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Magnetic drift processes in differentially rotating turbulence

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Abstract. The mean electromotive force is considered in a differentially rotating fluid taking into account stretching of the turbulent magnetic field. Calculations are performed by making use of the second order correlation approximation. Non-uniformity of the angular velocity leads to specific drift processes in the azimuthal direction. Due to this drift the magnetic field can rotate with a somewhat different angular velocity than the fluid. Differential rotation can also lead in a new instability of a non-axisymmetric mean field. Regardless of the law of the differential rotation this instability can result in an exponential amplification of the field.

Key words: accretion, accretion disks – Magnetohydrodynamics (MHD) – turbulence – stars: magnetic fields – ISM: magnetic fields

1. Introduction

It is widely believed that a turbulent dynamo may be responsible for the origin of magnetic fields in various astrophysical bodies. Under certain conditions turbulent fluid motions can work in such a way that a magnetic field is generated from a weak seed field (e.g. Moffatt 1978, Parker 1979). Fluid motions that lack mirror symmetry seem to be well suited for a generation of a large scale magnetic field (Krause & Rädler 1980). The mirror symmetry of turbulence can be broken, for example, by the Coriolis force if the fluid rotates. If rotation is rigid and the angular velocity $\boldsymbol{\Omega} = \text{const}$, the mean turbulent electromotive force can conventionally be represented in the form

$$\boldsymbol{\mathcal{E}} = \langle \boldsymbol{v} \times \boldsymbol{b} \rangle = \alpha \boldsymbol{B} - \eta \nabla \times \boldsymbol{B} \quad + \text{anisotropies}, \qquad (1)$$

where $\boldsymbol{v}$ and $\boldsymbol{b}$ are the fluctuating components of the velocity and magnetic field, respectively, and $\boldsymbol{B}$ is the mean magnetic field; $\langle ... \rangle$ denote ensemble averaging. The coefficients η and α describe respectively dissipative and non-dissipative contributions to the mean electromotive force. The α-term is caused by a departure from mirror symmetry and may be responsible for turbulent dynamo. This coefficient vanishes if the angular velocity goes to zero, however, rotation alone is not sufficient to induce the α-effect. The condition $\alpha \neq 0$ requires also the presence of large scale inhomogeneity or anisotropy of the fluid. The particular source of inhomogeneity may be different (density stratification, gradient of turbulence intensity etc., see e.g. Krause & Rädler 1980) but it is necessary because the pseudoscalar α can be formed from the axial vector $\boldsymbol{\Omega}$ only as a scalar production of $\boldsymbol{\Omega}$ and some polar vector. If inhomogeneity is weak, the mean electromotive force reduces to the dissipative term alone.

The situation may be quite different if the fluid rotates differentially. In this case, rotation can be itself the source of a large scale inhomogeneity since $\nabla\Omega \neq 0$. Differential rotation changes the intrinsic properties of turbulence because gradients of the angular velocity enter the momentum equation together with the Coriolis force. These gradients can induce various drift processes. Those are similar to temperature and partial number density gradients that can are known to lead to thermal and ordinary diffusion, respectively. Therefore, the behaviour of the mean field in a differentially rotating turbulent fluid may be rather complex and may differ essentially from that predicted by the simplified Eq. (1). The drift processes induced by a differential rotation can be important for the evolution of the magnetic field in many astrophysical bodies: galaxies, accretion discs, convective zones of stars.

In the present paper we consider the influence of differential rotation on the mean electromotive force. For the sake of simplicity we deal with the case where gradients of mean turbulence characteristics are small compared to gradients of the angular velocity. This allows to concentrate upon the new qualitative effects caused by a differential rotation. The effects of a differential rotation on $\boldsymbol{\mathcal{E}}$ in the presence of inhomogeneous turbulence will be considered in a forthcoming paper. The paper is organized as follows. In Sect. 2 we derive the general expression for the electromotive force in a differentially rotating fluid by making use of the second order correlation approximation (e.g. Krause & Rädler 1980). The effects of the non-mirror symmetry caused by differential rotation and their influence on $\boldsymbol{\mathcal{E}}$ are considered in Sect. 3. In Sect. 4 we discuss the obtained results.

