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The influence of boundary conditions on the excitation of disk dynamo modes

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Abstract. Calculations of mean field dynamos for galaxies have largely been for two rather disparate models. The thin disk model treats the ratio of disk height to radius explicitly as a small parameter, and applies zero tangential field boundary conditions at the disk surface. In contrast, the embedded disk model calculates the magnetic field in a spherical volume, whose radius is the disk radius and with the magnetic field fitting smoothly on to a *curl-free* exterior field at the surface of the *sphere.* The disk geometry is imposed by a flat distribution of the α-effect (and maybe also of the diffusivity η). For computational reasons this model has not been applied to very thin disks, so the regions of validity of the two models are almost disjoint. Comparison between their predictions is therefore difficult. In this paper we calculate, in linear theory, galactic dynamo modes according to both thin and embedded (or "thick") disk models for a simple underlying distribution of α-effect and differential rotation, using a common numerical scheme. For the smallest attainable ratio of disk height to radius, we find the critical dynamo numbers are similar, but that there are some significant differences in field topology.

Key words: Galaxies: magnetic fields – hydromagnetics – dynamo theory

1. Introduction

Mean field dynamo theory has produced relatively unambiguous results in two simple geometries – the Cartesian slab and the sphere/spherical shell. The reasons for choosing the latter are obvious – stars are (approximately) spherical. The slab has the advantage of geometrical and analytic simplicity. However, although valuable insight into qualitative behaviour of astrophysical dynamos may be obtained by judicious use of slab geometry, direct application of such a model to astrophysical systems is difficult or impossible. Spiral galaxies are highly flattened objects possessing coherent, large scale fields, eg Ruzmaikin et al. (1988). Their disks are strongly differentially rotating and highly turbulent, and so they are obvious candidates for dynamo action. However their geometry is awkward. Much work has been done assuming a *local* slab geometry, splitting the field into the product of a slowly varying function of s and a rapidly varying function of z (s, ϕ and z are cylindrical polar coordinates). The basic idea is that z_0/R is a small quantity, where z_0 is the disk semi thickness and R is the disk radius. This approach has been described in detail by Baryshnikova et al. (1987) and Ruzmaikin et al. (1988). Although considerable insight into the dynamo mechanism can be obtained in this way, there are serious limitations, see for example, Rädler & Bräuer (1987), Krause (1990), who point out *inter alia* that this method of solving the dynamo equations may not find certain modes. (But see Soward, 1978.) In particular, the boundary conditions applied to the field are that it matches to a vacuum at the disk surface, $z = z_0$. In the local slab geometry this implies the tangential field vanishes at this boundary.

In practice, galactic halos themselves contain turbulent conducting media, and these assumptions are thus highly questionable. Moreover there is now clear observational evidence for large scale magnetic fields extending away from the galactic disk into the halo region (Beck, 1991).

An alternative dynamo model has been developed in the last few years, that avoids some of these difficulties. This can be called the "embedded disk" or "thick disk" (TD) model in contrast to what might be referred to as the "zero boundary condition" (ZBC) model outlined above. In the original TD model, the dynamo equations are solved in a sphere, but with an α-effect that is confined to a disk. Boundary conditions are applied on the spherical surface $r = R$, and are that the interior field fit smoothly onto a vacuum (curl-free) field there. (See, eg, Elstner et al., 1990; Moss & Tuominen, 1990.) Modifications might, for example, include a z-dependent resistivity, η (eg Donner & Brandenburg, 1990).

The advantages of the TD model are the absence of any expansion in terms of the small parameter z_0/R. Moreover it is a truly global model. Disadvantages include its analytic intractability and that the disk does not fit "neatly" on to a finite difference grid in spherical polar coordinates r, θ and ϕ. The latter point does not, in practice, seem to be a serious problem except that with a straightforward difference scheme, in order to place a reasonable number of grid points in a thin disk where field gradients may be expected to be large, a large number of points must also be placed in the halo region where gradients can be expected to be relatively small. This computational inefficiency has limited the TD investigations to comparatively large values of z_0/R (greater than about 0.10 or 0.20 typically). Thus the TD and ZBC models are really useful in disjoint ranges of z_0/R, and it is not clear how well they would agree at a common value of this pa-

