Numerical simulations of gravitational waves from early-universe turbulence

APS April meeting (April 18–21 2020)

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April 20, 2020

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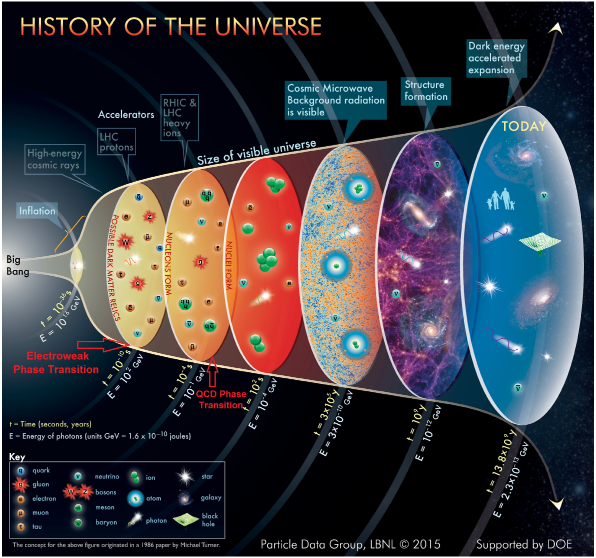
A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.* **114**, 130. arXiv:1807.05479 (2019)

A. Roper Pol *et al.*, *submitted to Phys. Rev. D* arXiv:1903.08585 (2019)

Introduction and Motivation

- Generation of cosmological gravitational waves (GWs) during phase transitions and inflation
    - **Electroweak phase transition** ~ 100 GeV
    - Quantum chromodynamic (QCD) phase transition ~ 100 MeV
    - Inflation


HISTORY OF THE UNIVERSE
Dark energy accelerated expansion
Cosmic Microwave Background radiation is visible
Structure formation
RHIC & LHC heavy ions
Accelerators
LHC protons
High-energy cosmic rays
Size of visible universe
TODAY
Inflation
Big Bang
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
t = 10^-36 s
E = 10^16 GeV
Electroweak Phase Transition
t = 10^-10 s
E = 10^2 GeV
t = 10^-4 s
E = 10^-1 GeV
QCD Phase Transition
t = 10^2 s
E = 10^-4 GeV
t = 3x10^5 y
E = 3x10^-10 GeV
t = 10^9 y
E = 10^-12 GeV
t = 13.8x10^9 y
E = 2.3x10^-13 GeV
t = Time (seconds, years)
E = Energy of photons (units GeV = 1.6 x 10^-10 joules)
Key
quark
gluon
electron
muon
tau
neutrino
bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
The concept for the above figure originated in a 1986 paper by Michael Turner.
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Introduction and Motivation

- Generation of cosmological gravitational waves (GWs) during phase transitions and inflation
  - Electroweak phase transition ~ 100 GeV
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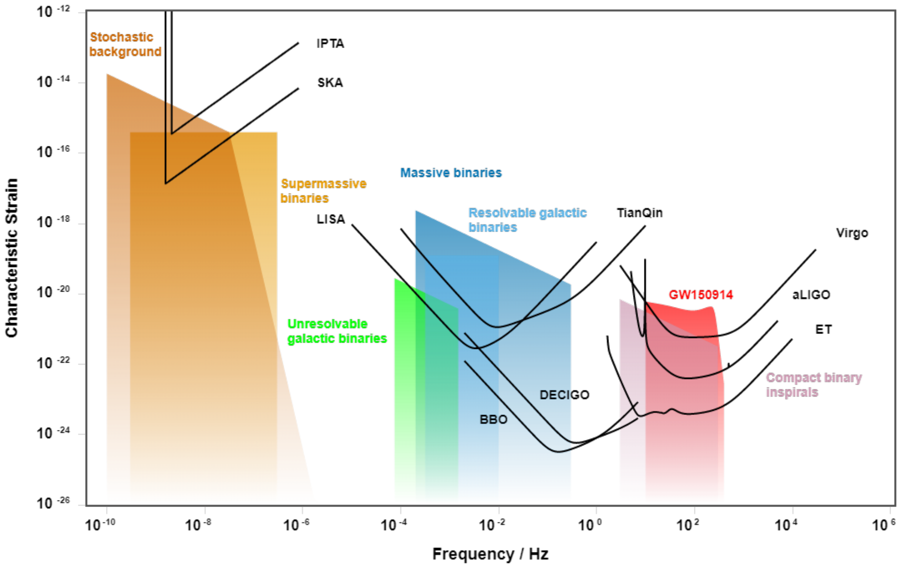
  - Inflation
- GW radiation as a probe of early universe physics
- Possibility of GWs detection with
  - **Space-based GW detector LISA**
  - Pulsar Timing Arrays (PTA)
  - B-mode of CMB polarization

