Mon. Not. R. Astron. Soc. **294,** 399–406 (1998)

Magnetic and vertical shear instabilities in accretion discs

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Accepted 1997 September 2. Received 1997 June 19; in original form 1997 March 10

ABSTRACT
The stability properties of magnetized discs rotating with angular velocity $\Omega = \Omega(s, z)$, dependent on *both* the radial and the vertical coordinates s and z, are considered. Such a rotation law is adequate for many astrophysical discs (e.g., galactic and protoplanetary discs, as well as accretion discs in binaries). In general, the angular velocity depends on height, even in thin accretion discs. A linear stability analysis is performed in the Boussinesq approximation, and the dispersion relation is obtained for short-wavelength perturbations. Any dependence of Ω on z can destabilize the flow. This concerns primarily small-scale perturbations for which the stabilizing effect of buoyancy is strongly suppressed due to the energy exchange with the surrounding plasma. For a weak magnetic field, instability of discs is mainly associated with vertical shear, whilst for an intermediate magnetic field the magnetic shear instability, first considered by Chandrasekhar and Velikhov, is more efficient. This instability is caused by the radial shear which is typically much stronger than the vertical shear. Therefore the growth time for the magnetic shear instability is much shorter than for the vertical shear instability. A relatively strong magnetic field can suppress both these instabilities. The vertical shear instability could be the source of turbulence in protoplanetary discs, where the conductivity is low.

Key words: accretion, accretion discs – instabilities – magnetic fields – MHD.

1 INTRODUCTION

The standard model of accretion discs requires sufficiently strong turbulence to enhance the efficiency of angular momentum transport, because the molecular viscosity alone is extremely inefficient. At present, there is no commonly accepted view point as to how a laminar flow is disrupted and turbulence generated. In general, turbulence may be generated due to various magnetohydrodynamic instabilities that can arise in differentially rotating non-uniform gaseous discs. Although the linear stability properties of discs are well studied the exact origin of turbulence is still controversial. Significant progress was made over the last few years in understanding that the magnetic field may be important for causing the onset of turbulence in some cases. In the presence of a magnetic field, one of the candidates for the origin of turbulence is the magnetic shear instability first considered by Velikhov (1959) and Chandrasekhar (1960). This instability was analysed in detail for stellar conditions (Fricke 1969; Acheson 1978, 1979; Balbus 1995; Urpin 1996). Here the instability can arise if the angular velocity decreases from the pole to the equator. This instability may also operate in protoplanetary discs, although it is possibly not the main mechanism for driving turbulence because of the low conductivity of the gas in these objects (Safronov 1969). There are strong arguments that the magnetic shear instability can arise in accretion discs where the necessary condition $\partial\Omega/\partial s < 0$ (i.e. a decrease of the angular velocity with cylindrical radius) is fulfilled (Balbus & Hawley 1991;, Kaisig, Tajima & Lovelace 1992; Kumar, Coleman & Kley 1994; Zhang, Diamond & Vishniac 1994). The instability exists not only for short-wavelength perturbations, but also for global modes with scales comparable to the disc height (Curry, Pudritz & Sutherland 1994; Curry & Pudritz 1995). Note that the magnetic shear instability can arise only in a restricted range of the magnetic field strength. The field should not be too strong, because this would suppress the instability (Urpin 1996; Kitchatinov & Rüdiger 1997). On the other hand, the field should not be too weak because the growth time becomes too long in this case. Several recent papers (Hawley, Gammie & Balbus 1995; Matsumoto & Tajima 1995; Brandenburg et al. 1995; Torkelsson et al. 1996) have dealt with the non-linear regime of the magnetic shear instability. Those simulations show that the generated turbulence may be sufficient to produce an enhanced effective viscosity. The effective Shakura–Sunyaev alpha viscosity parameter is on average of the order 0.01, but this value could be larger in the presence of an externally imposed magnetic field and for higher numerical resolution (Brandenburg et al. 1996; Hawley, Gammie & Balbus 1996; Stone et al. 1996). Also, deviations from Keplerian angular velocity could change this value (Abramowicz, Brandenburg & Lasota 1996).

The situation is much more uncertain in non-magnetized discs since, if rotation is strictly Keplerian, i.e. $\Omega \propto s^{-3/2}$, the disc seems to be linearly stable. This is in accordance with the well-known Rayleigh criterion of stability (Rayleigh 1880),

$$\partial(s^4\Omega^2)/\partial s > 0, \tag{1}$$

where Ω is the angular velocity and s is the cylindrical radius. Although this criterion is only valid for axisymmetric perturbations, Stewart (1975) showed that this is also true of non-axisymmetric perturbations. Of course, this consideration does not concern discs with unstable stratification, which can be subject to convective instability (Kley, Papaloizou & Lin 1993). The Rayleigh criterion does not apply, however, to real astrophysical discs where rotation depends on both the radial and the vertical coordinates. Hydrodynamical equilibrium in the radial and vertical directions can only be satisfied if Ω depends on both s and z, where z is the vertical coordinate (see, e.g., Urpin 1984; Kley & Lin 1992). The dependence of Ω on z is relatively weak for thin accretion discs, but is much more pronounced for thicker discs, such as galactic and protoplanetary discs. In any case, however, such a dependence changes drastically the stability properties, because the flow is then locally unstable to axisymmetric perturbations for any sign of $\partial\Omega/\partial z$. This type of instability is closely related to the well-known Goldreich–Schubert (1967) instability which can arise in the radiative zones of differentially rotating stars. The main difference from stellar conditions is a strong stabilizing influence from the Coriolis force, if the rotation law is close to a Keplerian one. In the absence of a magnetic field, instability associated with vertical shear may be sufficiently effective to drive turbulence and to enhance the effective viscosity.

In the present paper we consider the linear stability properties of magnetized discs with angular velocity being dependent on both the vertical and the radial coordinates. We treat the behaviour of different short-wavelength magnetohydrodynamic modes that can exist in such objects and determine the parameter domain in which these modes are unstable.

The paper is organized as follows. In Section 2, the main equations are presented and a dispersion relation is derived that describes the behaviour of short-wavelength perturbations in the Boussinesq approximation. The stability criteria for different modes are discussed in Section 3. Finally, our results are briefly summarized in Section 4.

