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Lyapunov exponents for hydromagnetic convection

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We estimate the two largest Lyapunov exponents in a three-dimensional simulation of hydromagnetic convection in which there is dynamo action. It turns out that these first two exponents (from a total of 8×63^3) are positive and of similar magnitude. Thus we conclude that the dynamo is chaotic. Furthermore, the consideration of local exponents helps in our understanding of the relevant dynamics. We find that the downdraft flows are more chaotic than the upward motions. Likewise, the velocity and magnetic fields have more chaotic dynamics than the temperature and density fields.

Although the energy output of the sun, the main basis of life on earth, is nearly constant, observations show that the solar surface (photosphere) is far from uniform. Typical features are the granulation pattern indicating turbulent motion in the convection zone underneath the photosphere, a rotation rate that decreases with latitude, and the most spectacular inhomogeneities, the sunspots, which are concentrations of strong magnetic fields.

It is generally accepted that the sun's magnetic field is generated by a turbulent dynamo process [1]. Such a self-sustained dynamo amplifies and maintains a magnetic field by converting kinetic energy of turbulent convective motions into magnetic energy. It has been demonstrated by three-dimensional (3D) simulations that a spontaneous onset and maintenance of dynamo action can result from turbulent motions of a conducting fluid [2–4]. The nature of such complex dynamics can be characterized by means of Lyapunov exponents. This concept has been applied to, for example, truncated dynamo models [5] and incompressible Navier-Stokes flow with external forcing [6].

Here, we use a direct simulation of dynamo action in the compressible hydromagnetic convection of Ref. [4]. It is not our aim to calculate the full spectrum of Lyapunov exponents and a dimension of the underlying attractor (cf. Ref. [6]), but rather to establish whether a system showing dynamo action behaves chaotically. Therefore we only estimate the two largest Lyapunov exponents.

Measuring Lyapunov exponents. An important classification of the dynamical behavior of a system with N degrees of freedom is by means of Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. The first exponent expresses the average exponential convergence or divergence of initially nearby trajectories of the state vector in phase space. If λ_1 is positive for certain control parameters (e.g., Rayleigh, Prandtl, or Taylor numbers) then the system is defined to be chaotic in this regime. λ_1 can asymptotically be determined from the separation $\epsilon(t)$ of two trajectories of the system [6]

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t-t_0} \ln \frac{\epsilon(t)}{\epsilon(t_0)}. \tag{1}$$

In order to keep the separation small enough we require $\epsilon(t_0) \ll 1$. Since $\epsilon(t)$ can grow rapidly in chaotic systems Benettin *et al.* [7] used a reorthonormalization at regular time intervals. λ_1 then follows from averaging the corresponding instantaneous exponents. Grappin and Léorat [6] applied this technique to their long-term simulations of stochastically forced Navier-Stokes flows with moderate resolution in 2D (64^2, 128^2) and low resolution in 3D (16^3).

In our case the flow is driven by a vertical temperature gradient large enough to cause irregular convective motions that in turn give rise to a dynamo effect [3]. A grid of 63^3 points is used in the numerical simulation (for further details of the method see Ref. [8]). Such a resolution prohibits long integrations due to constraints on the computer time available. Nevertheless, rough estimates of the two largest Lyapunov exponents over a few convective turnover times can be obtained, if our system is assumed to be ergodic. However, we should expect that successions of rapid and slow phases are typical [6], and this introduces some uncertainty. Hence, the stability of our estimates has to be checked.

To estimate the first and second Lyapunov exponents an initial state $\mathbf{q}^{(0)}$, as well as two perturbed states, $\mathbf{q}^{(1)} = \mathbf{q}^{(0)} + \delta\mathbf{q}^{(1)}$, and $\mathbf{q}^{(2)} = \mathbf{q}^{(0)} + \delta\mathbf{q}^{(2)}$, are integrated in time, where $\mathbf{q} = (u_x, u_y, u_z, \ln\rho, e, B_x, B_y, B_z)$ is the state vector. Here, u_x, u_y, and u_z are the three velocity components, $\ln\rho$ the logarithm of the density, e the internal energy (temperature), and B_x, B_y, and B_z are the three magnetic-field components. We take $\delta\mathbf{q}^{(1)} = (\delta u_x, \delta u_y, \delta u_z, 0,0,0,0,0)$, and $\delta\mathbf{q}^{(2)} = (\delta u_y, -\delta u_x, 0,0,0,0,0,0)$, i.e., the initial perturbations

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are orthogonal and made only in the velocity components. A solenoidal random velocity field is chosen for $\mathbf{q}^{(1)}$, but not $\mathbf{q}^{(2)}$ (note that the fluid is compressible).

