}_U = \frac{1}{6}\boldsymbol{\nabla} \iint (E^* - N)\, \hat{\Phi}^{(0)}\, \mathrm{d}^3k\, \mathrm{d}\omega\,, \tag{62}$$

$$\boldsymbol{g}_\Omega = \frac{1}{3}\boldsymbol{\nabla} \iint (N/k^2)\, \hat{\Psi}^{(0)}\, \mathrm{d}^3k\, \mathrm{d}\omega\,. \tag{63}$$

Using again the convolution theorem in combination with above-mentioned connection between N and $\widehat{G^{(\nu)}}$ and between E and $\widehat{G^{(\eta)}}$ we arrive at the equivalent relations

$$\boldsymbol{g} = \frac{1}{12\pi} \int \boldsymbol{\nabla}_\xi \Psi^{(0)}(\boldsymbol{x}; \boldsymbol{\xi}, 0)\, \frac{\mathrm{d}^3\xi}{\xi}\,, \tag{64}$$

$$\boldsymbol{g}_U = \frac{1}{6}\boldsymbol{\nabla} \iint \left(G^{(\eta)}(\boldsymbol{\xi}, \tau) - G^{(\nu)}(\boldsymbol{\xi}, \tau)\right) \Phi^{(0)}(\boldsymbol{x}; \boldsymbol{\xi}, \tau)\, \mathrm{d}^3\xi \mathrm{d}\tau\,, \tag{65}$$

$$\boldsymbol{g}_\Omega = \frac{1}{12\pi}\boldsymbol{\nabla} \iiint G^{(\nu)}(\boldsymbol{\xi} + \boldsymbol{\xi}', -\tau) \frac{\mathrm{d}^3\xi'}{\xi'} \Psi^{(0)}(\boldsymbol{x}; \boldsymbol{\xi}, \tau)\, \mathrm{d}^3\xi\, \mathrm{d}\tau\,, \tag{66}$$

where $\boldsymbol{\nabla}_\xi$ stands for a gradient in $\boldsymbol{\xi}$ space. Note that for homogeneous turbulence, in addition to $\boldsymbol{g}_U$ and $\boldsymbol{g}_\Omega$, also $\boldsymbol{g}$ vanishes since then $\hat{\Psi}^{(0)}$ is even in $\boldsymbol{k}$ and $\Psi^{(0)}$ even in $\boldsymbol{\xi}$.

These results confirm in some sense the statements made in the paper by Rädler (1976), formulated above in (5). They show however that the vectors $c_\gamma \gamma$ and $c_{\gamma\Omega}\gamma$ should not, as suggested there, be understood in the sense of $\boldsymbol{\nabla}\langle \boldsymbol{u}^2 \rangle$. These vectors rather correspond to $\boldsymbol{g}$ or $\boldsymbol{g}_\Omega$ as given in (61) and (64) and in (63) and (66). They should be interpreted in terms of correlations between $\boldsymbol{u}$ and $\boldsymbol{\nabla} \times \boldsymbol{b}$ and between $\boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{b}$.

5 Discussion

The most remarkable result of our calculations is that the mean electromotive force $\boldsymbol{\mathcal{E}}^{(0)}$ in a homogeneous isotropic magnetohydrodynamic turbulence may have a contribution proportional to the mean fluid velocity $\overline{\boldsymbol{U}}$, that is, $\boldsymbol{\mathcal{E}}^{(0)} = c_U \overline{\boldsymbol{U}} + \cdots$. We have labeled the occurrence of this contribution as $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect. The coefficient c_U turned out to be in general unequal to zero if only a non-zero correlation exists between the fluctuating parts of the fluid velocity and the electric current density, $\boldsymbol{u}$ and $\boldsymbol{j} = \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{b}$, or between the fluctuating parts of the vorticity and the magnetic field, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ and $\boldsymbol{b}$. As far as the second-order correlation approximation applies, c_U vanishes for $\eta = \nu$, and it changes its sign if ν/η varies and passes through $\nu/\eta = 1$.

When applying our result on the $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect on a specific situation we have to check carefully whether this situation corresponds to the assumptions used in our calculations. We have determined $\boldsymbol{\mathcal{E}}^{(0)}$ in frame of reference in which a mean flow, $\overline{\boldsymbol{U}}$, exists together with magnetohydrodynamic turbulence, and assumed that the turbulence is homogeneous and isotropic in the limit $\overline{\boldsymbol{U}} \to \boldsymbol{0}$. It is the deviation of the turbulence from isotropy, caused by $\overline{\boldsymbol{U}}$, which is crucial for the $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect. We could, of course, carry out the calculation also in a frame in which $\overline{\boldsymbol{U}}$ vanishes. Under the assumptions adopted so far the turbulence there has to be anisotropic. In this way a non-vanishing term $\boldsymbol{\mathcal{E}}^{(00)}$ would occur instead of $c_U \overline{\boldsymbol{U}}$ in (32), which has to be considered as another description of the $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect.

Our assumptions differ basically from that of "Galilean invariance" of the turbulence, which has been adopted in various investigations (e.g., Sridhar & Subramanian 2009). Galilean invariance means independence of the turbulence of the homogeneous part of $\overline{\boldsymbol{U}}$ and excludes the $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect. Galilean invariance is violated if the forcing is independent of $\overline{\boldsymbol{U}}$, that is, if the functions $\boldsymbol{h}$ and $\boldsymbol{f}$ in (9) and (10) are independent of $\overline{\boldsymbol{U}}$. This was the case in the numerical example considered in Sect. 3.3. On the other hand, turbulence driven by an instability of a flow, for example by thermal convection or by the magneto-rotational instability, should correspond to a forcing effectively advected with the flow and therefore be Galilean invariant. Astrophysical examples where Galilean invariance may be violated include turbulence driven by supernova explosions (assuming the stellar component to be decoupled from the gas) or by inflection point instabilities on solid surfaces such as that of neutron stars. However, more work is needed in order to clarify whether the $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect can be expected to operate in any of those environments. Furthermore, in general the $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effect is accompanied by the $\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ effects. It has then also to be investigated which of these effects dominate.