2 DISPERSION RELATION FOR SHORT-WAVELENGTH PERTURBATIONS

Consider a magnetized axisymmetric disc of finite vertical extent. The unperturbed angular velocity can generally depend on both s and z, so $\Omega = \Omega(s, z)$, where (s, φ, z) are cylindrical coordinates. The magnetic field, $\boldsymbol{B} = (B_s, B_\varphi, B_z)$, is assumed to be weak in the sense that the Alfvén speed, c_A, is small compared with the sound speed, c_s. This enables us to employ the Boussinesq approximation for a consideration of slow magnetoacoustic waves. We consider axisymmetric short-wavelength perturbations with the space–time dependence $\exp(\mathrm{i}\omega t - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$, where $\boldsymbol{k} = (k_s, 0, k_z)$ is the wavevector, and $\boldsymbol{k}\cdot\boldsymbol{r} \gg 1$. Small perturbations will be indicated by subscript 1, whilst unperturbed quantities will have no subscript, except for indicating vector components when necessary. In the unperturbed state, the disc is assumed to be in hydrostatic equilibrium in the s- and z-directions,

$$\frac{\nabla p}{\rho} = \boldsymbol{G} + \frac{1}{4\pi\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\ , \quad \boldsymbol{G} = \boldsymbol{g} + \Omega^2\boldsymbol{s}, \tag{2}$$

where $\boldsymbol{g}$ is the gravity of the central object. As usual, self-gravity of the disc is neglected. If $c_\mathrm{s} > c_\mathrm{A}$, the unperturbed Lorentz force is negligible compared with the pressure force in equation (2), thus the disc structure is mainly determined by the balance between gravity and centrifugal force.

We use the Boussinesq approximation since the e-folding time of short-wavelength instabilities associated with shear is typically much longer than the period of sound wave with the same wavelength. The linearized momentum, continuity and thermal balance equations governing the behaviour of small perturbations in this approximation read

$$\mathrm{i}\omega\boldsymbol{V}_1 + 2\boldsymbol{\Omega}\times\boldsymbol{V}_1 + \boldsymbol{e}_\varphi s(\boldsymbol{V}_1\cdot\nabla)\Omega = \frac{\mathrm{i}\boldsymbol{k}p_1}{\rho} - \beta\boldsymbol{G}T_1 + \frac{\mathrm{i}}{4\pi\rho}[\boldsymbol{k}(\boldsymbol{B}\cdot\boldsymbol{B}_1) - \boldsymbol{B}_1(\boldsymbol{k}\cdot\boldsymbol{B})], \tag{3}$$

$$\mathrm{i}\omega\boldsymbol{B}_1 = -\mathrm{i}\boldsymbol{V}(\boldsymbol{k}\cdot\boldsymbol{B}) + s\boldsymbol{e}_\varphi(\boldsymbol{B}_1\cdot\nabla\Omega), \tag{4}$$

$$\boldsymbol{k}\cdot\boldsymbol{V}_1 = 0, \tag{5}$$

$$\boldsymbol{k}\cdot\boldsymbol{B}_1 = 0, \tag{6}$$

$$\mathrm{i}\omega T_1 + \boldsymbol{V}_1\cdot(\Delta\nabla T) = -\chi k^2 T_1, \tag{7}$$

where $\boldsymbol{V}_1$, $\boldsymbol{B}_1$, p_1 and T_1 are perturbations of the hydrodynamic velocity, magnetic field, pressure and temperature, respectively; $\beta = -(\partial\ln\rho/\partial T)_p$ is the thermal expansion coefficient and χ is the thermal diffusivity; $(\Delta\nabla T) = \nabla T - \nabla_\mathrm{ad}T$ is the difference between the actual and adiabatic temperature gradients; we denote by $\boldsymbol{e}_\varphi$ the unit vector in the azimuthal direction. The set of equations (3)–(7) is written in the inertial frame and the term $2\boldsymbol{\Omega}\times\boldsymbol{V}_1$ in the momentum equation (3) originates from the inertial force. In equation (3) it is assumed that the density perturbation in the buoyancy force is mainly determined by the temperature perturbation, thus $\rho_1 = -\rho\beta T_1$, in accordance with the idea of the Boussinesq approximation. We also neglect viscous stresses in equation (3). This is justified for perturbations with $\omega \gg \nu k^2$, where ν is the kinematic viscosity. The magnetic field is assumed to be 'frozen' into the disc plasma and dissipative effects are neglected in equation (4). Note the absence of terms proportional to p_1 in the thermal balance equation (7), since their contribution is negligible in the Boussinesq approximation. In discs perturbations are generally non-adiabatic and the effect of the radiative heat transfer has to be taken into account in equation (5). In the calculations presented here the disc is assumed to be optically thick.

The general dispersion equation governing the behaviour of perturbations may be obtained from equations (3)–(5) in the standard way. Equating the determinant of the set of equations (3)–(7) to zero, we obtain

$$(\omega^2 - \omega_\mathrm{A}^2)^2 - (\omega^2 - \omega_\mathrm{A}^2)\left[\frac{\mathrm{i}\omega\ \omega_g^2}{\mathrm{i}\omega + \omega_\chi} + Q^2\right] - 4\Omega^2\omega_\mathrm{A}^2\frac{k_z^2}{k^2} = 0, \tag{8}$$

where

$$Q^2 = 4\Omega^2\frac{k_z^2}{k^2} + 2\Omega s\frac{k_z}{k^2}\left(k_z\frac{\partial\Omega}{\partial s} - k_s\frac{\partial\Omega}{\partial z}\right),$$

$$\omega_g^2 = -\beta(\Delta\nabla T)\cdot\left[\boldsymbol{G} - \frac{\boldsymbol{k}}{k^2}(\boldsymbol{k}\cdot\boldsymbol{G})\right],$$

and ω_g is the frequency of buoyancy waves; $\omega_\mathrm{A} = (\boldsymbol{k}\cdot\boldsymbol{B})/\sqrt{4\pi\rho}$ is the Alfvén frequency; and $\omega_\chi = \chi k^2$ is the inverse time-scale of dissipation due to the thermal conductivity. Since we use the Boussinesq approximation, the fast magnetoacoustic waves are excluded from our analysis and equation (8) describes only five low-frequency modes: two pairs of slow magnetoacoustic and buoyancy waves and a thermal mode associated with heat transport. In equation (8), the term proportional to Q^2 represents the effects associated with the angular velocity and its gradient, the term containing ω_g^2 is caused by the buoyancy force and the energy exchange between the perturbations and surrounding plasma, and the last term on the left-hand side describes coupling between the modes that originate from the joint action of the Coriolis and Lorentz forces.