In the course of the integration we estimate the largest instantaneous Lyapunov exponent

$$\lambda_1^{(\mathrm{inst})}(t) = \sum_{i=1}^{8} \frac{d}{dt} \ln \epsilon_i(t), \tag{2}$$

where $\epsilon_i(t)$ is the norm of the difference of two solutions (a) and (b) with

$$\epsilon_i(t) = \int \mid q_i^{(a)}(\mathbf{x},t) - q_i^{(b)}(\mathbf{x},t) \mid d^3x, \tag{3}$$

where $a, b = 0, 1, 2$. In practice, the integral in Eq. (3) is evaluated as a sum over all 63^3 meshpoints. In order to check the convergence we define the cumulative Lyapunov exponent

$$\lambda_1(t) = \frac{1}{t - t_0} \int_{t_0}^{t} dt' \lambda_1^{(\mathrm{inst})}(t'), \tag{4}$$

which approaches $\lambda_1 = \lim_{t\to\infty} \lambda_1(t)$, in accordance with Eq. (1). [The integral in Eq. (4) is evaluated as a sum over a finite number of time steps.] If three initial states are integrated we get the sum $\lambda_1 + \lambda_2$, analogous to Eqs. (2)–(4) from the growth rates of the *area* spanned by the three trajectories $\mathbf{q}^{(0)}$, $\mathbf{q}^{(1)}$, and $\mathbf{q}^{(2)}$. We also obtain three independent estimates for λ_1, which allows us to check the robustness of the values of λ_1 under different initial conditions.

Results. We restarted a simulation of Ref. [4] with magnetic Prandtl number Pr_M=4 at time $t = 608$, where spontaneous dynamo action with rapid amplification of magnetic-field energy approaches a saturation phase (see Fig. 1 in Ref. [4]). The Rayleigh number for this model is 10^6, the Taylor number 10^5, and the Prandtl number 0.2. The time unit adopted in the simulation is $(d/g)^{1/2}$, where d is the height of the layer and g is gravity. The time step (limited by the Courant condition) is about 0.003 time units and the convective turnover time d/u_t is around 20 time units, where u_t is the (turbulent) rms velocity.

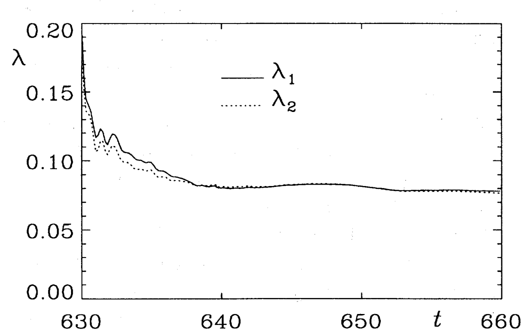

FIG. 1. The first and second cumulative Lyapunov exponents (solid and dotted lines) obtained at the time interval $625 < t < 660$.

The cumulative Lyapunov exponent $\lambda_n(t)$, given in Fig. 1, seems to settle after time $t \approx 640$. This is also the time when the generated magnetic field saturates (cf. Fig. 1 in Ref. [4]). We find $\lambda_1 \approx 0.08 \pm 0.01$, i.e., the e-folding time $\lambda_1^{-1} \approx 13$ is somewhat shorter than the convective turnover time. Moreover, λ_1 and λ_2 are of similar magnitude. Given that the $\lambda_n(t)$ spectrum is approximately linear with n (at least for small n; see, e.g., Refs. [6,9]), this suggests that a wide range of positive Lyapunov exponents exists, and that the Lyapunov dimension may be rather high. Note that Grappin and Léorat [6] found Lyapunov dimensions up to 123 for a system with only 3000 degrees of freedom and that the first exponents deviate only by a few percent.

The small initial perturbations chosen [$\epsilon_i(t_0) \approx 10^{-9}$] ensure an unbounded growth of $\epsilon_i(t)$ during our limited integration time, and a reorthonormalization procedure was not necessary. Our analysis of several simulations shows that the results obtained with this technique are similar over a rather broad range of initial perturbations [we also tried $\epsilon_i(t_0) \approx 10^{-6}$], revealing the numerical stability of the results.

In order to throw some light on the physical processes involved we also considered separately exponents for velocity (u_x, u_y, u_z), magnetic field (B_x, B_y, B_z), and thermal $(\ln\rho, e)$ components (cf. Fig. 2). Both kinetic and magnetic contributions show positive growth rates, and the sign of the growth rates of the thermal components oscillates, which suggests a weaker form of chaos from the thermal part of the system. This may be a consequence of the small Prandtl number of 0.2, which causes a more rapid relaxation of the temperature field than of the velocity and magnetic fields.