We point out that magnetohydrodynamic turbulence does not automatically imply non-zero correlations of $\boldsymbol{u}$ and $\boldsymbol{b}$, and of $\boldsymbol{u}$ and $\boldsymbol{j}$, or $\boldsymbol{\omega}$ and $\boldsymbol{b}$. It depends on the special circumstances whether, e.g., $\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$, $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ or $\langle \boldsymbol{\omega} \cdot \boldsymbol{b} \rangle$ are different from zero and what their signs are. Exploring the importance of the $\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ and $\langle \boldsymbol{u} \cdot \boldsymbol{j} \rangle$ effects in specific settings requires investigations on this topic, too.

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A Derivation of relation (22) for $\hat{Q}_{jk}$

Start from $Q_{jk}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)$ as given in (21) and introduce there the Fourier representations of $\hat{u}_j$ and $\hat{b}_k$ so that

$$Q_{jk}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = \iint \iint \langle \hat{u}_j(\boldsymbol{k}^\dagger, \omega^\dagger)\, \hat{b}_k(\boldsymbol{k}^\ddagger, \omega^\ddagger)\rangle \exp\big(\mathrm{i}\big((\boldsymbol{k}^\dagger + \boldsymbol{k}^\ddagger)\cdot\boldsymbol{x} + (\boldsymbol{k}^\dagger - \boldsymbol{k}^\ddagger)\cdot\boldsymbol{\xi}/2 - (\omega^\dagger + \omega^\ddagger)t - (\omega^\dagger - \omega^\ddagger)\tau/2\big)\big)\, \mathrm{d}^3k^\dagger\, \mathrm{d}\omega^\dagger\, \mathrm{d}^3k^\ddagger\, \mathrm{d}\omega^\ddagger\,. \tag{A1}$$

Change then the integration variables according to

$$\boldsymbol{k}^\dagger = \boldsymbol{k} + \boldsymbol{k}'/2\,, \quad \boldsymbol{k}^\ddagger = -\boldsymbol{k} + \boldsymbol{k}'/2\,, \quad \omega^\dagger = \omega + \omega'/2\,, \quad \omega^\ddagger = -\omega + \omega'/2\,, \tag{A2}$$

and find so

$$Q_{jk}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = \iint \iint \langle \hat{u}_j(\boldsymbol{k} + \boldsymbol{k}'/2, \omega + \omega'/2)\, \hat{b}_k(-\boldsymbol{k} + \boldsymbol{k}'/2, -\omega + \omega'/2)\rangle \exp\big(\mathrm{i}(\boldsymbol{k}'\cdot\boldsymbol{x} - \omega' t)\big)\, \mathrm{d}^3k'\, \mathrm{d}\omega' \exp\big(\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega\tau)\big)\, \mathrm{d}^3k\, \mathrm{d}\omega\,. \tag{A3}$$

This shows that $\hat{Q}_{jk}$ given by (22) is indeed the Fourier transform of $Q_{jk}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)$ with respect to $\boldsymbol{\xi}$ and τ.

B Concerning the derivations of (26) and (27)

For the derivation of (26) it is useful to know the relation[1]

$$\epsilon_{ijk}\Omega_k - (\epsilon_{ikl}k_j - \epsilon_{jkl}k_i)\frac{k_k\Omega_l}{k^2} = \epsilon_{ijk}k_k\frac{(\boldsymbol{k}\cdot\boldsymbol{\Omega})}{k^2}\,, \tag{B1}$$

which applies for arbitrary vectors $\boldsymbol{k}$ and $\boldsymbol{\Omega}$. From this we have

$$\epsilon_{ijk}\Omega_j\hat{u}_k - k_i\epsilon_{jkl}\hat{u}_j\frac{k_k\Omega_l}{k^2} = \epsilon_{ijk}\hat{u}_jk_k\frac{(\boldsymbol{k}\cdot\boldsymbol{\Omega})}{k^2} \tag{B2}$$

for any vector $\hat{\boldsymbol{u}}$ satisfying $\hat{\boldsymbol{u}}\cdot\boldsymbol{k} = 0$.

In view of (27) it is of interest that

$$\begin{aligned}
&\frac{\partial\hat{u}(\boldsymbol{k}+\boldsymbol{k}'/2)}{\partial k_i}\,\hat{b}(-\boldsymbol{k}+\boldsymbol{k}'/2) = \Big(\frac{1}{2}\frac{\partial}{\partial k_i} + \frac{\partial}{\partial k'_i}\Big)\big(\hat{u}(\boldsymbol{k}+\boldsymbol{k}'/2)\,\hat{b}(-\boldsymbol{k}+\boldsymbol{k}'/2)\big)\,,\\
&\hat{u}(\boldsymbol{k}+\boldsymbol{k}'/2)\,\frac{\partial\hat{b}(-\boldsymbol{k}+\boldsymbol{k}'/2)}{\partial k_i} = \Big(\frac{1}{2}\frac{\partial}{\partial k_i} - \frac{\partial}{\partial k'_i}\Big)\big(\hat{u}(\boldsymbol{k}+\boldsymbol{k}'/2)\,\hat{b}(-\boldsymbol{k}+\boldsymbol{k}'/2)\big)\,.
\end{aligned} \tag{B3}$$

[1] The corresponding relation (A1) in Rädler et al. (2003) contains a sign error.