In a stratified inviscid flow the buoyancy force acts as a stabilizing factor if the temperature gradient is subadiabatic. However, stabilization may be substantially reduced if heat is transported sufficiently fast. This effect is of particular importance for short-wavelength perturbations because the 'thermal frequency', ω_χ, increases rapidly with the wavevector $\boldsymbol{k}$. In the limiting case of very short wavelengths ($k \rightarrow \infty$), when perturbations are practically isothermal, the stabilizing effect of buoyancy is completely suppressed. Conversely, this stabilizing effect is maximal for adiabatic perturbations with $\omega \gg \omega_\chi$.

For the particular case of $B = 0$ and small buoyant response, equation (8) yields the criterion of the shear instability first derived by Goldreich & Schubert (1967), $Q^2 < 0$. If rotation is constant on cylinders the necessary condition of instability reduces to the well-known Rayleigh criterion (1). If $B \neq 0$, but the vertical shear and thermal effects are negligible, equation (8) reduces to the equation considered by Balbus & Hawley (1991) in their analysis of stability of magnetized discs.

3 CRITERIA OF INSTABILITY

In the present paper we deal with the stability properties of slow magnetoacoustic and buoyancy waves. The dispersion relation (8) in the general form can be solved only numerically. Therefore we consider different particular cases. The stability properties of waves are very sensitive to the efficiency of heat transport and, hence, to the relationship between ω and ω_χ. We consider two limiting cases of $\omega \gg \omega_\chi$ (adiabatic limit) and $\omega \ll \omega_\chi$ (isothermal limit). Which limit is more suitable for discs can easily be understood from the following estimate of ω_χ. Since $\chi = \kappa/\rho c_p$, where κ is the radiative thermal conductivity and c_p is the specific heat at a constant pressure, one has

$$\omega_\chi = \frac{\kappa}{\rho c_p H_T^2}(kH_T)^2.$$

Here H_T is the temperature scaleheight. The thermal conductivity can be expressed in terms of the heat flux, $\boldsymbol{F}$, by making use of its definition, $\boldsymbol{F} = -\kappa \nabla T$, thus $\kappa = FH_T/T$. In turn, the estimate of $\boldsymbol{F}$ can be obtained from the thermal balance. If the disc is approximately Keplerian, then

$$\nabla \cdot \boldsymbol{F} = \frac{9}{4}\rho \nu_{\rm t} \Omega^2,$$

where $\nu_{\rm t}$ is the turbulent viscosity. Thus, one has $F \approx (9/4)\rho\nu_{\rm t}\Omega^2 H_T$. By making use of these estimates the expression for ω_χ can be transformed to

$$\omega_\chi = \frac{9}{4}\frac{\nu\Omega^2}{c_p T}(kH_T)^2.$$

Taking into account that in a fully ionized plasma $c_pT = \frac{5}{2}c_{\rm s}^2$, and representing $\nu_{\rm t}$ in standard form for accretion discs, $\nu_{\rm t} = \alpha c_{\rm s} H$ where H is the disc scaleheight and α is the dimensionless turbulent viscosity parameter, one obtains

$$\omega_\chi \approx \alpha \frac{H\Omega^2}{c_{\rm s}}(kH_T)^2.$$

In thin accretion discs $c_{\rm s}/H \sim \Omega$, thus we finally have

$$\frac{\omega_\chi}{\Omega} \approx \alpha (kH_T)^2. \tag{9}$$

The thermal frequency turns out to be relatively large in accretion discs. Even if the growth time of the instability is comparable with the Keplerian period, there exists only a relatively narrow range of wavevectors $[1/\sqrt{\alpha} \gg (kH_T) \gg 1]$ for which waves can be considered as adiabatic. Outside this range, at $kH_T \gg 1/\sqrt{\alpha}$, waves are close to isothermality. If the growth time is longer than the Keplerian period the adiabatic approximation applies even in a more narrow domain of wavevectors. Therefore, the short-wavelength perturbations in discs are probably isothermal rather than adiabatic. Nevertheless, we consider both cases.

3.1 Instability of adiabatic perturbations

In the limiting case $\omega \gg \omega_\chi$, equation (8) simplifies to

$$(\omega^2 - \omega_{\rm A}^2)^2 - (\omega^2 - \omega_{\rm A}^2)(\omega_g^2 + Q^2) - 4\Omega^2\omega_{\rm A}^2 \frac{k_z^2}{k^2} = 0. \tag{10}$$

The solutions of this equation can be written in the form

$$\omega_1^2 = \omega_{\rm A}^2 + \frac{1}{2}(\omega_g^2 + Q^2) \pm \sqrt{\frac{1}{4}(\omega_g^2 + Q^2)^2 + 4\Omega^2\omega_{\rm A}^2\frac{k_z^2}{k^2}}, \tag{11}$$

$$\omega_2^2 = \omega_{\rm A}^2 + \frac{1}{2}(\omega_g^2 + Q^2) \mp \sqrt{\frac{1}{4}(\omega_g^2 + Q^2)^2 + 4\Omega^2\omega_{\rm A}^2\frac{k_z^2}{k^2}}, \tag{12}$$

where the upper or lower signs should be taken if $\omega_{\rm A}^2 + (\omega_g^2 + Q^2)/2 > 0$ or $\omega_{\rm A}^2 + (\omega_g^2 + Q^2)/2 < 0$, respectively. The modes given by equations (11) and (12) are, respectively, the buoyant and slow magnetoacoustic waves modified by rotation. If the unperturbed magnetic field vanishes one has $\omega_1^2 = \omega_g^2 + Q^2$ and $\omega_2^2 = 0$. The stability properties of these modes may be very different.