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rameter. For example, one of the predictions of Ruzmaikin et al. is that, although in spherical dynamos nonaxisymmetric modes are inhibited by differential rotation (eg Rädler, 1986), this may not be true for thin galactic disks. Such a result may, of course, be important for the explanation of nonaxisymmetric (eg BSS) fields in some spiral galaxies. The argument why differential rotation inhibits nonaxisymmetric modes is as follows. In a spherical dynamo, differential rotation winds up a nonaxisymmetric field, forcing field lines of opposing direction close to one another, so that reconnection is facilitated. Differential rotation does not have this effect on axisymmetric fields, but simply generates azimuthal field from poloidal. Thus axisymmetric fields are easier to maintain against decay. In a thin ZBC disk the important length scale for both axisymmetric and nonaxisymmetric fields is z_0, not R. This is directly imposed by the boundary conditions. If z_0/R is small enough, the decay of both types of field is governed by the length z_0, and not by the winding up by differential rotation. Thus there is no discrimination against nonaxisymmetric fields.

Models presented by Ruzmaikin et al. (1988) support this argument although even they never find nonaxisymmetric modes that have faster linear growth rates than axisymmetric. (Note that, in nonlinear theory, away from the bifurcation from the trivial solution, the relevant sizes of linear growth rates is irrelevant in determining the *stable* modes, eg Krause & Meinel (1988), Brandenburg et al. (1989).) It is not obvious that the disk height plays such a central role in the TD model. The disk scale z_0, imposed via the α-effect, clearly will still be important, but the magnetic field is not now constrained *a priori* to have the scale z_0 by the boundary conditions applied at the disk surface.

Thus it is of interest to compare predictions of the two models. As studied at present the approaches are so different (expansion in powers of z_0/R and WKB approximation versus grid point method in r and θ) that it is difficult to distinguish between the consequences of the basic models and those of the various assumptions and approximations made in implementing them. Assuming that α-effect and angular velocity Ω are respectively antisymmetric and symmetric about the plane $z = 0$, we present computations to determine the first bifurcations from the trivial solution of modes symmetric (S) and antisymmetric (A) with respect to the plane $z = 0$. We use a common numerical scheme for both the TD and ZBC models and study both axisymmetric ($m = 0$) and nonaxisymmetric ($m = 1$) modes. We are thus able to compare directly the effects of the differing assumptions about the boundary conditions on the magnetic field.

2. Computational method

We solve the linear mean field dynamo equation

$$\partial \boldsymbol{B}/\partial t = \mathrm{curl}\,[\boldsymbol{u} \times \boldsymbol{B} + \alpha \boldsymbol{B} - \mathrm{curl}\,(\eta\,\mathrm{curl}\,\boldsymbol{B})], \tag{1}$$

by step by step integration on a uniform r, θ grid of size NI $\times$ NJ ($0 \le r \le 1$, $0 \le \theta \le \pi/2$). $\boldsymbol{u} = \Omega r \hat{\boldsymbol{\phi}}$ is an azimuthal velocity corresponding to a prescribed angular velocity Ω, and r and $z = r\cos\theta$ are now dimensionless coordinates, scaled with the radius R. α and Ω take the form

$$\alpha = \alpha_0 \cos\theta f^2 (3 - 2f), \tag{2}$$

where $f = (z_0 - z)/z_0$ for $0 \le z \le z_0$ and $f = 0$ for $z > z_0$;

$$\Omega = \Omega_0 \left[1 + \left(\frac{r}{r_0}\right)^2\right]^{-1/2}, \tag{3}$$

and we take $r_0 = 0.5$. We define the usual dynamo parameters

$$C_\alpha = \alpha_0 R/\eta, \quad C_\Omega = \Omega_0 R^2/\eta. \tag{4}$$

In linear theory the units of $\boldsymbol{B}$ are arbitrary.

The numerical codes are a modification of the nonlinear axisymmetric code described in Brandenburg et al. (1989) and a modification of the nonaxisymmetric code described in section 4 of Jennings et al. (1990). When we study nonaxisymmetric ($m = 1$) modes we assume $\boldsymbol{B} \sim \exp(i\phi)$, so that Eq. (1) still reduces to a partial differential equation in r, θ and t. In the TD computations we fit the field inside the sphere $r = R$ onto a curl-free external field. In the ZBC case the NI $\times$ NJ grid points are similarly distributed throughout $r \le R$, but we impose the condition of zero field in $z > z_0$ by making the radial component of $\boldsymbol{B}$ go continuously to zero on $z = z_0$, and the θ and ϕ components go discontinuously to zero there. This implementation slightly overestimates the gradients of B_θ and B_ϕ near $z = z_0$ (see Figs. 1 and 2), but experimentation suggests that growth rates, critical C_α values ($C_{\alpha,c}$) and general field topology are little affected. NI and NJ take values between 51 and 201, depending on the value of z_0. We felt that $z_0 = 0.1$ is the smallest value for which reasonably accurate results could be obtained.