2. The turbulent electromotive force

In the second order correlation approximation the behaviour of mean quantities is governed by equations including non-linear effects in fluctuating terms, whilst the linearized equation are used for the fluctuating quantities. This approximation strictly

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applies in the case of turbulence with small ordinary and magnetic Reynolds numbers, Re and Re_m, respectively (note, however, that in this case the field generation is possible only if the magnetic Reynolds number based on the large-scale motion is large). This approximation may also be sufficiently accurate for an ensemble of magnetohydrodynamic waves with relatively high frequencies and small amplitudes when the so called Strouhal number is small, or $S = vt_0/\ell < 1$. Here t_0 and ℓ are the correlation time and the length-scale of turbulence, respectively. In our considerations below we always assume $S < 1$.

The magnetic field $\boldsymbol{\mathcal{B}}$ and the velocity $\boldsymbol{u}$ can be separated into the mean and fluctuating parts, $\boldsymbol{\mathcal{B}} = \boldsymbol{B} + \boldsymbol{b}$ and $\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{v}$, where $\boldsymbol{B}$ and $\boldsymbol{V}$ are the mean field and velocity, respectively. The induction equation for the fluctuating magnetic field reads

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{b}) + \nabla \times (\boldsymbol{v} \times \boldsymbol{B}). \tag{2}$$

In this equation, we neglect the non-linear term in fluctuating quantities in accordance with the spirit of the second order correlation approximation. We also neglect dissipative effects assuming that $\mathrm{Re}_m \gg 1$ for small-scale motions (but $S \ll 1$). We consider the rotating fluid with $\boldsymbol{V} = s\Omega \boldsymbol{e}_\varphi$ where $\Omega = \Omega(s, z)$ is the angular velocity; (s, φ, z) are the cylindrical coordinates; $\boldsymbol{e}_s$, $\boldsymbol{e}_\varphi$, and $\boldsymbol{e}_z$ are the unit vectors in the corresponding directions. Substituting $\boldsymbol{V} = s\Omega \boldsymbol{e}_\varphi$ into Eq. (2), we have

$$\frac{\partial \boldsymbol{b}}{\partial t} + \Omega \frac{\partial \boldsymbol{b}}{\partial \varphi} - s\boldsymbol{e}_\varphi (\boldsymbol{b} \cdot \nabla \Omega) = \boldsymbol{A}, \tag{3}$$

where

$$\boldsymbol{A} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla \cdot \boldsymbol{v}), \tag{4}$$

and

$$\frac{\partial \boldsymbol{b}}{\partial \varphi} = \boldsymbol{e}_s \frac{\partial b_s}{\partial \varphi} + \boldsymbol{e}_\varphi \frac{\partial b_\varphi}{\partial \varphi} + \boldsymbol{e}_z \frac{\partial b_z}{\partial \varphi}. \tag{5}$$

Introducing the shifted azimuthal coordinate ψ, with $d\psi = d\varphi - \Omega dt$, and making use of Fourier transformation in t for fluctuating quantities as in

$$\boldsymbol{b}(\boldsymbol{r}, t) = \int_{-\infty}^{+\infty} d\omega e^{i\omega t} \hat{\boldsymbol{b}}(\boldsymbol{r}, \omega), \tag{6}$$

where the subscript ω labels the Fourier amplitude, we can express $\hat{\boldsymbol{b}}(\boldsymbol{r}, \omega)$ in terms of $\hat{\boldsymbol{v}}(\boldsymbol{r}, \omega)$ from Eq. (3)

$$\hat{\boldsymbol{b}} = -\frac{i}{\omega}\hat{\boldsymbol{A}} - \frac{s}{\omega^2}\boldsymbol{e}_\varphi (\nabla\Omega \cdot \hat{\boldsymbol{A}}), \tag{7}$$

$$\hat{\boldsymbol{A}} = (\boldsymbol{B} \cdot \nabla)\hat{\boldsymbol{v}} - (\hat{\boldsymbol{v}} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla \cdot \hat{\boldsymbol{v}}). \tag{8}$$