The fluctuations of the instantaneous Lyapunov exponent (upper panel of Fig. 2) are much smaller in our model than, for example, in simple shell models of turbulence [9]. However, such shell models do not reflect the full spatio-temporal pattern. In order to investigate spatial nonuniformity we define local exponents

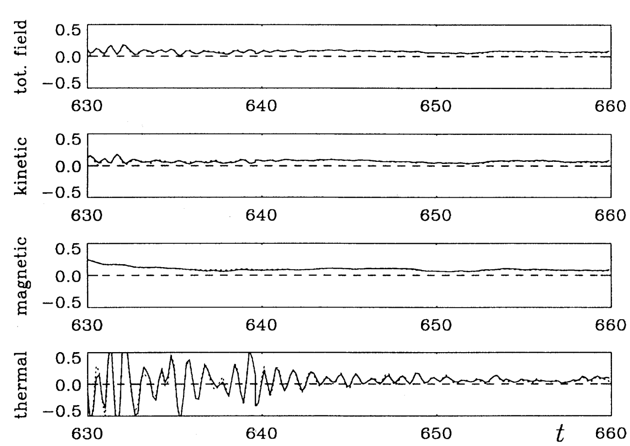

FIG. 2. Instantaneous Lyapunov exponents (upper panel) and the contributions from the kinetic, magnetic, and thermal variables. The dotted lines refer to the second exponent.

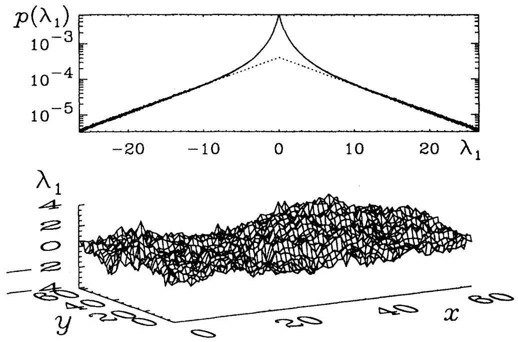

FIG. 3. Histogram (upper panel) and spatial structure (lower panel) of local exponents $\lambda_1(\mathbf{x})$. The data are accumulated over the time range $640 < t < 660$. Note the spatial nonuniformity.

$$\lambda_1^{(\mathrm{loc})}(\mathbf{x}) = \sum_{i=1}^{8} \frac{1}{t_1 - t_0} \ln \frac{\epsilon_i(\mathbf{x}, t_1)}{\epsilon_i(\mathbf{x}, t_0)}. \quad (5)$$

A histogram of the distribution $p(\lambda_1)$ of local and instantaneous exponents indicates a marked inhomogeneity in the turbulence (upper panel of Fig. 3). Note especially that there are stable regions of considerable extent with $\lambda_1(\mathbf{x}) < 0$. The higher moments of this distribution (the skewness is -0.005 and the kurtosis or excess is 11.7) quantify the deviation from the normal distribution. Such nonuniform dynamics result mainly from spatial inhomogeneities (lower panel of Fig. 3). It is the spatial averaging in Eq. (3) that, in our model, smooths spatial inhomogeneities. This could be the reason for the surprisingly uniform behavior of $\lambda_n(t)$ in Fig. 1. Note

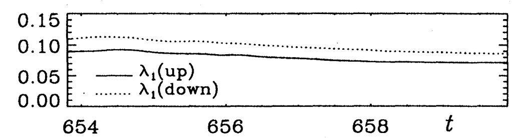

FIG. 4. Contributions from regions of upward and downward flow to the cumulative Lyapunov exponent.

that the opportunity of such extensive spatial averaging compensates for the disadvantage of relatively short integration times mentioned earlier, if we assume ergodicity.

It is known in compressible convection that downdraft flows are much faster and more vigorous than upward motions [10], and most of the dynamo action comes from these downdraft regions [4]. It is therefore interesting to see whether the local exponents reflect such behavior. In Fig. 4 we have plotted the instantaneous exponents $\lambda_1^{(\mathrm{up})}$ and $\lambda_1^{(\mathrm{down})}$ of the two subsystems, in which the vertical velocity component u_z is directed upwards and downwards, respectively. We see that $\lambda_1^{(\mathrm{down})}$ is slightly larger than $\lambda_1^{(\mathrm{up})}$, but the difference is less than anticipated from the results of Ref. [4]. The lack of a strong difference here may be partly due to an enhanced divergence of phase-space trajectories related to the fluid expansion of rising fluid elements.

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