It is evident from the definition of the buoyant mode (11) that it is unstable only if

$$\omega_g^2 + Q^2 + 2\omega_{\rm A}^2 < 0. \tag{13}$$

Since $\omega_{\rm A}^2 > 0$, the necessary condition for instability of this mode is $\omega_g^2 + Q^2 < 0$. Generally, both ω_g^2 and Q^2 can be negative. For example, $\omega_g^2 < 0$ if the temperature gradient exceeds its adiabatic value. In this case, the standard convective instability can arise in discs. Convection is probably important in protoplanetary discs or in the outermost region of accretion discs where gas is not fully ionized and the opacity is high (see, e.g., Kley et al. 1993). However, ω_g^2 may also be negative if the temperature gradient is subadiabatic but $\Delta\nabla T$ is not parallel to the 'effective gravity', $\boldsymbol{G}$. This obliqueness can be caused, in principle, either by the dependence of Ω on z or by radiative heat transport in the radial direction. In this case the condition $\omega_g^2 < 0$ requires

$$(\boldsymbol{G} \cdot \Delta\nabla T) - (\boldsymbol{n} \cdot \boldsymbol{G})(\boldsymbol{n} \cdot \Delta\nabla T) > 0, \tag{14}$$

where $\boldsymbol{n} = \boldsymbol{k}/k$. Introducing the angle ψ between the vectors $\boldsymbol{G}$ and $\boldsymbol{n}$ and representing $\Delta\nabla T$ as a sum of components parallel and perpendicular to $\boldsymbol{G}$, $\Delta\nabla T = (\Delta\nabla T)_\parallel + (\Delta\nabla T)_\perp$, the inequality (14) can be rewritten in the form

$$\sin^2\psi(\Delta\nabla T)_\parallel - \sin\psi(\Delta\nabla T)_\perp > 0. \tag{15}$$

Obviously, this condition can be fulfilled even for stratification that would be stable according to the standard Schwarzschild criterion of convection, $(\Delta\nabla T)_{\|} < 0$. Due to the obliqueness of $\boldsymbol{G}$ and $\Delta\nabla T$, one has $\omega_g^2 < 0$ for perturbations with a small (but non-zero) angle ψ. Since $(\Delta\nabla T)_{\perp} \sim (H/s)(\Delta\nabla T)_{\|}$ it is easy to estimate that the value of ω_g^2 for such perturbations is relatively small,

$$\omega_g^2 \sim -\Omega^2 (H/s)^2. \tag{16}$$

For other perturbations ω_g^2 is positive and its value may be comparable with Ω^2.

The term Q^2 in equation (13) can generally also be negative. The inequality $Q^2 < 0$ can be rewritten in the form

$$\frac{k_z^2}{k^2}\frac{1}{s^3}\frac{\partial}{\partial s}(s^4\Omega^2) - \frac{k_z k_s}{k^2} 2\Omega s \frac{\partial \Omega}{\partial z} < 0. \tag{17}$$

For thin accretion discs, the radial dependence of Ω is approximately given by the Keplerian law, $\Omega \propto s^{-3/2}$, and, hence, the first term on the right-hand side of this inequality is positive. The sign of the second term depends on the direction of a wavevector $\boldsymbol{k}$ and, therefore, only this term may cause a destabilizing effect in Keplerian discs. For any dependence of Ω on z there exists, evidently, a certain domain of angles of the vector $\boldsymbol{k}$ where $Q^2 < 0$. For example, in thin accretion discs one has

$$\partial\Omega/\partial z \approx q\Omega z/s^2 \tag{18}$$

(see, e.g., Urpin 1984; Kley & Lin 1992), where $q = q(s) \sim 1$ is the parameter in the series expansion of $\Omega(s, z)$ around the Keplerian rotation. The term associated with the vertical shear dominates the right-hand side of equation (17) if

$$k_s > |(k_z/2q)(s/z)|. \tag{19}$$

Thus, one has $Q^2 < 0$ for perturbations with wavelengths much shorter in the radial direction than in the vertical. Evidently, $Q^2 > 0$ in the central plane of the disc where $\partial\Omega/\partial z = 0$.

Thus, both ω_g^2 and Q^2 may be negative separately, but the sum $\omega_g^2 + Q^2$ is always positive in thin discs and, therefore, the buoyant mode is stable. However, this mode can be unstable in thick discs where the growth rate of instability may be comparable with Ω. Note that even if $\omega_g^2 + Q^2 < 0$, a relatively strong magnetic field stabilizes the buoyant mode, in accordance with criterion (13).

The slow magnetoacoustic mode (12) is stable when the instability criterion (13) is fulfilled and the buoyant mode is unstable. In this case, the quantity $\omega_g^2 + Q^2$ is negative, thus $\omega_g^2 + Q^2 = -|\omega_g^2 + Q^2|$ and, hence,

$$\omega_2^2 = \sqrt{\frac{1}{4}(\omega_g^2 + Q^2)^2 + 4\Omega^2\omega_A^2\frac{k_z^2}{k^2}} - \frac{1}{2}|\omega_g^2 + Q^2| + \omega_A^2.$$

It is easy to check that $\omega_2^2 > 0$ under the inequality (13).

In contrast to the buoyant mode, however, the instability of the slow magnetoacoustic wave can develop at

$$\omega_g^2 + Q^2 + 2\omega_A^2 > 0, \tag{20}$$

when the frequency is determined by equation (12) with the upper sign. The condition for instability in this case reads

$$\omega_g^2 + Q^2 < -\omega_A^2 + 4\Omega^2\frac{k_z^2}{k^2}. \tag{21}$$

Combining the inequalities (20) and (21), we obtain the domain of values of $\omega_g^2 + Q^2$ where the slow magnetoacoustic mode is unstable:

$$4\Omega^2\frac{k_z^2}{k^2} - \omega_A^2 > \omega_g^2 + Q^2 > -2\omega_A^2. \tag{22}$$

For vertically directed perturbations, $\omega_g^2 \ll \Omega^2$ and condition (22) reduces to

$$4\Omega^2 - \omega_A^2 > Q^2 > -2\omega_A^2. \tag{23}$$

Evidently, this condition is weaker than the Rayleigh condition (1), particularly, if the magnetic field is weak and $\Omega \gg \omega_A$. Thus, for the rotation law $\Omega \propto s^{-\gamma}$, the criterion (23) is satisfied for $2 > \gamma > 0$, whereas the Rayleigh criterion requires $\gamma > 2$.

