

ROBUSTNESS OF TRUNCATED $\alpha\Omega$ DYNAMOS WITH A DYNAMIC α

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Abstract. In a recent work (Covas *et al.*, 1996), the behaviour and the robustness of truncated $\alpha\Omega$ dynamos with a dynamic α were studied with respect to a number of changes in the driving term of the dynamic α equation, which was considered previously by Schmalz and Stix (1991) to be of the form $\sim A_\phi B_\phi$. Here we review and extend our previous work and consider the effect of adding a quadratic quenching term of the form $\alpha|\mathbf{B}|^2$. We find that, as before, such a change can have significant effects on the dynamics of the related truncated systems. We also find intervals of (negative) dynamo numbers, in the system considered by Schmalz and Stix (1991), for which there is sensitivity with respect to small changes in the dynamo number and the initial conditions, similar to what was found in our previous work. This latter behaviour may be of importance in producing the intermittent type of behaviour observed in the Sun.

1. Introduction

Given the importance of hydrodynamic dynamos in accounting for solar and stellar variabilities, a large number of studies have been made of their dynamical modes of behaviour using a range of dynamo models. However, the complexity and the numerical costs of employing the full magneto-hydrodynamical partial differential equations (Gilman, 1983) have resulted in a great deal of effort going into the study of simpler related systems, including mean field and truncated models.

Both the mean field models (cf., Krause and Rädler, 1980; Brandenburg *et al.*, 1989; Brandenburg, Moss and Tuominen, 1989; Tavakol *et al.*, 1995) and truncated systems (Zeldovich, Ruzmaikin, and Sokoloff, 1983; Weiss, Cattaneo, and Jones, 1984; Feudel, Jansen, and Kurths, 1993) have been shown to be capable of producing a spectrum of dynamical modes of behaviour, including equilibrium, periodic and chaotic states.

Here we shall concentrate on truncated models and recall that there have been two approaches in the study of such models: the quantitative study of the truncated equations resulting from concrete partial differential equations (e.g., Schmalz and Stix, 1991) and the qualitative approach involving the use of normal form theory to construct robust minimal low order systems which capture important aspects of the dynamo dynamics (Tobias, Weiss, and Kirk, 1995).

Given the approximate nature of the truncated models, an important question arises as to the extent to which the results produced by such models are an artefact

of their details. This is potentially of importance since, on the basis of results from dynamical systems theory, structurally stable systems are not everywhere dense in the space of dynamical systems (Smale, 1966), and therefore small changes in the models can potentially produce qualitatively important changes in their dynamics (Tavakol and Ellis, 1988; Coley and Tavakol, 1992; Tavakol *et al.*, 1995).

To partially answer this question, we take a somewhat complementary approach to the previous works, by looking at the robustness of the truncated models studied by Schmalz and Stix with respect to a number of reasonable changes to their components. This work extends and reviews our previous work (Covas *et al.*, 1996).

2. Mean Field Dynamo Models with a Dynamic α

As an example of how the details of truncated models can change their behaviour, we take the truncated dynamic α models considered in Schmalz and Stix. The starting point of this work is the mean field induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} + \alpha \mathbf{B} - \eta_t \nabla \times \mathbf{B}), \quad (1)$$

in the usual notation with the turbulent magnetic diffusivity η_t and the coefficient α arising from the correlation of small-scale (turbulent) velocity and magnetic fields (Krause and Rädler, 1980).

For the sake of comparison we take an axisymmetrical configuration with one spatial dimension x corresponding to a latitude coordinate (and measured in terms of the stellar radius R), together with a longitudinal velocity with a constant radial gradient (the vertical shear ω_0). The magnetic field is given by

$$\mathbf{B} = \left(0, B_\phi, \frac{1}{R} \frac{\partial A_\phi}{\partial x} \right), \quad (2)$$

where A_ϕ is the ϕ -component (latitudinal) of the magnetic vector potential and B_ϕ the ϕ -component of \mathbf{B} . These assumptions allow Equation (1) to be split into

$$\frac{\partial A_\phi}{\partial t} = \frac{\eta_t}{R^2} \frac{\partial^2 A_\phi}{\partial x^2} + \alpha B_\phi, \quad (3)$$

$$\frac{\partial B_\phi}{\partial t} = \frac{\eta_t}{R^2} \frac{\partial^2 B_\phi}{\partial x^2} + \frac{\omega_0}{R} \frac{\partial A_\phi}{\partial x}. \quad (4)$$