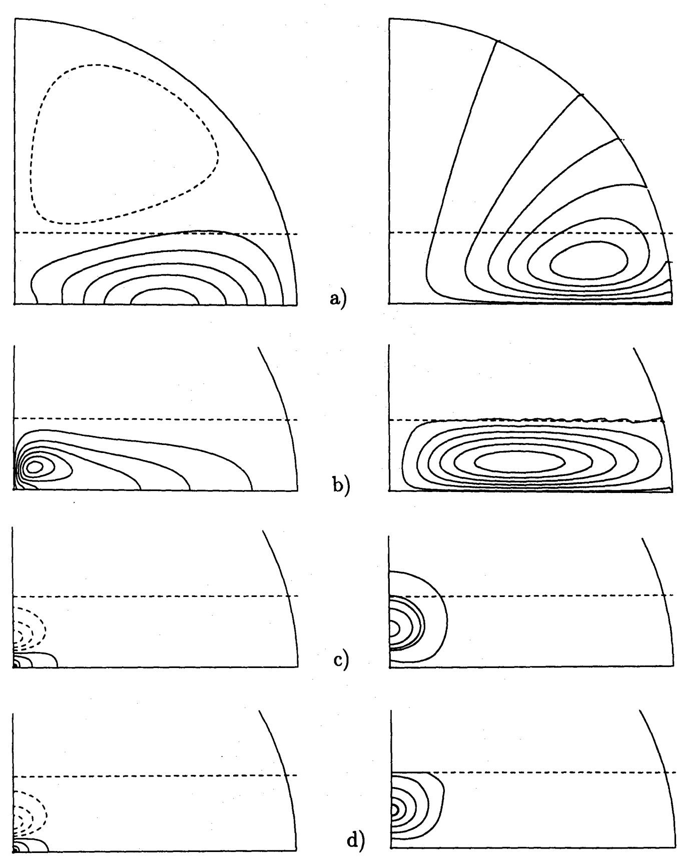

Fig. 1. Eigenmodes, showing on the left contours of B_ϕ in arbitrarily chosen meridional planes, for $z_0 = 0.25$. a) S0, TD; b) S0, ZBC; c)S1, TD; d)S1, ZBC. The right hand plots show poloidal field lines for the $m = 0$ modes, and those for the $m = 1$ modes are described in the text

