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Non-linear magnetic diffusivity in mean-field electrodynamics

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ABSTRACT

We consider non-linear transport and drift processes caused by an inhomogeneous magnetic field in a turbulent fluid. The coefficients of magnetic diffusivity and drift velocity are calculated by making use of the second-order correlation approximation. Transport processes in the presence of a sufficiently strong magnetic field become anisotropic with larger diffusion rate and turbulent electrical resistivity across the field than along the field. Non-linear effects also lead to a drift of the magnetic field away from the regions with a higher magnetic energy.

Key words: MHD – turbulence – stars: magnetic fields – ISM: magnetic fields.

1 INTRODUCTION

Over the past three decades, large-scale dynamo theory (e.g. Moffatt 1978; Parker 1979) has been used to explain the origin of magnetic fields in astrophysical bodies. The original question was whether fluid motions can work in such a way that a magnetic field is generated from a week seed magnetic field. Fluid motions that show some swirl (helicity) seem to be best suited to accomplishing this (Krause & Rädler 1980). However, over the last few years questions have been raised as to whether any part of this process can still be valid when the magnetic energy becomes comparable to the kinetic energy (Vainshtein & Cattaneo 1992; Kulsrud & Anderson 1992). Nevertheless, there are now several simulations that show the possibility of generation of strong large-scale magnetic fields in the contexts of the geodynamo (Glatzmaier & Roberts 1995, 1996) and accretion disc turbulence (Brandenburg et al. 1995, 1996). Obviously, the presence of such a strong magnetic field can modify (or even quench) the kinematic dynamo effect and change the transport properties of turbulence. The main purpose of this paper is to consider the influence of a finite-amplitude magnetic field on the magnetic diffusivity of a turbulent fluid.

The central point of any mean field theory is to express the mean turbulent electromotive force

$$\mathcal{E} = \langle \boldsymbol{v} \times \boldsymbol{b} \rangle, \tag{1}$$

where $\boldsymbol{v}$ and $\boldsymbol{b}$ are the turbulent velocity and magnetic field, respectively, in terms of large-scale quantities; and $\langle \ldots \rangle$ denote ensemble averaging. The tensor of turbulent magnetic diffusivity is determined by the coefficients in front of the three components of $\nabla \times \boldsymbol{B}$ in this expression, where $\boldsymbol{B}$ is the mean magnetic field. To obtain the equation for $\mathcal{E}$, we use a 'two-scale' approximation in which the temporal and spatial scales associated with the turbulence, τ and ℓ, are assumed to be small compared with the scales characterizing the mean field, t_0 and L.

The paper is organized as follows. In Section 2 we discuss the basic equations governing the behaviour of turbulence in the presence of finite-amplitude mean magnetic fields and formulate our assumptions. Calculations of the non-linear contribution to the mean electromotive force and anisotropic magnetic diffusivity are presented in Section 3. In Section 4, we discuss the results.

2 THE BASIC EQUATIONS

To derive $\mathcal{E}$, we use the second-order correlation approximation (see Moffatt 1978 and Krause & Rädler 1980 for more details). In this approximation, the behaviour of mean quantities is governed by equations that include non-linear effects in the mean field, although linearized equations are used for the fluctuating quantities. The second-order correlation approximation strictly applies in the case of turbulence with small ordinary and magnetic Reynolds numbers (note, however, that in this case the field can be generated only if the magnetic Reynolds number with respect to the large-scale motion is large enough). Generally, such turbulence can be induced by weak instabilities, for example by convection when the difference between the real and adiabatic temperature gradients is small. This approximation may also be applicable in the case of an ensemble of magnetohydrodynamic waves with relatively high frequencies and small amplitudes when the so-called Strouhal number is small, $v\tau/\ell < 1$.