It is seen from inequality (22) that a relatively strong magnetic field can suppress the shear instability of magnetoacoustic waves. The instability is suppressed if

$$\omega_A^2 > 4\Omega^2\frac{k_z^2}{k^2} - Q^2 - \omega_g^2. \tag{24}$$

For the most unstable perturbations with $k_s \approx 0$ and for the case of a power-law cylindrical rotation, $\Omega \propto s^{-\gamma}$, this condition reads

$$\omega_A^2 > 2\gamma\Omega^2,$$

and, hence, an estimate of the critical magnetic field, B_{cr}, that stabilizes rotation may be given by

$$B_{cr} \sim \sqrt{\frac{2}{\pi}\gamma\rho\Omega^2\lambda^2}, \tag{25}$$

where $\lambda = 2\pi/k$ is the wavelength of perturbations. The critical field depends on the wavelength of perturbations: the longer the wavelength, the higher the magnetic field required for stabilization. Note that the energy of the magnetic field that can stabilize the mode with wavelength λ is approximately a factor of $4\pi^2(H/\lambda)^2 \gg 1$ smaller than the thermal energy of the plasma, ρc_s^2, for thin accretion discs.

The growth time of the adiabatic instability may be rather short. An estimate of the growth time is particularly simple if the magnetic field is so weak that $\omega_g^2 + Q^2 \gg 4|\Omega\omega_A k_z/k|$. Expanding the square root in equation (12) in a power series of ω_A^2 and keeping terms only to the order of ω_A^2, one obtains

$$\omega_2^2 \approx \frac{\omega_A^2}{\omega_g^2 + Q^2}\left[\omega_g^2 + 2\Omega s\frac{k_z}{k^2}\left(k_z\frac{\partial\Omega}{\partial s} - k_s\frac{\partial\Omega}{\partial z}\right)\right]. \tag{26}$$

For perturbations with $k_s \approx 0$ and for a power-law rotation constant on cylinders, $\Omega \propto s^{-\gamma}$, the frequency is given by

$$\omega_2^2 \approx -\frac{\gamma}{2-\gamma}\omega_A^2. \tag{27}$$

The growth time is typically much longer than the period of rotation. The only exception is perturbations with $\omega_A \sim \Omega$, which can grow on time-scales comparable with the period.

3.2 Instability of isothermal perturbations

In the case $\omega \ll \omega_\chi$ the solutions of equation (8) that correspond to the buoyant and slow magnetoacoustic modes are, respectively,

$$\omega_1^2 = \omega_A^2 + \frac{1}{2}Q^2 \pm \sqrt{\frac{1}{4}Q^4 + 4\Omega^2\omega_A^2\frac{k_z^2}{k^2}}, \tag{28}$$

$$\omega_2^2 = \omega_A^2 + \frac{1}{2}Q^2 \mp \sqrt{\frac{1}{4}Q^4 + 4\Omega^2\omega_A^2\frac{k_z^2}{k^2}}, \tag{29}$$

where the upper or lower signs have to be taken if $\omega_A^2 + Q^2/2 > 0$ or $\omega_A^2 + Q^2/2 < 0$, respectively. Of course, the first mode can be called buoyant only conditionally since the effect of buoyancy is completely suppressed for this mode as well as for the magnetoacoustic one (29). In the non-magnetic case, equation (28) describes inertial waves, $\omega_1^2 = Q^2$, which can exist in a rotating fluid, whereas the second mode is degenerate, $\omega_2 = 0$.

According to definition (28), the buoyant mode is unstable if

$$\omega_A^2 + Q^2/2 < 0. \qquad (30)$$

Evidently, this condition can hold only if the necessary condition $Q^2 < 0$ (see equation 17) is fulfilled. In the case of rotation constant on cylinders, this necessary condition reduces to the Rayleigh criterion (1). If the angular velocity is dependent on the vertical coordinate then the condition $Q^2 < 0$ can be satisfied at any dependence $\Omega(z)$ by a particular choice of $\boldsymbol{k}$. Thus, as was mentioned, for thin accretion discs with a radial dependence of Ω given by the Keplerian law ($\propto s^{-3/2}$) and with $\partial\Omega/\partial z$ given by equation (18), the condition $Q^2 < 0$ is fulfilled if the radial and vertical components of the wavevector are related by inequality (19).

In a weak magnetic field, $Q^2 \gg \omega_A^2$, the growth rate of the first mode is

$$\mathrm{i}\omega = \pm\sqrt{-Q^2}. \qquad (31)$$

In the case of rotation close to Keplerian, the quantity Q^2 is negative only for small values of k_z/k_s (see equation 19). It reaches its minimum, $Q^2_{\min} \approx -q^2(z/s)^2\Omega^2$, approximately at

$$\frac{k_z}{k_s} \approx q\frac{z}{s}. \qquad (32)$$

Substituting $Q^2_{\min}$ into equation (31), we obtain the order of magnitude estimation of the maximum growth rate of the buoyant mode,

$$\mathrm{i}\omega \sim \Omega|qz/s|. \qquad (33)$$

Thus, the growth time is of the order of the time-scale of the vertical shear. Since the radial wavelength of unstable perturbations should be much shorter than the vertical one, the instability associated with the vertical shear leads, probably, to the generation of strongly anisotropic turbulence.

Like the case of adiabatic perturbations, a relatively strong magnetic field satisfying the condition $\omega_A^2 > |Q^2|$ can suppress the instability. In the case of an approximately Keplerian rotation with vertical shear as given by (18), the stabilizing magnetic field, $B_{\rm cr}$, is

$$B_{\rm cr} \sim \sqrt{\frac{q^2}{2\pi}\rho\lambda^2\Omega^2\left(\frac{z}{s}\right)^2}, \qquad (34)$$

The energy of the stabilizing field (34) is a factor of $(16\pi^2/q^2)(H/\lambda)^2(s/z)^2$ smaller than the thermal energy of plasma.