Furthermore α is divided into a static (kinematic) and a dynamic (magnetic) part as follows: $\alpha = \alpha_0 \cos x - \alpha_M(t)$, with its time-dependent part $\alpha_M(t)$ satisfying an evolution equation of the form

$$\frac{\partial \alpha_M}{\partial t} = \mathcal{O}(\alpha_M) + \mathcal{F}(\mathbf{B}), \quad (5)$$

where $\mathcal{O}(\alpha_M)$ is the damping term and $\mathcal{F}(\mathbf{B})$ the driving term, which is a pseudo-scalar and quadratic in the magnetic field (see Covas *et al.* (1996) for details of this and the truncations of Equations (3), (4), and (5)). We consider the interval $0 \leq x \leq \pi$ (which corresponds to the full range of latitudes) and take $A = B = C = 0$ at $x = 0$ and $x = \pi$ as boundary conditions. We restrict the allowed modes to the antisymmetric subset, that is, those modes that satisfy the condition $\mathbf{B} = 0$ at $x = \pi/2$.

Schmalz and Stix then fix the functional form of \mathcal{F} and study the resulting modal truncations. Now given the fact that the exact form of $\mathcal{F}(\mathbf{B})$ is not known precisely, we shall examine in the next section how robust their results are with respect to changes in $\mathcal{F}(\mathbf{B})$.

3. Robustness with Respect to Changes in the Driving Term

In their studies, Schmalz and Stix take the following form for the feedback term:

$$\mathcal{F}(\mathbf{B}) \sim A_\phi B_\phi ; \quad (6)$$

they look at the various N -modal truncations of these equations to find out what happens to the dynamical behaviour of the resulting systems as N is increased. In all of the following discussion, we assume pure antisymmetric modes.

To determine the nature of the dynamics, we shall employ the spectrum of Lyapunov exponents, and distinguish signatures of the types $(-, -, -, \dots)$, $(0, -, -, \dots)$, and $(+, 0, -, \dots)$ as corresponding to equilibrium, periodic, and chaotic regimes, respectively.

For the sake of comparison we summarise, schematically, the results of the integration of the systems of Schmalz and Stix (see also Covas *et al.*, 1996), in Figure 1. Here the largest Lyapunov exponent is depicted by a solid line and its negative, zero and positive values indicate equilibrium, periodic and chaotic regimes respectively (the second Lyapunov exponent is plotted as a dashed line).

Since in many astrophysical settings (including that of the Sun) the sign of the dynamo number, D , is not known, we also study the effects of changing its sign. The results for negative dynamo numbers are given in Figure 2.

An interesting mode of behaviour occurs in the spiky regions of Figure 2 which corresponds to the presence of ‘multiple attractors’ (of more than one attractor consisting of equilibrium and periodic states) over substantial intervals of D .

To clarify the consequences of this behaviour, we have plotted in Figure 3 the behaviour of the $N = 4$ truncation as a function of small changes in the dynamo number and the initial conditions. It clearly demonstrates that in these regimes small changes in either D or the initial conditions may produce drastic changes in the dynamical behaviour of the system. This form of *fragility* could be of significance in producing seemingly intermittent types of behaviour.

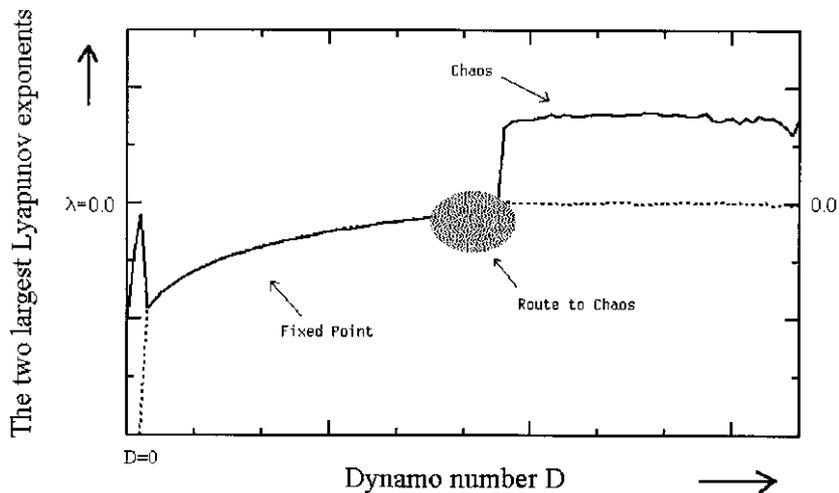


Figure 1. Schematic graph of the typical asymptotic behaviour of the two largest Lyapunov exponents for the $D > 0$ case with a driving term $\mathcal{F}(\mathbf{B}) \sim A_\phi B_\phi$. The route to chaos seems to vary as the truncation order increases.