The magnetic field $\mathcal{B}$ and the governing equation can be separated into mean and fluctuating parts with $\mathcal{B} = \boldsymbol{B} + \boldsymbol{b}$, where $\boldsymbol{B}$ is the mean field. We assume that there is no mean flow; thus only the fluctuating component of velocity, $\boldsymbol{v}$, is non-vanishing. The induction equation for the fluctuating magnetic field reads

$$\frac{\partial \boldsymbol{b}}{\partial t} - \nu_{\mathrm{m}} \Delta \boldsymbol{b} = \boldsymbol{A},$$

$$\boldsymbol{A} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla \cdot \boldsymbol{v}), \tag{2}$$

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where ν_{m} is the magnetic diffusivity. In this equation, we neglect the non-linear term in fluctuating quantities according to the spirit of the second-order correlation approximation. By making use of the Fourier transformation for fluctuating quantities, as in

$$\boldsymbol{b}(t,\boldsymbol{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\hat{\boldsymbol{b}}(\omega,\boldsymbol{k}), \tag{3}$$

where the hat denotes the Fourier amplitude, we can express $\hat{\boldsymbol{b}}(\omega,\boldsymbol{k})$ in terms of $\hat{\boldsymbol{v}}(\omega,\boldsymbol{k})$ using equation (2), so

$$\hat{\boldsymbol{b}}(\omega,\boldsymbol{k}) = \frac{\hat{\boldsymbol{A}}(\omega,\boldsymbol{k})}{\mathrm{i}\omega + \omega_{\mathrm{m}}}, \tag{4}$$

where $\omega_{\mathrm{m}} = \nu_{\mathrm{m}}k^2$.

The momentum equation for the fluctuating velocity reads

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \frac{\nabla p}{\rho} = \frac{1}{4\pi\rho}[(\nabla\times\boldsymbol{b})\times\boldsymbol{B} + (\nabla\times\boldsymbol{B})\times\boldsymbol{b}], \tag{5}$$

where ν is the viscosity and p is the fluctuating component of the pressure. According to the spirit of the second-order correlation approximation, we will also neglect the non-linear term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ in this equation. In the Fourier representation, equation (5) has the form

$$(\mathrm{i}\omega + \omega_\nu)\hat{\boldsymbol{v}} - \frac{\mathrm{i}\boldsymbol{k}\hat{p}}{\rho} = \frac{1}{4\pi\rho}[-\mathrm{i}(\boldsymbol{k}\times\hat{\boldsymbol{b}})\times\boldsymbol{B} + (\nabla\times\boldsymbol{B})\times\hat{\boldsymbol{v}}], \tag{6}$$

where $\omega_\nu = \nu k^2$. Equation (6) can be solved by making use of a perturbation procedure similar to the approach of 'original turbulence' introduced by Rüdiger (1990). However, instead of introducing a rather uncertain non-potential part in the random force ($\hat{\vec{f}}^{s}$ in the notations of Rüdiger & Kitchatinov 1993), we calculate this force assuming that the perturbations of turbulent velocity caused by the magnetic field are small and employ a standard perturbation procedure. For this purpose, we split the velocity and other Fourier amplitudes into two components, $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_0 + \hat{\boldsymbol{v}}_1$, with $\hat{\boldsymbol{v}}_0$ being the turbulent velocity in the absence of the magnetic field, and $\hat{\boldsymbol{v}}_1$ being a small perturbation to $\hat{\boldsymbol{v}}_0$ caused by $\boldsymbol{B}$. The unperturbed velocity, $\hat{\boldsymbol{v}}_0$, is governed by equation (6) with $\boldsymbol{B} = 0$. As we neglect thermal effects in our model, the velocity field in the 'zeroth' approximation is represented by an ensemble of isothermal sound waves with the dispersion relation $\omega_{1,2} = \pm\omega_{\mathrm{s}} + \mathrm{i}\omega_\nu/2$, where $\omega_{\mathrm{s}} = kc_{\mathrm{s}}$, c_{s} is the sound speed, $\omega_{\mathrm{s}} \gg \omega_\nu$. We assume that the 'zeroth' approximation describes isotropic turbulence with mirror symmetry, whereas $\hat{\boldsymbol{v}}_1$ may have a non-mirror symmetric component that can influence the large-scale magnetic field. One has for $\hat{\boldsymbol{v}}_1$

$$(\mathrm{i}\omega + \omega_\nu)\hat{\boldsymbol{v}}_1 - \frac{\mathrm{i}\boldsymbol{k}\hat{p}_1}{\rho} = \boldsymbol{F}, \tag{7}$$

where

$$\boldsymbol{F} = \frac{1}{4\pi\rho}[(\nabla\times\boldsymbol{B})\times\hat{\boldsymbol{b}}_0 - \mathrm{i}(\boldsymbol{k}\times\hat{\boldsymbol{b}}_0)\times\boldsymbol{B}], \tag{8}$$

and $\hat{\boldsymbol{b}}_0$ is given by equation (4) with $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_0$. It is seen from equations (7) and (8) that a non-mirror symmetric component of the velocity field may be caused by the Lorentz force.