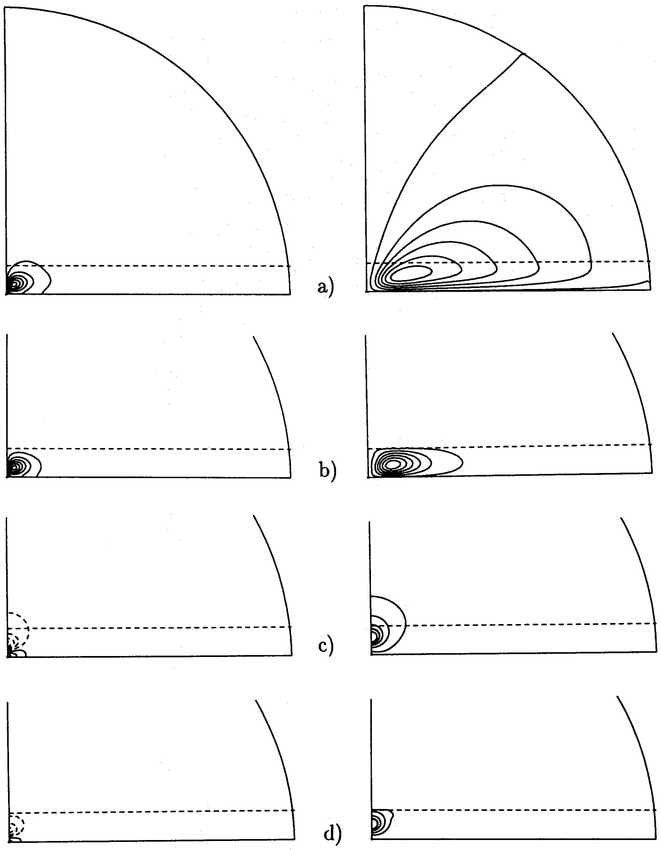

Fig. 2. As Fig. 1, but for $z_0 = 0.10$

3. Results and Discussion

Table 1 presents values of C_α at the first bifurcation for A0, S0, A1 and S1 modes. For the ZBC nonaxisymmetric case we only give results for $C_\Omega = 10^3$. Calculations with a slightly different model lead us to expect that the $C_{\alpha,c}$ values at $C_\Omega = 0$ for the S1 and A1 modes do not differ systematically between the TD and ZBC models, and that at $C_\Omega = 10^3$ the $C_{\alpha,c}$ values for the S1 and A1 modes are very similar. (For example, for $z_0 = 0.15$, $C_{\alpha,c}^{(A1)} = 90.2$ and $C_{\alpha,c}^{(S1)} = 90.3$.) Where we give only results for the S1 mode we expect the $C_{\alpha,c}$ value for the A1 mode to be very similar.

Table 1. Values of $C_{\alpha,c}$ at the first bifurcation for A0, S0, A1 and S1 modes. For each case the smallest value of $C_{\alpha,c}$ is typed in bold face

		TD				ZBC		
z_0	C_Ω	S1	A1	S0	A0	S1	S0	A0
.50	0.	26.4	26.5	**25.6**	25.3	–	**26.0**	26.3
.25	0.	52.9	53.4	**50.8**	**50.6**	–	**52.3**	52.5
.15	0.	88.4	89.5	**84.7**	–	–	87.3	**86.8**
.50	10^3	32.8	31.6	**13.4**	36.1	32.8	32.0	**15.0**
.25	10^3	54.8	54.3	**28.6**	53.6	53.8	**48.3**	54.6
.15	10^3	89.0	88.5	**77.0**	86.3	90.3	**86.9**	88.0
.10	10^3	133.9	–	**126.8**	127.6	135.9	**131.2**	131.6

Table 2. Δ as defined by Eq. (5) for the modes with $C_\Omega = 10^3$

z_0	Δ_{TD}	Δ_{ZBC}
0.50	0.42	0.37
0.25	0.31	0.05
0.15	0.07	0.02
0.10	0.03	0.02

The quantity Δ is defined by

$$\Delta = [C_{\alpha,c}^{(S1)} - C_{\alpha,c}^{(S0)}]/[C_{\alpha,c}^{(S1)} + C_{\alpha,c}^{(S0)}], \tag{5}$$

and is a measure of the relative ease of exciting the S0 rather than the S1 mode. Values of Δ for the modes with $C_\Omega = 10^3$ are given in Table 2.

When $C_\Omega = 10^3$, we see that for $z_0 = 0.5$, then $\Delta_{TD} \approx \Delta_{ZBC}$, whereas for intermediate z_0 then $\Delta_{TD} \gg \Delta_{ZBC}$. When $z_0 = 0.10$, the smallest value studied, then $\Delta_{TD} \approx \Delta_{ZBC}$ again. For $C_\Omega = 10^3$, $z_0 = 0.25$ it is actually easier to excite $m = 1$ ZBC than TD modes, but in all cases the differences in $C_{\alpha,c}$ values are quite small between corresponding $m = 1$ modes. Thus it appears that both for thick and relatively thin disks there is little relative difference between the relative ease of excitation of $m = 0$ and $m = 1$ modes when comparing the TD and ZBC models, whereas for intermediate values in the ZBC case nonaxisymmetric modes are relatively favoured. However in no case are nonaxisymmetric modes *easier* to excite than axisymmetric.