Introduction and Motivation



Introduction and Motivation

- Generation of cosmological gravitational waves (GWs) during phase transitions and inflation
  - Electroweak phase transition $\sim$ 100 GeV
  - Quantum chromodynamic (QCD) phase transition $\sim$ 100 MeV
  - Inflation
- GW radiation as a probe of early universe physics
- Possibility of GWs detection with
  - Space-based GW detector LISA
  - Pulsar Timing Arrays (PTA)
  - B-mode of CMB polarization
- Magnetohydrodynamic (MHD) sources of GWs:
  - Hydrodynamic turbulence from phase transition bubbles nucleation
  - Primordial magnetic fields
- Numerical simulations using PENCIL CODE to solve:
  - Relativistic MHD equations
  - Gravitational waves equation

Gravitational waves equation

GWs equation for an expanding flat Universe

- Assumptions: isotropic and homogeneous Universe
- Friedmann–Lemaître–Robertson–Walker (FLRW) metric $\gamma_{ij} = a^2 \delta_{ij}$
- Tensor-mode perturbations above the FLRW model:

$$g_{ij} = a^2 \left(\delta_{ij} + h_{ij}^{\rm phys} \right)$$

- GWs equation is[1]

$$\left(\partial_t^2 - c^2 \nabla^2 \right) h_{ij} = \frac{16 \pi G}{\mathbf{a} c^2} T_{ij}^{\rm TT}$$

  - h_{ij} are rescaled $h_{ij} = a h_{ij}^{\rm phys}$
  - Comoving spatial coordinates $\nabla = a \nabla^{\rm phys}$
  - Conformal time $\mathrm{d}t = a \, \mathrm{d}t^{\rm phys}$
  - Comoving stress-energy tensor components $T_{ij} = a^4 T_{ij}^{\rm phys}$
  - Radiation-dominated epoch such that $a'' = 0$

[1] L. P. Grishchuk, *Sov. Phys. JETP*, 40, 409-415 (1974)

Normalized GW equation[2]

$$\left(\partial_t^2 - \nabla^2\right) h_{ij} = 6\, T_{ij}^{\mathrm{TT}} / t$$

Properties

- All variables are normalized and non-dimensional
- Conformal time is normalized with t_*
- Comoving coordinates are normalized with c/H_*
- Stress-energy tensor is normalized with $\mathcal{E}_{\mathrm{rad}}^* = 3H_*^2 c^2/(8\pi G)$
- Scale factor is $a_* = 1$, such that $a = t$

[2] A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.* **114**, 130. arXiv:1807.05479 (2019)

Gravitational waves equation

Properties

- Tensor-mode perturbations are gauge invariant
- h_{ij} has only two degrees of freedom: h^{+}, $h^{\times}$
- The metric tensor is traceless and transverse (TT gauge)

Contributions to the stress-energy tensor

$$T^{\mu\nu} = \left(p/c^2 + \rho\right) U^\mu U^\nu + p g^{\mu\nu} + F^{\mu\gamma} F^{\nu}{}_{\gamma} - \frac{1}{4} g^{\mu\nu} F_{\lambda\gamma} F^{\lambda\gamma}$$

- From fluid motions
  $T_{ij} = \left(p/c^2 + \rho\right) \gamma^2 u_i u_j + p\delta_{ij}$
  Relativistic equation of state:
  $p = \rho c^2/3$
- From magnetic fields:
  $T_{ij} = -B_i B_j + \delta_{ij} B^2/2$

MHD equations

Conservation laws

$$T^{\mu\nu}{}_{;\nu} = 0$$

Relativistic MHD equations are reduced to[3]

MHD equations

$$\frac{\partial \ln \rho}{\partial t} = -\frac{4}{3}\left(\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \ln \rho\right) + \frac{1}{\rho}\left[\boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) + \eta \boldsymbol{J}^2\right]$$

$$\frac{D\boldsymbol{u}}{Dt} = \frac{4}{3}\left(\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \ln \rho\right) - \frac{\boldsymbol{u}}{\rho}\left[\boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) + \eta \boldsymbol{J}^2\right] - \frac{1}{4}\nabla \ln \rho + \frac{3}{4\rho}\boldsymbol{J} \times \boldsymbol{B} + \frac{2}{\rho}\nabla \cdot (\rho \nu \boldsymbol{S})$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{J})$$

for a flat expanding universe with comoving and normalized $p = a^4 p_{\rm phys}, \rho = a^4 \rho_{\rm phys}, B_i = a^2 B_{i,_{\rm phys}}, u_i$, and conformal time t.