Then the fluctuating magnetic field is given by

$$\boldsymbol{b}(\boldsymbol{r}, t) = -i \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} e^{i\omega t} \left(\hat{\boldsymbol{A}} - \frac{i}{\omega} \boldsymbol{e}_\varphi s \nabla\Omega \cdot \hat{\boldsymbol{A}} \right). \tag{9}$$

The first term in brackets represents the contribution to the fluctuating field caused by turbulent motions in the presence of a large scale frozen-in magnetic field, and the second term describes the effect of differential rotation. We are now in a position to express the turbulent electromotive force in terms of the velocity correlation tensor. Substituting $\boldsymbol{b}$ from Eq. (9) into Eq. (1) and taking into account that

$$\langle \hat{v}_i(\boldsymbol{r}, \omega) \hat{v}_j(\boldsymbol{r}, \omega') \rangle \propto \delta(\omega + \omega') \tag{10}$$

for quasi-stationary turbulence (see, e.g., Rüdiger & Kitchatinov 1993), we obtain

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_2, \tag{11}$$

where

$$\boldsymbol{\mathcal{E}}_1 = i \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \langle \hat{\boldsymbol{v}}(\boldsymbol{r}, \omega) \times \hat{\boldsymbol{A}}(\boldsymbol{r}, -\omega) \rangle \ , \tag{12}$$

$$\boldsymbol{\mathcal{E}}_2 = s\boldsymbol{e}_\varphi \times \int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2} \ \langle \hat{\boldsymbol{v}}(\boldsymbol{r}, \omega) [\nabla\Omega \cdot \hat{\boldsymbol{A}}(\boldsymbol{r}, -\omega)] \rangle \ . \tag{13}$$

If rotation is rigid, the electromotive force is represented by $\boldsymbol{\mathcal{E}}_1$ alone. Differential rotation produces an additional contribution to the electromotive force. The density stratification or the gradient of the turbulence intensity can influence the effect of differential rotation. However, the main purpose of the present study is to consider new qualitative effects caused by differential rotation. Therefore, in what follows, we will neglect inhomogeneities of density and turbulence intensity.

Calculating the electromotive force we assume that rotation is relatively slow, so $\Omega t_0 < 1$, and we restrict ourselves only to terms linear in Ω and $\nabla\Omega$. For this reason, we split the velocity and other Fourier amplitudes into two components, $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_0 + \hat{\boldsymbol{v}}_1$, with $\hat{\boldsymbol{v}}_0$ being the turbulent velocity in the absence of rotation, and $\hat{\boldsymbol{v}}_1$ being a small departure to $\hat{\boldsymbol{v}}_0$ caused by rotation. We assume that the "zeroth" order approximation describes isotropic turbulence with mirror symmetry where the correlation tensor given by

$$\langle v_{0i}(\boldsymbol{r}, \omega) v_{0j}(\boldsymbol{r}, -\omega) \rangle = \tfrac{1}{3} v_\omega^2(\boldsymbol{r}) \delta_{ij}. \tag{14}$$

For instance, the velocity field $\hat{\boldsymbol{v}}_0$ can be represented by an ensemble of sound or any other waves existing in a non-rotating fluid. Note that our consideration does not apply to inertial waves (see, e.g., Moffatt 1970) which can exist only in a rotating fluid.

Calculations of $\boldsymbol{\mathcal{E}}_2$ require only the zeroth order approximation in the turbulent velocity, thus we have for homogeneous turbulence

$$\boldsymbol{\mathcal{E}}_2 = -s\ell^2 \boldsymbol{e}_\varphi \times [\nabla(\boldsymbol{B} \cdot \nabla\Omega) - (\boldsymbol{B} \cdot \nabla)\nabla\Omega], \tag{15}$$

where

$$\ell^2 = \frac{1}{3} \int \frac{d\omega}{\omega^2} \hat{\boldsymbol{v}}^2(\boldsymbol{r}, \omega) \tag{16}$$

is the square of the correlation length of the turbulence. We assume that the spectral power of $\boldsymbol{v}_0$ goes to zero at low frequencies, thus there is no singularity in the integral (16). The component of the electromotive force (15) operates in the meridional plane and cannot produce an α-effect. It only leads to a drift of the magnetic field with the drift velocity being dependent on the rotation law.