The solution of equation (7) can easily be obtained if we adopt the equation of state of an ideal isothermal gas, $\hat{p}_1/p = \hat{\rho}_1/\rho$, and take into account the continuity equation, $\hat{\rho}_1/\rho = (\boldsymbol{k}\cdot\hat{\boldsymbol{v}}_1)/\omega$. Then,

$$\hat{\boldsymbol{v}}_1 = \frac{1}{\mathrm{i}\omega + \omega_\nu}\left[\boldsymbol{F} - \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{F})c_{\mathrm{s}}^2}{\mathrm{i}\omega(\mathrm{i}\omega + \omega_\nu) + \omega_{\mathrm{s}}^2}\right]. \tag{9}$$

Note that equation (9) applies only for a relatively weak magnetic field so that $|\hat{\boldsymbol{v}}_1| \ll |\hat{\boldsymbol{v}}_0|$. This condition holds if the dimensionless parameter $c_{\mathrm{A}}\tau/\ell$ is small compared with unity, where $c_{\mathrm{A}} = B/\sqrt{4\pi\rho}$ is the Alfvén speed.

3 THE TURBULENT ELECTROMOTIVE FORCE

Substituting $\boldsymbol{b}$ and $\boldsymbol{v}$ in terms of their Fourier integrals into equation (1) and taking into account that

$$\langle\hat{v}_i(\omega,\boldsymbol{k})\hat{v}_j(\omega',\boldsymbol{k}')\rangle \propto \delta(\omega+\omega')\delta(\boldsymbol{k}+\boldsymbol{k}') \tag{10}$$

for a quasi-stationary and quasi-homogeneous turbulence (see e.g. Rüdiger & Kitchatinov 1993), we obtain

$$\mathcal{E} = -\int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}}{\mathrm{i}\omega - \omega_{\mathrm{m}}}\langle\hat{\boldsymbol{v}}(\omega,\boldsymbol{k})\times\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{B})\hat{\boldsymbol{v}}(-\omega,-\boldsymbol{k}) - [\hat{\boldsymbol{v}}(-\omega,-\boldsymbol{k})\cdot\nabla]\boldsymbol{B} - \mathrm{i}\boldsymbol{B}[\boldsymbol{k}\cdot\hat{\boldsymbol{v}}(-\omega,-\boldsymbol{k})]\}\rangle. \tag{11}$$

In the case of isotropic acoustic turbulence with

$$\langle\hat{v}_i(\omega,\boldsymbol{k})\hat{v}_j(\omega',\boldsymbol{k}')\rangle = R(\omega,\boldsymbol{k})k_ik_j\delta(\omega+\omega')\delta(\boldsymbol{k}+\boldsymbol{k}'), \tag{12}$$

(see e.g. Krause & Rädler 1980), equation (11) yields the diffusive component of the electromotive force alone,

$$\mathcal{E} = -\eta\nabla\times\boldsymbol{B}, \tag{13}$$

where

$$\eta = \frac{1}{3}\int_{-\infty}^{+\infty}\frac{\omega_{\mathrm{m}}k^2R(\omega,\boldsymbol{k})}{\omega^2+\omega_{\mathrm{m}}^2}\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k} \tag{14}$$

is the scalar turbulent magnetic diffusivity.

The electromotive force (1) can now be calculated by substituting $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_0 + \hat{\boldsymbol{v}}_1$ into equation (11) and keeping only the lowest terms in $\hat{\boldsymbol{v}}_1$. The spectral function, $R(\omega,\boldsymbol{k})$, is assumed to be isotropic. In linear theory, the terms in equation (11) allow a relatively simple qualitative interpretation and correspond to an alpha effect, turbulent diffusion and drift processes caused by the non-uniformity of large-scale quantities, respectively. If non-linear effects are taken into account, the situation becomes quite different and any term in equation (11) can contribute to the rate of diffusion or to the drift velocity. Assuming the turbulence to be homogeneous and isotropic in the zeroth approximation, we can represent $\mathcal{E}$ as a sum of three components,

$$\mathcal{E} = -(\eta + \gamma_1B^2)\nabla\times\boldsymbol{B} + \gamma_2\boldsymbol{B}[\boldsymbol{B}\cdot(\nabla\times\boldsymbol{B})] - \gamma_3\nabla B^2\times\boldsymbol{B}. \tag{15}$$