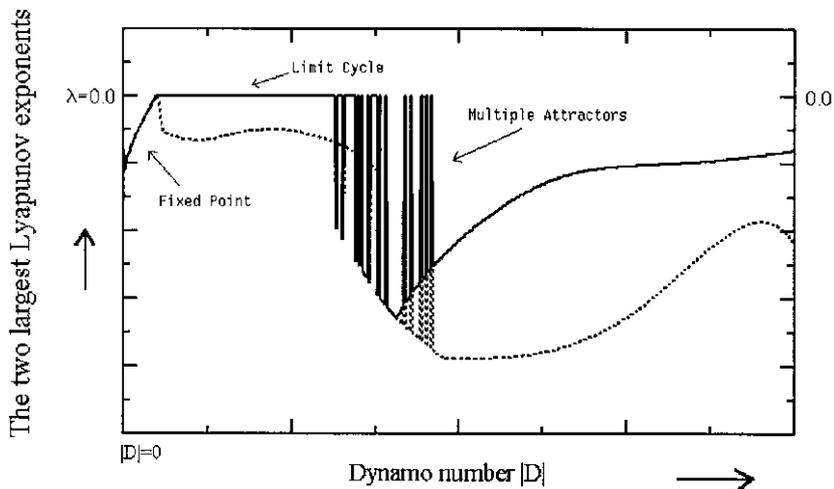


Figure 2. Schematic graph of the typical asymptotic behaviour of the two largest Lyapunov exponents for the $D < 0$ case with a driving term $\mathcal{F}(\mathbf{B}) \sim A_\phi B_\phi$.

To study the robustness of this system we considered a modified feedback term in the form

$$\mathcal{F}(\mathbf{B}) = f_1 A_\phi B_\phi + f_2 \alpha |\mathbf{B}|^2, \quad (7)$$

where $f_1 = (\mu_0 \rho)^{-1}$ and $f_2 = (\mu_0 \rho \beta)^{-1}$ (β is the combined (turbulent plus ohmic) diffusion of the field, ρ the density of the medium and μ_0 the magnetic constant). The additional second term on the right hand side of Equation (7) is reminiscent

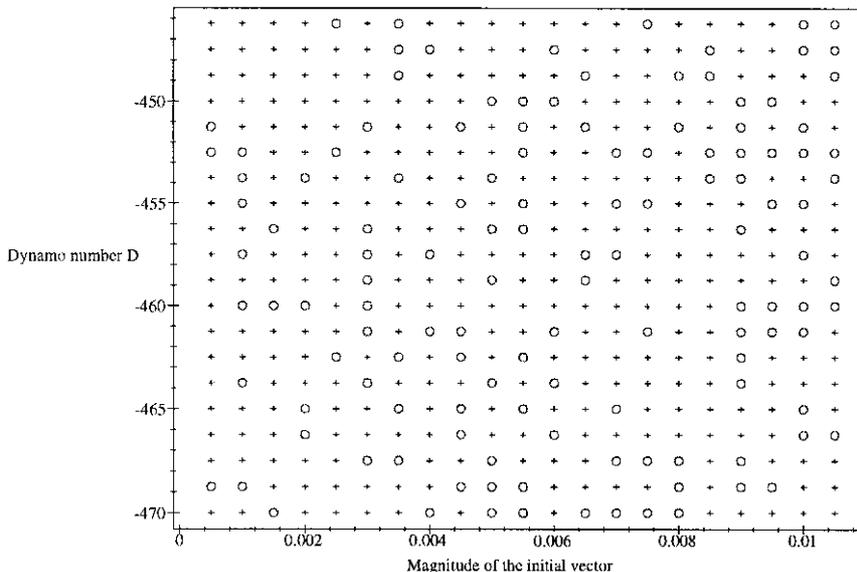


Figure 3. Fragility in the dynamics with respect to small changes in the dynamo number D and the magnitude of the initial vector $(A_\phi, B_\phi, \alpha_M)$. Crosses and circles represent equilibrium and periodic behaviour, respectively.

of the type appearing in a more physically motivated form of \mathcal{F} given by Kleorin and Ruzmaikin (1982), Zeldovich, Ruzmaikin and Sokoloff (1983) and Kleorin, Rogachevskii, and Ruzmaikin (1995) and discussed in Covas *et al.* (1996).