3. The effect of non-mirror symmetry

Calculations of $\boldsymbol{\mathcal{E}}_1$ require a more detailed consideration taking into account a small departure from $\hat{\boldsymbol{v}}_0$. Generally, rotation can influence the properties of turbulence providing the non-mirror symmetry to its distribution. If one neglects this influence and substitutes the correlation tensor in the form (14) into Eq. (12) then, to zeroth order in Ω and $\nabla\Omega$, this part of the electromotive force vanishes. Note that we neglected the dissipative term in the induction Eq. (2) in our simplified analysis. Thus the term representing the turbulent diffusion should not appear in $\boldsymbol{\mathcal{E}}_1$. To obtain the expression for $\boldsymbol{\mathcal{E}}_1$ with the accuracy to linear terms in Ω and $\nabla\Omega$, we should take into account the influence of rotation on the turbulent velocity.

The momentum equation for the fluctuating velocity reads

$$\frac{\partial \boldsymbol{v}}{\partial t}+\Omega\frac{\partial \boldsymbol{v}}{\partial \varphi}+2\boldsymbol{\Omega}\times\boldsymbol{v}+\boldsymbol{e}_\varphi s(\boldsymbol{v}\cdot\nabla\Omega)+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\frac{\nabla p}{\rho}, \quad (17)$$

where p is the fluctuating component of the pressure. In this equation we neglect viscosity, which is usually unimportant in this kind of consideration. Following the spirit of the second order correlation approximation, we will also neglect the non-linear advective term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ in Eq. (2). Introducing the shifted time coordinate τ and making Fourier transformation in τ, we obtain

$$i\omega\hat{\boldsymbol{v}}+2\boldsymbol{\Omega}\times\hat{\boldsymbol{v}}+\boldsymbol{e}_\varphi s(\hat{\boldsymbol{v}}\cdot\nabla\Omega)=-\frac{\nabla\hat{p}}{\rho}. \quad (18)$$

Eq. (18) can be solved by making use of the method similar to the approach of "original turbulence" suggested by Rüdiger (1989; see also Rüdiger & Kitchatinov 1993). However, instead of introducing a rather uncertain non-potential part of the random force ($\hat{\boldsymbol{f}}^s$ in the notation of Rüdiger & Kitchatinov 1993), which is not influenced by rotation according to the assumption, we calculate this force assuming that rotation is relatively slow and the perturbations of turbulent velocity caused by rotation are small. Under this assumption, calculations can be done by making use of a standard perturbation procedure. The unperturbed velocity, $\boldsymbol{v}_{0\omega}$, is governed by the momentum Eq. (17) with $\boldsymbol{\Omega}=0$. The perturbation $\hat{\boldsymbol{v}}_1$ may generally have non-mirror symmetry caused by rotation. Solving the momentum equation for $\hat{\boldsymbol{v}}_1$, we have

$$\hat{\boldsymbol{v}}_1=\frac{i}{\omega}\left[\frac{\nabla\hat{p}_1}{\rho}+2\boldsymbol{\Omega}\times\hat{\boldsymbol{v}}_0+\boldsymbol{e}_\varphi s(\hat{\boldsymbol{v}}_0\cdot\nabla\Omega)\right]. \quad (19)$$

Note that Eq. (19) applies only for a slow rotation with $\Omega t_0<1$ when the perturbation of velocity is small, $\hat{v}_0>\hat{v}_1$.