The transport coefficients in this equation are given by

$$\gamma_1 = \frac{10}{3}\xi_1 + \xi_2 + 2\xi_3 + 7\xi_4,$$

$$\gamma_2 = 2\xi_1 - \xi_2 + 2\xi_3 + 7\xi_4,$$

$$\gamma_3 = \xi_1 + \tfrac{1}{2}(\xi_2 + \xi_3 + \xi_4),$$

where the functions ξ_i are determined by the spectrum of the turbulence, $R(\omega,\boldsymbol{k})$,

$$\xi_i = \frac{1}{2\pi\rho}\int_{-\infty}^{\infty}\frac{k^4R(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}}{(\omega^2+\omega_\nu^2)(\omega^2+\omega_{\mathrm{m}}^2)^2}f_i, \tag{16}$$

with

$$f_1 = \tfrac{1}{5}\,\omega_{\mathrm{m}}(\omega^2 - \omega_\nu\omega_{\mathrm{m}}),$$

$$f_2 = \frac{2\omega_m \omega_s^2[(\omega^2 - \omega_s^2)(\omega^2 - \omega_\nu \omega_m) - \omega^2 \omega_\nu(\omega_\nu + \omega_m)]}{15[(\omega^2 - \omega_s^2)^2 + \omega^2 \omega_\nu^2]},$$

$$f_3 = \frac{\omega_s^2 \omega^2[(\omega^2 - \omega_s^2)(\omega_\nu + \omega_m) + \omega_\nu(\omega^2 - \omega_\nu \omega_m)]}{5[(\omega^2 - \omega_s^2)^2 + \omega^2 \omega_\nu^2]},$$

$$f_4 = \frac{1}{10}\omega_\nu(\omega^2 + \omega_m^2).$$

The coefficient γ_1 in equation (15) describes an enhancement of the scalar turbulent resistivity caused by the magnetic field, γ_2 represents a change of the resistivity component parallel to the magnetic field. Thus the electric resistivity (and, hence, the magnetic diffusivity) becomes anisotropic in the presence of the field. The coefficient γ_3 describes the drift of the magnetic field caused by a non-uniform distribution of the magnetic energy in the fluid. If $\gamma_3 > 0$, the field drifts away from the regions with higher B^2, and the velocity of this drift increases proportionally to ∇B^2. Note that expression (15) is the most general polar linear combination of terms that are third order in $\boldsymbol{B}$ that can be formed from the axial vector $\boldsymbol{B}$ and the polar vector ∇.

It may be convenient to describe diffusive processes in terms of components parallel and perpendicular to the magnetic field, $\eta_\|$ and $\eta_\perp$. Introducing the parallel and perpendicular components of $\nabla \times \boldsymbol{B}$ by

$$(\nabla \times \boldsymbol{B})_\| = \frac{\boldsymbol{B}}{B^2}[\boldsymbol{B} \cdot (\nabla \times \boldsymbol{B})], \quad (\nabla \times \boldsymbol{B})_\perp = \nabla \times \boldsymbol{B} - (\nabla \times \boldsymbol{B})_\|, \tag{17}$$

we can represent $\mathcal{E}$ as

$$\mathcal{E} = -\eta_\|(\nabla \times \boldsymbol{B})_\| - \eta_\perp(\nabla \times \boldsymbol{B})_\perp - \gamma_3 \nabla B^2 \times \boldsymbol{B}, \tag{18}$$

where $\eta_\| = \eta + (\gamma_1 - \gamma_2)B^2$ and $\eta_\perp = \eta + \gamma_1 B^2$ describe turbulent diffusion along and across the magnetic field, respectively. As generally $\gamma_2 \neq 0$, we have $\eta_\| \neq \eta_\perp$, so the magnetic field causes anisotropy of a turbulent fluid. Owing to this anisotropy, different components of the magnetic field can diffuse with different rates.