The slow magnetoacoustic mode (29) is stable within the domain of instability of the buoyant mode (30). However, it can be unstable for

$$\omega_A^2 + Q^2/2 > 0. \qquad (35)$$

In this case, the frequency is given by equation (29) with the upper sign, and the instability condition $\omega_2^2 < 0$ requires

$$Q^2 < 4\Omega^2\frac{k_z^2}{k^2} - \omega_A^2.$$

Combining this condition with inequality (35), we obtain the domain of unstable parameters for the second mode,

$$4\Omega^2\frac{k_z^2}{k^2} - \omega_A^2 > Q^2 > -2\omega_A^2. \qquad (36)$$

For perturbations directed approximately vertically, this domain coincides with the domain of instability (23) in the adiabatic limit.

The instability can be prevented by a sufficiently strong magnetic field, in the same way as all other instabilities of low-frequency waves. For the most unstable perturbations with $k_z \gg k_s$, the value of the magnetic field stabilizing the shear flow is given by equation (25).

In the isothermal limit, the growth rate is small if $\omega_A \ll \Omega$ but it may be comparable to Ω in a relatively strong (but not stabilizing) field. In the case ($Q^2 \gg 4\Omega|\omega_A k_z/k|$), the expression for ω_2^2 can be obtain by analogy with equation (26),

$$\omega_2^2 \approx 2\Omega s\frac{k_z}{k^2}\frac{\omega_A^2}{Q^2}\left(k_z\frac{\partial\Omega}{\partial s} - k_s\frac{\partial\Omega}{\partial z}\right). \qquad (37)$$

For the rotation law $\Omega \propto s^{-\gamma}$ this expression reduces to equation (27). For a stronger magnetic field, the dependence of ω on the magnetic field strength and the direction of the wavevector is more complicated.

We now discuss the dependence of the growth rates of the instability of the two modes on $\boldsymbol{k}$. However, we should keep in mind that a larger growth rate of one of the two modes does not necessarily imply a more efficient turbulent transport associated with that mode. The efficiency of transport driven by instabilities depends on a number of factors (range of the unstable wavelengths, non-linear behaviour, etc.) and is beyond the scope of the present paper.

The only dependence on the magnitude of $\boldsymbol{k}$ is through ω_A, because Q^2 depends only on the ratios k_s/k and k_z/k. It is therefore natural to discuss the dispersion relation in terms of the angle $\theta = \cos^{-1}(\Omega\cdot\boldsymbol{n})/\Omega$, so then $k_z = k\cos\theta$ and $k_s = k\sin\theta$. It is then convenient to define the parameter $\omega_{A0} = Bk/\sqrt{4\pi\rho}$ to be equal to the maximal Alfvén frequency at a given magnetic field. If $\boldsymbol{B}$ is aligned with $\boldsymbol{\Omega}$ and $\cos\theta_B \equiv \boldsymbol{B}\cdot\boldsymbol{\Omega}/B\Omega = 1$ then $\omega_A = \omega_{A0}\cos\theta$.

In the following we present the dependence of ω^2 on θ for different values of ω_{A0}. Larger values of ω_{A0} correspond to larger field strengths or to larger values of k. Thus, even for very weak magnetic fields there will always be a wavenumber large enough so that the growth rate of the magnetoacoustic mode is larger than the growth rate of the buoyant mode provided k does not exceed the magnetic dissipative length-scale, $\sim\sqrt{\Omega/\eta}$, where η is the magnetic diffusivity. Moreover, magnetoacoustic waves with k approaching the value $\sim\Omega/c_A$ (but still satisfying inequality 36) can grow on the time-scale $\sim\Omega$.

In Fig. 1 we plot ω^2/Ω^2 versus θ for different values of ω_{A0}/Ω, assuming a vertical magnetic field. In this and the following two plots we assume that Ω is close to the Keplerian angular velocity, i.e. $\Omega \propto s^{-3/2}$, and that the vertical dependence of Ω is given by equation (18). The ratio z/s is taken to be equal to 0.1, which appears more or less suitable for typical accretion discs.

The buoyant modes are only excited (i.e. $\omega^2 < 0$) near $\theta = 94^\circ.3$. For $\omega_{A0}/\Omega \gtrsim 0.04$ the slow magnetoacoustic mode with $\theta = 0$ is more easily excited. For $\omega_{A0}/\Omega \gtrsim 2.2$ the field is strong enough to stabilize both modes.

In Fig. 2 we show the minimum values of ω^2/Ω^2 at $\theta = 94^\circ.3$ and $\theta = 0$ as a function of ω_{A0}/Ω, assuming again a vertical magnetic field, i.e. $\theta_B = 0$. Note that for $\theta = 94^\circ.3$ and $\omega_{A0}/\Omega \lesssim 0.7$ the most unstable mode is the buoyant mode. For $\omega_{A0}/\Omega \gtrsim 0.7$ and $\theta = 94^\circ.3$ the most unstable mode has turned into the slow magnetoacoustic mode. However, for $0.7 \lesssim \omega_{A0}/\Omega \lesssim 1.7$ this mode has a lower growth rate than at $\theta = 0$. Finally, for $\omega_{A0}/\Omega \gtrsim 1.7$ the slow magnetoacoustic mode becomes suppressed.

Evidently, the growth rate of the buoyant mode associated with the vertical shear is typically smaller than that of the magnetoacoustic mode caused by the stronger radial shear. The growth rate of the buoyant mode exceeds that of the magnetoacoustic one