Restricting ourselves to the terms linear in Ω and $\nabla\Omega$, we obtain

$$\boldsymbol{\mathcal{E}}_1=i\int\frac{d\omega}{\omega}\langle\hat{\boldsymbol{v}}_0(\boldsymbol{r},\omega)\times\hat{\boldsymbol{A}}_1(\boldsymbol{r},-\omega) +\hat{\boldsymbol{v}}_1(\boldsymbol{r},\omega)\times\hat{\boldsymbol{A}}_0(\boldsymbol{r},-\omega)\rangle. \quad (20)$$

Substituting $\hat{\boldsymbol{v}}_1$ given by Eq. (19) into Eq. (20), we represent the expression for $\boldsymbol{\mathcal{E}}_1$ in terms of various correlation functions of second order. Apart from the unperturbed velocity correlators, $\langle\hat{v}_{0i}(\boldsymbol{r},\omega)\hat{v}_{0j}(\boldsymbol{r},-\omega)\rangle$, which are assumed to be given by Eq. (14), $\boldsymbol{\mathcal{E}}_1$ also contains correlators of the type $\langle\hat{v}_{0i}\hat{p}_1\rangle$. All these correlators have to vanish because of the properties of symmetry. Actually, they are components of a polar vector since $\hat{\boldsymbol{v}}$ is a polar vector and $\hat{p}$ is a scalar. On the other hand, these correlators should be proportional to Ω or $\nabla\Omega$ because $\hat{p}_1$ is a small perturbation induced by rotation. In our simplified model turbulence is assumed to be homogeneous. Therefore there are no other vectors characterizing the fluid except $\boldsymbol{\Omega}$ and $\nabla\Omega$. Hence, all correlators $\langle\hat{v}_{0i}\hat{p}_1\rangle$ should be equal to zero.

Using Eq. (14), non-zero terms in $\boldsymbol{\mathcal{E}}_1$ can be represented in the form

$$\boldsymbol{\mathcal{E}}_1=\ell^2\{2\boldsymbol{B}\times(\nabla\times\boldsymbol{\Omega})-\nabla\Omega\times(\partial\boldsymbol{B}/\partial\varphi)+4\Omega\nabla B_z -s\boldsymbol{e}_\varphi\times[(\boldsymbol{B}\cdot\nabla)\nabla\Omega-(\nabla\Omega\cdot\nabla)\boldsymbol{B}]+4(\boldsymbol{B}\cdot\nabla)\boldsymbol{\Omega}\}. \quad (21)$$

Substituting this expression into $\boldsymbol{\mathcal{E}}$, excluding the gradient terms which lead to polarization of the fluid but do not influence the magnetic evolution, and taking into account that $\nabla\times\boldsymbol{\Omega}=-\boldsymbol{e}_\varphi(\partial\Omega/\partial s)$, we finally have after some arrangements

$$\boldsymbol{\mathcal{E}}=\ell^2\left\{\frac{\partial\Omega}{\partial s}\,\boldsymbol{e}_\varphi\times\boldsymbol{B}-s\boldsymbol{e}_\varphi\times(\boldsymbol{B}\cdot\nabla)\nabla\Omega -\nabla\Omega\times\frac{\partial\boldsymbol{B}}{\partial\varphi}-s\nabla\Omega(\nabla\times\boldsymbol{B})_\varphi\right\}. \quad (22)$$

All terms on the r.h.s. of Eq. (22) contain derivatives of the angular velocity; hence the electromotive force vanishes for the rigid rotation. Note that our expression for $\boldsymbol{\mathcal{E}}$ does not contain dissipative effects, because they were excluded from the consideration at the very beginning.

4. Discussion

In this section we discuss the physical meaning of different terms in Eq. (22) and their contribution to the evolution of the mean field. In the induction equation for the mean field, the electromotive force $\boldsymbol{\mathcal{E}}$ should be added to the electromotive force induced by the mean motion, $\boldsymbol{V}\times\boldsymbol{B}$.