Following Rüdiger & Kitchatinov (1993), we can rewrite $\mathcal{E}$ in terms of a scalar diffusivity and a turbulent 'ambipolar' drift velocity, $\boldsymbol{U}_a$. Substituting

$$\boldsymbol{B}[\boldsymbol{B} \cdot (\nabla \times \boldsymbol{B})] = B^2 \nabla \times \boldsymbol{B} - \boldsymbol{B} \times [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}] \tag{19}$$

into equation (15), we have

$$\mathcal{E} = -\eta_\| \nabla \times \boldsymbol{B} + \boldsymbol{U}_d \times \boldsymbol{B}, \tag{20}$$

where the drift velocity $\boldsymbol{U}_d$ is given by

$$\boldsymbol{U}_d = -\gamma_3 \nabla B^2 + \boldsymbol{U}_a, \qquad \boldsymbol{U}_a = \gamma_2(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}. \tag{21}$$

The velocity $\boldsymbol{U}_a$, being perpendicular to both the magnetic field and the electrical current, governs a drift of the magnetic field that resembles very closely the ambipolar diffusion in a partially ionized plasma. However, the physical mechanism causing this drift is quite different. In a partially ionized plasma, the ambipolar drift is caused by an average motion of charged particles relative to neutrals and, because the magnetic field is associated with charged particles, it drifts relative to the plasma. In a turbulent fluid, the drift is entirely determined by the interaction of small-scale fluctuations with non-uniformities of the mean field. The mean field deflects turbulent motions in such a way that they generate the mean electromotive force that carries lines of the mean field in the direction of the Lorentz force. Turbulent currents play the role of charged particles in a partially ionized plasma, and interaction of these currents with the mean field leads to its drift relative to the fluid.

Obviously the representations (15), (18) and (20) are equivalent but describe the behaviour of the magnetic field in different terms.

4 DISCUSSION

Contrary to the widely accepted point of view, the influence of non-linear effects on turbulent transport processes cannot be described solely in terms of magnetic quenching of the α coefficient. Our calculations confirm this conclusion, first obtained by Roberts (1971), who proposed the general expression (15) for $\mathcal{E}$ but did not estimate the coefficients. The similar structure of the expression for $\mathcal{E}$ has also been obtained by Rüdiger & Kitchatinov (1993) in their model of 'original turbulence'. In addition to the anisotropy of the turbulent diffusivity, which seems to be quite natural for transport processes in the presence of a magnetic field (see e.g. Landau & Lifshitz 1979), non-linear effects result in a qualitatively new drift process given by the last term on the right-hand side of equation (15). This drift can be especially pronounced in regions with a strong inhomogeneity of the magnetic field. This effect is reminiscent of turbulent diamagnetism, which causes the large-scale field to be expelled from regions where the turbulence intensity is large. However, the non-linear behaviour of the magnetic field can be rather complex if one takes account of all non-linear terms in equation (15).

Note that our conclusion concerning the dependence of the turbulent magnetic diffusivity on $\boldsymbol{B}$ is at variance with earlier results by Gruzinov & Diamond (1994, 1995), who considered the non-linear turbulent transport coefficients for an incompressible fluid. Deriving their expressions for the diffusivity, the authors assumed that the second-order correlator of a fluctuating magnetic field, $\langle b_i(\boldsymbol{k}) b_j(\boldsymbol{k})\rangle$ does not depend on the mean magnetic field and, as a consequence, they concluded that η does not change in the presence of the mean magnetic field. Our direct calculations as well as the results by Rüdiger & Kitchatinov (1993) show that this assumption is incorrect.

The expressions for turbulent transport coefficients have an especially simple form if the sound frequency, ω_s, is much greater than any of the dissipative frequencies, ω_ν or ω_m. In this case, the spectral function of acoustic turbulence, $R(\omega, \boldsymbol{k})$, is to have a maximum near the acoustic frequency, $\omega \sim \omega_s$, and rapidly tends to 0 at $\omega \to 0$, because $\omega = 0$ corresponds to acoustic waves with an infinitely large wavelength. At $\omega \gg \max(\omega_\nu, \omega_m)$, the integrals (16) can be easily transformed by making use of one of the representations of δ functions,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2 + \varepsilon^2} = \pi \delta(x). \tag{22}$$