Figure 4 summarises the results for this modified form of \mathcal{F} and as can be seen, the inclusion of $\alpha|\mathbf{B}|^2$ in Equation (7) has drastic effects on the dynamics of the system, for both positive and negative dynamo numbers. In particular, it strongly suppresses the chaotic behaviour found by Schmalz and Stix.

4. Comparison with Other Functional Forms

In Covas *et al.* (1996), we considered the effects of using more physically motivated choices for the driving term given by Kleorin and Ruzmaikin (1982), Zeldovich, Ruzmaikin, and Sokoloff (1983) and Kleorin, Rogachevskii, and Ruzmaikin (1995) in the form

$$\mathcal{F}(\mathbf{B}) = g_1 \mathbf{J} \cdot \mathbf{B} + g_2 \alpha |\mathbf{B}|^2, \quad (8)$$

where $g_1 = -(\mu_0 \rho)^{-1}$ and $g_2 = (\mu_0 \rho \beta)^{-1}$ are physical constants.

The results of considering the cases with $g_1 \neq 0, g_2 = 0$ and $g_1 \neq 0, g_2 \neq 0$ can be summarised as follows. In the former case, the behaviour seems to mirror that seen for the case with $\mathcal{F}(\mathbf{B}) \sim A_\phi B_\phi$, in the sense that for $D > 0$ there is

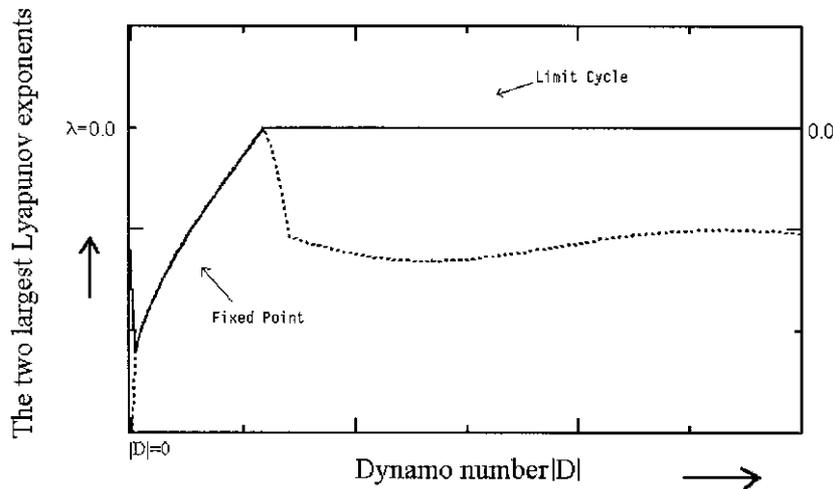


Figure 4. Schematic graph of the typical asymptotic behaviour of the two largest Lyapunov exponents for the case with a driving term $\mathcal{F}(\mathbf{B}) = f_1 A_\phi B_\phi - f_2 \alpha |\mathbf{B}|^2$. The overall behaviour is independent of the sign of D .

an asymptotic dominance of ‘multiple attractor’ regimes, while for $D < 0$ this behaviour is replaced by chaotic behaviour. In the latter case, however, where the term proportional to $\alpha |\mathbf{B}|^2$ is included, there is a suppression of both chaotic and ‘multiple attractor’ regimes, and the behaviour looks similar to that depicted in Figure 4. We also changed the damping operator so that it was of the form derived by Kleerorin, Rogachevskii, and Ruzmaikin (1995). It was found, however, that this change did not produce any qualitative changes.

5. Conclusions

We have studied the robustness of truncated $\alpha\Omega$ dynamos including a dynamic α equation, with respect to a change in the driving term of the $\alpha |\mathbf{B}|^2$ type. We find that this changes the results of Schmalz and Stix drastically by suppressing the possibility of chaotic behaviour. Our results presented here and those in Covas *et al.* (1996) show that changes in the driving term have important effects on the dynamical behaviour of the resulting systems. Furthermore, the sign of the dynamo number also plays an important role, being capable of radically changing the behaviour of the system. Typically we find that the behaviour of the system for $\mathcal{F}(\mathbf{B}) \sim A_\phi B_\phi$ and $D > 0$ is similar to the one with $\mathcal{F}(\mathbf{B}) \sim \mathbf{J} \cdot \mathbf{B}$, but with $D < 0$ and *vice-versa*. The change of sign causes the behaviour to change from typically chaotic to ‘multiple attractor’ solutions. These correspond to regimes where equilibrium and periodic regimes are present simultaneously. As a result, small changes in D or the initial conditions can substantially change the behaviour

of the system. This type of behaviour can be of importance in producing intermittent type behaviour observed in solar variability.

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