The behaviour of the mean field is determined by $\nabla\times\boldsymbol{\mathcal{E}}$ which can be represented as

$$\nabla\times\boldsymbol{\mathcal{E}}=\boldsymbol{e}_\varphi\ell^2 L(B_s,B_z)-\frac{W_i}{s}\frac{\partial\boldsymbol{B}}{\partial\varphi}-\boldsymbol{e}_\varphi\frac{W_a}{s}\frac{\partial B_\varphi}{\partial\varphi}+ \ell^2\left\{\boldsymbol{e}_s\frac{\partial\Omega}{\partial z}\frac{\partial^2 B_z}{\partial s\partial\varphi}+\boldsymbol{e}_z\frac{\partial\Omega}{\partial s}\frac{\partial^2 B_s}{\partial z\partial\varphi}\right\}+ \ell^2\left\{\boldsymbol{e}_s\frac{\partial\Omega}{\partial s}\frac{\partial^2 B_s}{\partial s\partial\varphi}+\boldsymbol{e}_z\frac{\partial\Omega}{\partial z}\frac{\partial^2 B_z}{\partial z\partial\varphi}+\boldsymbol{e}_\varphi(\nabla\Omega\cdot\nabla)\frac{\partial B_\varphi}{\partial\varphi}\right\}. \quad (23)$$

Here we have introduced the drift velocities

$$W_i=-s\ell^2\left(\frac{\partial^2\Omega}{\partial s^2}+\frac{\partial^2\Omega}{\partial z^2}\right),\; W_a=\ell^2\frac{\partial\Omega}{\partial s}. \quad (24)$$

The linear operator $L(B_s,B_z)$ acts only on the poloidal field components,

$$L(B_s,B_z)\;=\;\left(\frac{\partial\Omega}{\partial z}\frac{\partial}{\partial s}-\frac{\partial\Omega}{\partial s}\frac{\partial}{\partial z}\right)s(\nabla\times\boldsymbol{B})_\varphi$$

$$-\frac{\partial}{\partial z}\left(s\frac{\partial^2\Omega}{\partial s\partial z}B_s\right)-\frac{\partial}{\partial s}\left(s\frac{\partial^2\Omega}{\partial s\partial z}B_z\right)$$
$$+s\boldsymbol{B}\cdot\nabla\left(\frac{1}{s}\frac{\partial\Omega}{\partial s}\right)-\frac{\partial}{\partial s}\left(s\frac{\partial^2\Omega}{\partial s^2}B_s\right)$$
$$-\frac{\partial}{\partial z}\left(s\frac{\partial^2\Omega}{\partial z^2}B_z\right). \quad (25)$$

This operator describes the generation of a toroidal magnetic field from a poloidal one. Note that the stretching term due to the mean motion leads to a qualitatively similar but usually much more efficient generation of azimuthal field.

The second and third terms on the r.h.s. of Eq. (23) have an obvious interpretation: they describe the drift of different components of the magnetic field in the φ-direction. All three field components experience the drift with the velocity W_i. This velocity depends generally on s and z, so, at any point, the magnetic field rotates with a somewhat different angular velocity than the fluid. Of course, the drift in the φ-direction can manifest itself only for non-axisymmetric magnetic configurations. If Ω depends on the cylindrical radius alone, and if $\partial^2\Omega/\partial s^2 > 0$, then $W_i < 0$, and the field lags behind the fluid. Conversely, if $\partial^2\Omega/\partial s^2 < 0$ then the field advances the fluid. Since both Ω and W_i do not depend on φ in our simplified analysis, the azimuthal shift between the field and the fluid, $\Delta\varphi$, increases linearly with time,

$$\Delta\varphi=\ell^2\left(\frac{\partial^2\Omega}{\partial s^2}+\frac{\partial^2\Omega}{\partial z^2}\right)t, \quad (26)$$

and can reach a large value if the fluid has spun for a sufficiently long time.

The drift associated with the velocity W_i is the same for all components of the magnetic field. On the other hand, the drift in the φ-direction caused by the term proportional to W_a is anisotropic: only the φ-component experiences this drift. Apart from being involved in the mean motion and the isotropic azimuthal drift with the velocities $\boldsymbol{V}$ and W_i, respectively, the azimuthal field moves in the φ-direction with the velocity W_a. This anisotropic drift is determined by the first derivative of the angular velocity. Note that the two drift velocities, W_i and W_a, can generally be comparable in astrophysical bodies.