Introducing the magnetic Prandtl number, $P = \nu_m/\nu = \omega_m/\omega_\nu$, we have then for ξ_1

$$\xi_1 = \frac{P}{10\pi\rho}\int_{-\infty}^{+\infty} \frac{\omega_\nu R(\omega, \boldsymbol{k})}{\omega^2 + \omega_\nu^2} \cdot \frac{k^4(\omega^2 - P\omega_\nu^2)}{(\omega^2 + P^2\omega_\nu^2)^2} \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}$$
$$\approx \frac{P}{10\rho}\int_{-\infty}^{+\infty} \frac{k^4}{\omega^2} R(\omega, \boldsymbol{k}) \delta(\omega)\, \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}. \tag{23}$$

The quantity $k^4 R(\omega, \boldsymbol{k})/\omega^2$ characterizes the spectrum of displacements and has to vanish for displacements with infinite wavelength (or, in the case of acoustic turbulence, with $\omega \to 0$), thus $\xi_1 = 0$ in the limit of small viscosity. By analogy, it is easy to show that ξ_4 is proportional to the same spectral integral and hence should also vanish. Calculating the integral ξ_2, we can substitute

$$\frac{\omega_\nu}{(\omega^2+\omega_\nu^2)[(\omega^2-\omega_s^2)^2+\omega^2\omega_\nu^2]}=\frac{1}{\omega_s^2(2\omega^2-\omega_s^2)}\times\left[\frac{\omega^2\omega_\nu}{(\omega^2-\omega_s^2)^2+\omega^2\omega_\nu^2}-\frac{\omega_\nu}{\omega^2+\omega_\nu^2}\right]. \quad (24)$$

Then, ξ_2 can be transformed into

$$\xi_2=\frac{P}{15\pi\rho}\int_{-\infty}^{+\infty}\frac{k^4R(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}}{(\omega^2+P^2\omega_\nu^2)^2(\omega_s^2-2\omega^2)}\times\left[\frac{\omega_\nu}{\omega^2+\omega_\nu^2}-\frac{\omega^2\omega_\nu}{(\omega^2-\omega_s^2)^2+\omega^2\omega_\nu^2}\right]\times\{\omega^2(\omega^2-\omega_s^2)-\omega_\nu^2[P(\omega^2-\omega_s^2)+(1+P)\omega^2]\}. \quad (25)$$

Taking the limit $\omega_\nu\to 0$ and using the representation (22), we obtain

$$\xi_2=\frac{P}{15\pi\rho}\int_{-\infty}^{+\infty}\frac{k^4(\omega^2-\omega_s^2)R(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}}{\omega^2(\omega_s^2-2\omega^2)}\times[\delta(\omega)-\omega\delta(\omega^2-\omega_s^2)]. \quad (26)$$

The integral containing $\delta(\omega)$ vanishes because R/ω^2 tends to 0 at $\omega\to 0$, and the integral proportional to $\delta(\omega^2-\omega_s^2)$ gives zero because of the properties of the delta function. Therefore, $\xi_2=0$ and a non-zero contribution to the kinetic coefficients can be provided only by ξ_3. This integral can be calculated in the same fashion as ξ_2. Substituting expression (24) and taking the limit $\omega_\nu\to 0$, we obtain

$$\xi_3=\frac{1}{10\rho}\int_{-\infty}^{+\infty}\frac{k^4R(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}}{\omega^2(2\omega^2-\omega_s^2)}[(1+P)(\omega^2-\omega_s^2)+\omega^2]\times[\omega\delta(\omega^2-\omega_s^2)-\delta(\omega)]. \quad (27)$$

Using the properties of the delta function, we finally have

$$\xi_3\approx\frac{1}{10\rho c_s^2}\int_{-\infty}^{+\infty}k^2R(\omega_s,\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}. \quad (28)$$

Then the transport coefficients are

$$\gamma_1\approx\gamma_2\approx 4\gamma_3. \quad (29)$$

The electromotive force (15) can now be rewritten as

$$\mathcal{E}=-\eta\nabla\times\boldsymbol{B}+\gamma_1\left\{B^2\nabla\times\boldsymbol{B}+\boldsymbol{B}[\boldsymbol{B}\cdot(\nabla\times\boldsymbol{B})]-\tfrac{1}{4}\nabla B^2\times\boldsymbol{B}\right\}. \quad (30)$$

From the definition of the spectral function, we have

$$v^2=\int_{-\infty}^{+\infty}k^2R(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}, \quad (31)$$

where v is the characteristic turbulence velocity. Therefore, the order of magnitude estimate of the quantity γ_1B^2 characterizing the enhancement of diffusivity in the presence of the magnetic field reads

$$\gamma_1B^2\sim\frac{v^2B^2}{5\rho c_s^2\bar{\omega}_s}\sim\frac{2}{5}v\ell\frac{v}{c_s}\left(\frac{c_A}{c_s}\right)^2, \quad (32)$$

where $\bar{\omega}_s$ and ℓ are respectively the characteristic frequency and wavelength of the turbulence and c_A is the Alfvén velocity.