In the left hand columns of Figs. 1 and 2 we present contours of equal toroidal field strength in arbitrarily chosen meridional planes ϕ = constant. For axisymmetric modes, the right hand columns show poloidal field lines. For $m = 1$ modes we plot the trajectories obtained by following the vector $(B_r, B_\theta, 0)$ in these planes, starting from arbitrarily chosen points in the plane; these are *not* magnetic field lines. These figures show that the contours of B_ϕ (equally spaced) have a generally similar structure in the ZBC and TD models. The magnetic fields projected on to meridional planes show the larger differences, with the TD field extending well into the halo region (cf Donner & Brandenburg, 1990). As found in previous studies, the strong z-dependence in α itself imposes a length scale in the z-direction of order z_0, at least in these linear calculations, and this seems to be responsible for the convergence of the $C_{\alpha,c}$ values. The nonaxisymmetric ZBC models in particular display a concentration of the magnetic field to a region $s \lesssim z_0$. In Fig. 3 we show the radial dependence of B_r, B_θ and B_ϕ along a direction near to the equatorial plane $\theta = \pi/2$. However, more realistic models might include a "halo α" also resulting from turbulent motions in the halo region (Sokoloff & Shukurov, 1990), and it is not clear *a priori* how well these features would then survive.

The following estimate can be made of when the effect of the vertical disk structure becomes important. Differential rotation can be expected to inhibit nonaxisymmetric fields when the field is strongly wound up. The diffusion-limited winding number can be estimated by $n_c \approx (\Delta\Omega R^2/2\pi\eta)^{1/3}$ (Rädler, 1986; Moss et al., 1990). In the context of a disk model, this means that the length scale in the radial (s-) direction is then reduced to $O(R/n_c)$. Putting $\Delta\Omega \sim \Omega \sim 3\,10^{-16}\mathrm{sec}^{-1}$, $R \sim 10\,\mathrm{kpc}$, $\eta \sim 10^{26}\mathrm{cm}^2\mathrm{sec}^{-1}$ (roughly consistent with our value $C_\Omega = 10^3$) gives $n_c = O(10)$. We expect that the vertical disk structure becomes important when $z_0 \lesssim R/n_c$, ie if $z_0/R \lesssim 0.1$.

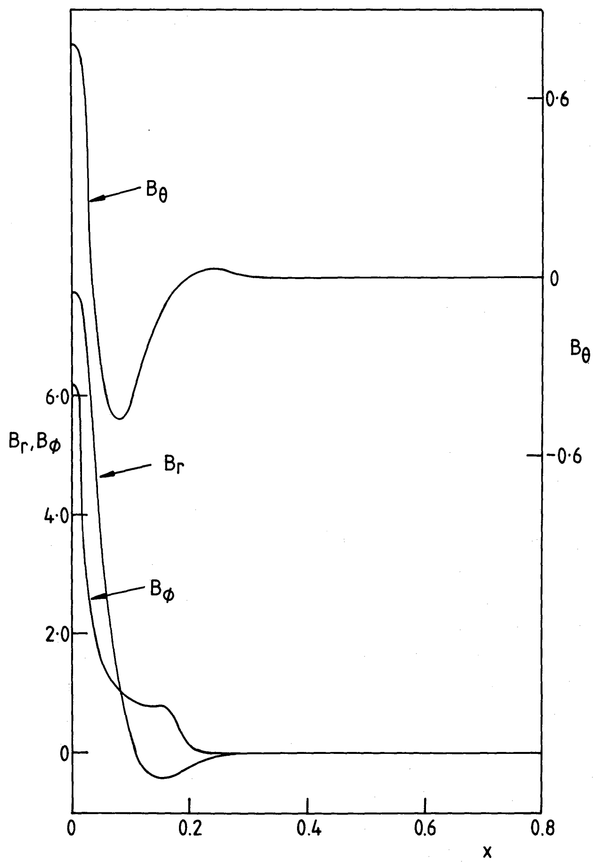

Fig. 3. B_r, B_θ and B_ϕ (in arbitrary units) as a function of radius for arbitrary ϕ and θ near to $\pi/2$ for ZBC model, $z_0 = 0.15$, S1 mode (resolution NI=201, NJ=101)

We note that, for the weakly supercritical values of C_α that we investigated, the dependence of the field growth rate, λ, on C_α is quite steep. If λ is dimensionless (unit η/R^2), then for $z_0 = 0.1$ we find $d\lambda/d\ln C_\alpha = O(1000)$. With the above values for R and η the unit of λ is 10^{-19} sec^{-1}, and a growth time $\sim 10^9$ years corresponds to $\lambda \sim 300$. Then $\Delta\lambda \sim \lambda$ for $\Delta \ln C_\alpha \sim 0.3$. Thus the growth times are relatively sensitive to the ill-known parameter C_α and, for example, arguments for or against the importance of nonlinear effects based on the absolute size of growth rates for any assumed value of C_α should be regarded with caution.

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