In a turbulent fluid differential rotation can induce not only drift processes, but it can also lead, under some conditions, to coupling between different components of the magnetic field. The fourth term on the r.h.s. of Eq. (23) provides an example of such coupling. Like other effects caused by differential rotation this coupling can manifest itself only if the magnetic field is non-axisymmetric. Coupling results in an exchange of energy between B_s and B_z on a time scale $\sim\Omega^{-1}(L/\ell)^2$, where L is an outer length scale, which can be the size of the body, for example. Since $L\gg\ell$, this time scale is long compared to the rotation period. Generally, the azimuthal field is also involved in the exchange of energy due to the stretching term in the mean field induction equation, although it does not exert a feedback on the meridional field. Therefore the link between the field components owing to differential rotation is quite different from that caused by the alpha-effect.

The electromotive force induced by differential rotation may drive a particular sort of waves in a fluid. These waves exist due to the joint action of coupling and anomalous diffusion of a non-axisymmetric field represented by the last term in Eq. (23). For the purpose of illustration we consider the waves in the idealized case $\nabla\Omega$=const and neglect dissipative effects. Under these assumptions, the mean field induction equation for the poloidal field components reads

$$\frac{\partial B_s}{\partial t}=\ell^2\frac{\partial\Omega}{\partial z}\frac{\partial^2 B_z}{\partial s\partial\varphi}+\ell^2\frac{\partial\Omega}{\partial s}\frac{\partial^2 B_s}{\partial s\partial\varphi}, \quad (27)$$

$$\frac{\partial B_z}{\partial t}=\ell^2\frac{\partial\Omega}{\partial s}\frac{\partial^2 B_s}{\partial z\partial\varphi}+\ell^2\frac{\partial\Omega}{\partial z}\frac{\partial^2 B_z}{\partial z\partial\varphi}. \quad (28)$$

Oscillations of the poloidal field components are accompanied by oscillations of the azimuthal field due to the stretching effect. However, B_φ does not influence the behaviour of B_s and B_z, so the dispersion properties of waves are entirely determined by Eqs. (27) and (28). In our consideration the coefficients in Eqs. (27) and (28) are constant, so the solution can be taken in the form $\exp(\gamma t-i\boldsymbol{q}\cdot\boldsymbol{r}-im\psi)$, where $\boldsymbol{q}=(q_s,0,q_z)$ is the poloidal wave vector. Substituting this solution into Eqs. (27) and (28) we obtain the dispersion equation,

$$\gamma=-m\ell^2(\boldsymbol{q}\cdot\nabla\Omega). \quad (29)$$

Obviously, the waves should have similar properties in the case of a more complex rotation law, provided the wavelength is small compared to the the outer scale, i.e. $qL\gg 1$. The only difference is that the field can experience a drift, if the rotation law is sufficiently complex. Thus one should add the drift terms on the r.h.s. of Eqs. (27) and (28). The azimuthal drift of a non-axisymmetric magnetic field adds an imaginary contribution to γ, so

$$\gamma=-im\ell^2\left(\frac{\partial^2\Omega}{\partial s^2}+\frac{\partial^2\Omega}{\partial z^2}\right)-m\ell^2(\boldsymbol{q}\cdot\nabla\Omega). \quad (30)$$

Due to the drift, the solution takes the form of travelling waves. In both cases the instability of the mean field arises if $m(\boldsymbol{q}\cdot\nabla\Omega)<0$. This instability is driven by differential rotation and is qualitatively different from that associated with the alpha-effect. However, this instability resembles in some ways the dynamo effect due to Rädler (1968; see also Krause & Rädler 1980), which is proportional to $\boldsymbol{\Omega}\times\boldsymbol{J}$, where $\boldsymbol{J}$ is the electric current. The two effects have in common that the important term in the electromotive force involves derivatives of $\boldsymbol{B}$, and not $\boldsymbol{B}$ itself.