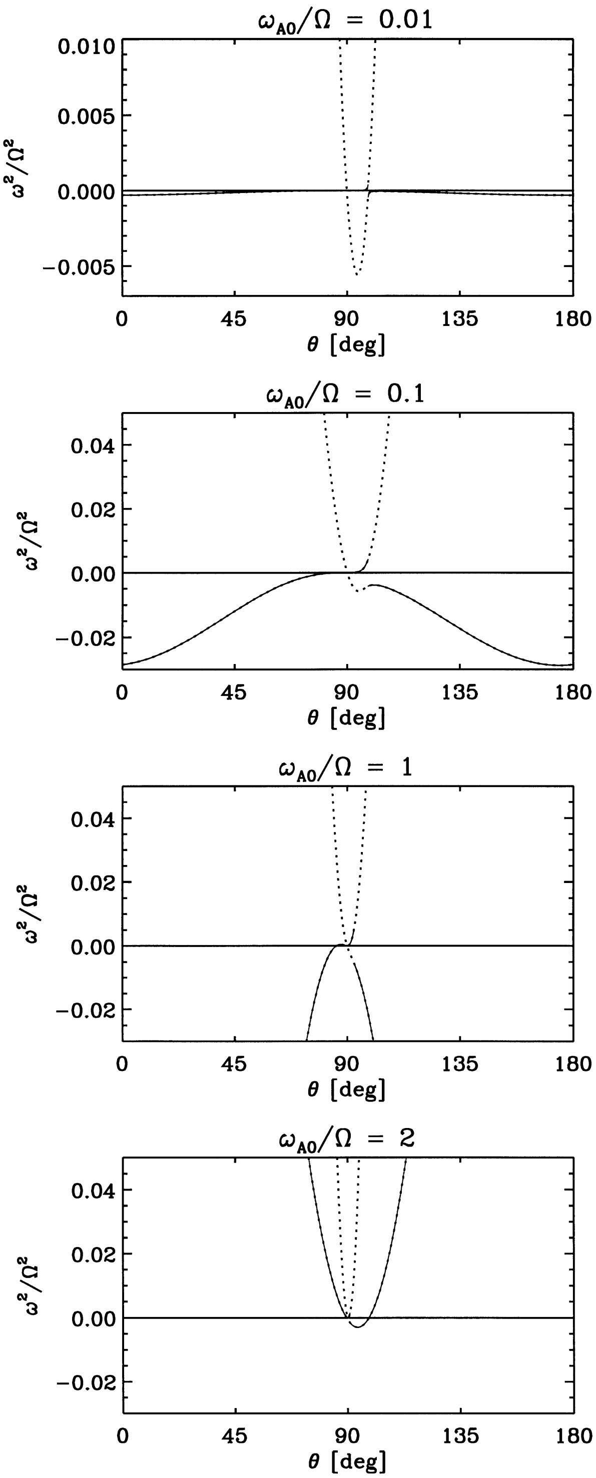

Figure 1. Dependence of ω^2/Ω^2 on the angle θ for different values of ω_{A0}/Ω and for keplerian rotation, $\gamma = 3/2$, in the case of the vertical magnetic field, i.e. $\theta_B = 0$. The slow magnetoacoustic modes are shown as solid lines, whilst the buoyant modes are shown as dotted lines.

approximately at $\omega_{A0}/\Omega < 0.04$. For larger values of ω_{A0} the magnetoacoustic mode grows substantially faster. For thin accretion discs the maximal growth rate is of the order of 0.1Ω for the buoyant mode and 0.75Ω for the magnetoacoustic mode if $\omega_{A0}/\Omega \sim 1$ (Balbus & Hawley 1991). Note that the growth rate of

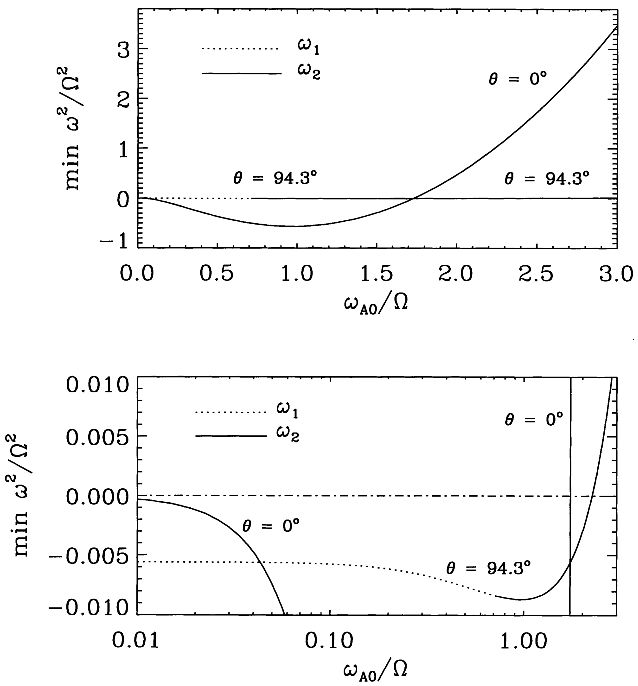

Figure 2. Dependence of ω^2/Ω^2 on ω_{A0}/Ω for $\theta_B = 0$ at $\theta = 94\overset{\circ}{.}3$ and $\theta = 0$. As in Fig. 1 the solid lines refer to the slow magnetosonic mode and the dotted lines to the buoyant mode. In the lower panel a narrower range of ω^2/Ω^2 is shown, so one can see that the buoyant mode is preferred in the range $0 \le \omega_{A0}/\Omega \le 0.04$. Also in the range $1.73 \le \omega_{A0}/\Omega \le 2.24$ the mode at $\theta = 94\overset{\circ}{.}3$ is preferred, but now it is the slow magnetoacoustic mode; cf. Fig. 1.

the magnetoacoustic mode is large only in the relatively narrow range $0.3 < \omega_{A0}/\Omega < 1.5$. On the other hand, the buoyant mode can be unstable only for perturbations with wavevectors close to the radial direction, whereas the magnetoacoustic waves can arise for a wider range of θ. However, this range narrows with increasing magnetic field strength or decreasing wavelength. In those cases where the modes are stable, their frequencies may be of the order of, or even higher than, the Keplerian frequency.

The stability properties of discs depend on the direction of the applied magnetic field. In Fig. 3 we show the dispersion curves for three different values of θ_B. In the case of a radially directed magnetic field (third panel in Fig. 3), the instability of the buoyant mode is overwhelmed by the magnetoacoustic mode. The only exception is the region $\omega_{A0}/\Omega < 0.05$, where the growth rate of buoyant waves is larger. For higher values of ω_{A0} the magnetoacoustic wave grows more rapidly. The instability of the buoyant mode is suppressed when ω_{A0}/Ω is larger than ≈ 0.05. The stabilizing magnetic field for the magnetoacoustic mode is much stronger: $\omega_{A0}/\Omega > 5$. This value is slightly higher than in the case of a vertical field. Note also that the range of ω_{A0}/Ω, where the growth rate of the magnetoacoustic mode is comparable with Ω, is substantially wider for the radial field and spreads from ≈ 0.6 to ≈ 5. On the other hand, the range of unstable θ seems to be narrower for the radial field than for the vertical one. Thus, only perturbations with approximately vertical wavevectors may be unstable when the field approaches its stabilizing value. As in the case of a vertical field, when the modes are stable the frequency of oscillation of both modes is comparable with, or even higher than, the Keplerian frequency.