Note that the non-linear magnetic diffusivity may well exceed the ordinary turbulent diffusivity given by equation (14). Assuming that $\omega_s\gg\max(\omega_\nu,\omega_m)$ and using again the representation (22), we have

$$\eta\approx\frac{\pi}{3}\int_{-\infty}^{+\infty}k^2R(0,\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}+O(\nu_m). \quad (33)$$

As already mentioned, the spectral function tends to 0 at $\omega\to 0$ for the acoustic turbulence and, calculating η, one should generally take into account small terms proportional to ν_m. Therefore, η may be small compared with the non-linear diffusivity (32) if the magnetic field is sufficiently strong. In our simplified analysis, we used the assumption of statistically steady turbulence with the correlation properties given by (12) and did not specify the mechanism of maintaining the turbulence. Of course, viscous dissipation would cause the turbulence to die out and upset this assumption. However, the viscous decay time-scale (which is of the order of $1/\nu\ell^2$) must be rather long for small viscosity. Therefore, our consideration applies if non-linear diffusive processes are much faster than the viscous dissipation, $\gamma_1B^2\gg\nu$. Note, however, that in the case of a very small viscosity the turbulence will be highly intermittent; thus the second-order moments will describe it only qualitatively.

As $\gamma_1=\gamma_2$, the parallel diffusivity is scarcely influenced by the non-linear interactions and remains unchanged in our model, so $\eta_\parallel=\eta$. Hence, the component of the electric current parallel to the magnetic field diffuses with the same rate as in the case $\boldsymbol{B}\to 0$. On the other hand, the perpendicular diffusivity experiences a strong enhancement, so the component of the current perpendicular to $\boldsymbol{B}$ decays on a short time-scale. This resembles very closely the properties of transport processes in a partially ionized plasma where the magnetic field suppresses perpendicular diffusion. However, in that case the field experiences a Hall drift, which is perpendicular to both the magnetic field and the electric current. In a turbulent fluid this drift is negligible. Instead, the non-uniformity of the magnetic field causes a drift in the direction of decreasing magnetic pressure. Thus the turbulence tends to make the field distribution more homogeneous. The rate of drift is generally comparable to the rate of diffusion.

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REFERENCES

Brandenburg A., Nordlund A., Stein R., Torkelsson U., 1995, ApJ, 446, 741
Brandenburg A., Nordlund A., Stein R., Torkelsson U., 1996, ApJ, 458, L45
Glatzmaier G., Roberts P., 1995, Nat, 377, 203
Glatzmaier G., Roberts P., 1996, Sci, 274
Gruzinov A., Diamond P., 1994, Phys. Rev. Lett., 72, 1651
Gruzinov A., Diamond P., 1995, Phys. Plasma, 2
Krause F., Rädler K.-H., 1980, Mean-field magnetohydrodynamics and dynamo theory. Akademie-Verlag, Berlin
Kulsrud R. M., Anderson S. W., 1992, ApJ, 396, 606
Landau L., Lifshitz E., 1979, Physical kinetics, Pergamon Press, London
Moffatt H. K., 1978, Magnetic field generation in electrically conducting fluids. Cambridge University Press, Cambridge

Parker E. N., 1979, Cosmical magnetic fields. Clarendon Press, Oxford
Roberts P. H., 1971, in Reid W. H., ed., Lectures in Applied Mathematics 14, Mathematical Problems in the Geophysical Sciences. American Mathematical Society, Providence RI, p. 129
Rüdiger G., 1990, Geophys. Astrophys. Fluid Dyn., 50, 53
Rüdiger G., Kitchatinov L., 1993, A&A, 269, 581
Vainshtein S., Cattaneo F., 1992, ApJ, 393, 165

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