The electromotive force driven by differential rotation can play an important role in the evolution of the magnetic field even if one takes into account dissipative effects. The rate of dissipative processes is determined by the turbulent magnetic diffusivity η,

$$\eta=\frac{1}{3}\int_{-\infty}^{+\infty}\frac{\omega_m E(\boldsymbol{k},\omega)}{\omega^2+\omega_m^2}d\omega d\boldsymbol{k}, \quad (31)$$

where $\omega_m=\nu_m k^2$ with ν_m being the magnetic diffusivity, and $E(\boldsymbol{k},\omega)$ is the spectrum function of turbulence (see, e.g., Krause

& Rädler 1980, Kitchatinov et al. 1994). The spectrum function is related to the turbulent velocity by

$$v^2 = \int_{-\infty}^{+\infty} E(\boldsymbol{k},\omega) d\omega d\boldsymbol{k}. \quad (32)$$

For acoustic turbulence the spectrum function usually goes to 0 when $\omega \to 0$. Therefore, when calculating η one must take into account the contribution of magnetic diffusivity. Since in our model the magnetic Reynolds number is large, $\mathrm{Re}_m \gg 1$, we can restrict ourselves to terms linear in ν_m and neglect in the integral (31) ω_m^2 compared to ω^2. Introducing the average dissipative frequency ω_d characterizing the ensemble of acoustic waves, we have

$$\eta \approx \frac{1}{3}\omega_d \int_{-\infty}^{+\infty} \frac{E(\boldsymbol{k},\omega)}{\omega^2} d\omega d\boldsymbol{k} \sim \omega_d \ell^2. \quad (33)$$

The effects caused by differential rotation dominates the electromotive force if

$$m\ell^2 \mid \boldsymbol{q} \cdot \nabla\Omega \mid > \eta \left(q^2 + \frac{m^2}{s^2} \right). \quad (34)$$

For $q \sim m/s$ we simply have $s \mid \nabla\Omega \mid > \omega_d$. This condition can hold for many cases of astrophysical interest.

5. Conclusions

In the present paper we have established the general form of the expression for the electromotive force governing the effects of differential rotation on the mean magnetic field in a turbulent medium. Here we have adopted acoustic turbulence as a model for the small scale motions, so the form of the expression for the electromotive force may change in the case of ordinary vortical turbulence. The basic elements from which this expression is constructed are linear combinations of the vectors $\boldsymbol{B}$, $\boldsymbol{\nabla}\Omega$ and $\boldsymbol{e}_\phi$ together with the $\boldsymbol{\nabla}$ operator. The mean electromotive force contains only terms proportional to $\boldsymbol{\nabla}\Omega$ and not to $\boldsymbol{\Omega}$, because the turbulence is assumed to be homogeneous, so there are neither α nor $\Omega \times \boldsymbol{J}$ effects. In tensor form, components of the polar vector $\boldsymbol{\mathcal{E}}$ can be formed from the spatial derivatives of $\boldsymbol{V}$ and $\boldsymbol{B}$, for instance as a linear combination of

$$\varepsilon_{ijp} V_{p,k} B_{j,k}, \quad (35)$$

and the corresponding terms with transposed indices. Taking into account that $V_p = \varepsilon_{pmn}\Omega_m x_n$ in a rotating fluid one can obtain the general expression for electromotive force in terms of $\Omega_{i,j}$ and $B_{k,l}$.

The significance of the new effect found here cannot be fully assessed unless we can calculate all other effects that would come into play, such as anisotropic turbulent diffusion and α-effect for the same turbulence model. Nevertheless, the new terms point towards the possibility of an instability that could destabilise even the field-free state for sufficiently strong differential rotation. This could then constitute an additional source of large scale field generation. However, our preliminary estimates suggest that $s|\boldsymbol{\nabla}\Omega/\Omega|$ has to be of order unity for interesting things to happen. For ordinary vortical turbulence with $\Omega t_0 \geq 1$, the basic turbulence is to be affected by the differential rotation and our perturbative approach will break down. Nevertheless, it is likely that the general form of the expression is not going to change and that our results may still be valid qualitatively. Our results would therefore provide a good framework for analysing numerical simulations of differentially rotating turbulence to get independent quantitative estimates for the various turbulent transport coefficients.

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