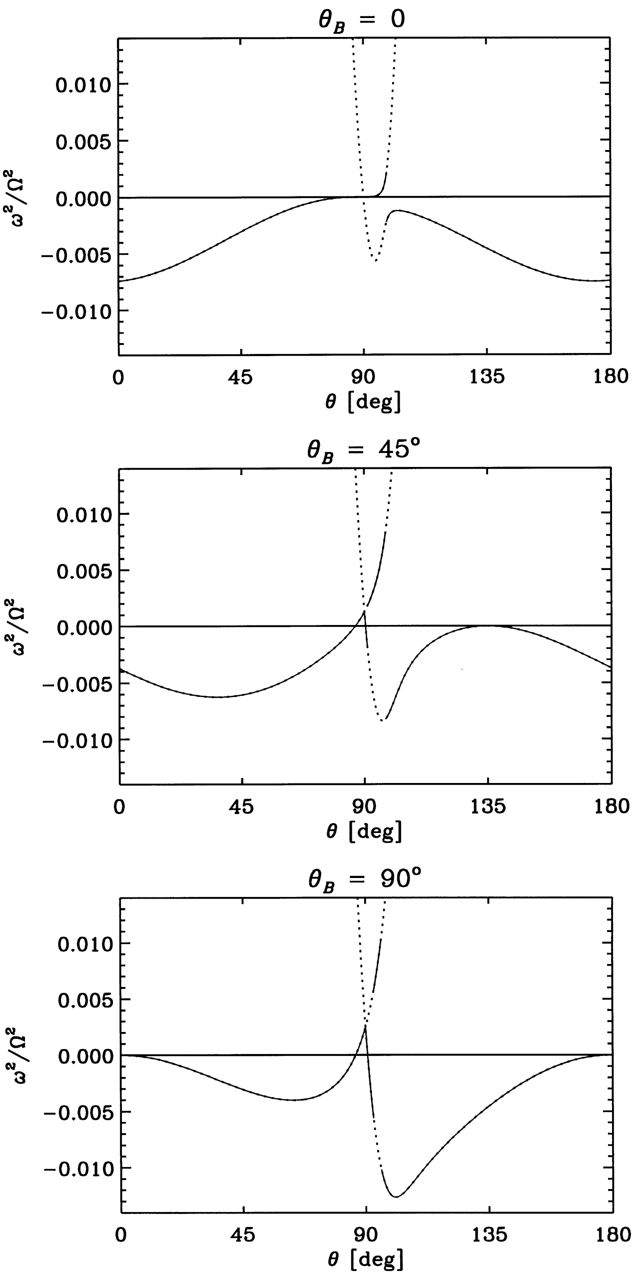

Figure 3. The same as in Fig. 1, but for three different orientations of the magnetic field, θ_B, assuming $\omega_{A0}/\Omega = 0.05$. In the second and third panels there are isolated regions ($91.^{\circ}1 < \theta < 97.^{\circ}7$ and $92.^{\circ}9 < \theta < 95.^{\circ}7$) on the upper branch which belong to the magnetoacoustic mode.

4 CONCLUSION

We have shown that magnetized and non-magnetized discs are unstable with respect to different kinds of shear instabilities for short-wavelength perturbations. In the non-magnetic case the instability is associated with the buoyant mode and is caused by the vertical shear. It can arise for any dependence of the angular velocity on z: both positive and negative values of $\partial\Omega/\partial z$ can lead to instability. The most rapidly growing perturbations in a thin accretion disc have growth rates of the order of $q\Omega(z/s)$. For these perturbations the ratio of the vertical and radial components of the wavevector is small, $k_z/k_s \sim qz/s$. In other words, the radial wavelength has to be shorter than the vertical wavelength by approximately a factor of s/z. Perturbations with suitable wavevectors arise faster near the disc surface than near the central plane. In the absence of a magnetic field the vertical shear instability does not arise at $z = 0$ where $\partial\Omega/\partial z = 0$. The e-folding time of the most unstable perturbations is relatively short and, perhaps, the considered instability may be a candidate for the origin of turbulence in non-magnetized discs.

In the presence of a magnetic field the instability of discs may be caused mainly by the slow magnetoacoustic mode (with the exception of the case of a weak magnetic field when the buoyant mode grows faster). The instability of magnetoacoustic waves is associated with the radial shear which, in thin accretion discs, is much stronger than the vertical shear. Therefore, the growth time of this instability is typically shorter. For a small Alfvén frequency, $\omega_A \ll \Omega$, the growth rate is of the order of ω_A, but it may be comparable to the Keplerian period for the most rapidly growing perturbations with $\omega_A \sim \Omega$. However, a relatively strong magnetic field should completely suppress the instability of magnetoacoustic waves. The strength of the field that can stabilize the flow depends on the wavelength of the perturbation and is determined by equation (25). For all short-wavelength perturbations with $kH \gg 1$, the energy of the stabilizing field is much less than the thermal energy of the plasma.

In the present paper, we have addressed the behaviour of only axisymmetric perturbations. It is clear, however, that the results obtained can apply to non-axisymmetric perturbations with azimuthal wavelength much longer than the vertical or radial ones, $\min(k_r, k_z) \gg k_\varphi$. The turbulence that could be generated by shear instabilities may be strongly anisotropic because the instability criterion is sensitive to the direction of the wavevector.

Note that the linear stability analysis presented here allows us only to calculate the growth rate of different modes and the range of unstable wavevectors. Obviously, the results obtained do not allow us to estimate the efficiency of the turbulence driven by the considered instabilities or to calculate the corresponding transport coefficients. This problem requires fully non-linear, three-dimensional calculations and we are planning to address this in a forthcoming publication.

ACKNOWLEDGMENTS

One of the authors (VU) gratefully acknowledges financial support by the Russian Foundation of Basic Research (Grant 94-02-06540) and by INTAS (Grant 94-3097).

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