[3] A. Brandenburg, K. Enqvist, and P. Olesen, *Phys. Rev. D* **54**, 1291 (1996)

Linear polarization modes $+$ and $\times$

Linear polarization basis (defined in Fourier space)

$$e_{ij}^{+} = (\boldsymbol{e}_1 \times \boldsymbol{e}_1 - \boldsymbol{e}_2 \times \boldsymbol{e}_2)_{ij}$$

$$e_{ij}^{\times} = (\boldsymbol{e}_1 \times \boldsymbol{e}_2 + \boldsymbol{e}_2 \times \boldsymbol{e}_1)_{ij}$$

Orthogonality property

$e_{ij}^{A} e_{ij}^{B} = 2\delta_{AB}$, where $A, B = +, \times$

$+$ and $\times$ modes

$$\tilde{h}^{+} = \frac{1}{2} e_{ij}^{+} \tilde{h}_{ij}^{\mathrm{TT}}, \qquad \tilde{T}^{+} = \frac{1}{2} e_{ij}^{+} \tilde{T}_{ij}^{\mathrm{TT}}$$

$$\tilde{h}^{\times} = \frac{1}{2} e_{ij}^{\times} \tilde{h}_{ij}^{\mathrm{TT}}, \qquad \tilde{T}^{\times} = \frac{1}{2} e_{ij}^{\times} \tilde{T}_{ij}^{\mathrm{TT}}$$

Solving the GW equation

- Compute Fourier transform of stress-energy tensor $\tilde{T}_{ij}$
- Project into TT gauge $\tilde{T}_{ij}^{\mathrm{TT}} = \left(P_{il}P_{jm} - \frac{1}{2}P_{ij}P_{lm}\right) \tilde{T}_{lm}^{\mathrm{TT}}$
- Compute $\tilde{T}^{+}$ and $\tilde{T}^{\times}$ modes
- Discretize time using δt from MHD simulations
- Assume $\tilde{T}^{+,\times}/t$ to be constant between subsequent timesteps (robust as $\delta t \to 0$)
- GW equation solved analytically between subsequent timesteps in Fourier space[4]

$$\begin{pmatrix} \omega \tilde{h} - 6\omega^{-1}\tilde{T}/t \\ \tilde{h}' \end{pmatrix}_{+,\times}^{t+\delta t} = \begin{pmatrix} \cos\omega\delta t & \sin\omega\delta t \\ -\sin\omega\delta t & \cos\omega\delta t \end{pmatrix} \begin{pmatrix} \omega \tilde{h} - 6\omega^{-1}\tilde{T}/t \\ \tilde{h}' \end{pmatrix}_{+,\times}^{t}$$

[4] A. Roper Pol *et al.*, *Geophys. Astrophys. Fluid Dyn.*, **114**, 130 arXiv:1807.05479 (2019)

GWs energy density:

$$\Omega_{\mathrm{GW}} = \mathcal{E}_{\mathrm{GW}}/\mathcal{E}^0_{\mathrm{crit}}, \quad \mathcal{E}^0_{\mathrm{crit}} = \frac{3H_0^2 c^2}{8\pi G}$$

$$\Omega_{\mathrm{GW}} = \int_{-\infty}^{\infty} \Omega_{\mathrm{GW}}(k)\,\mathrm{d}\ln k$$

$$\boldsymbol{\Omega}_{\mathrm{GW}}(\mathbf{k}) = (a_*/a_0)^4 \frac{k}{6H_0^2} \int_{4\pi} \left(\left| \dot{\tilde{h}}_+^{\mathrm{phys}} \right|^2 + \left| \dot{\tilde{h}}_\times^{\mathrm{phys}} \right|^2 \right) k^2 \, \mathrm{d}\Omega_k$$

$$H_0 = 100\, h_0 \ \mathrm{km\ s^{-1}\ Mpc^{-1}}$$

$$\frac{a_0}{a_*} \approx 1.254 \cdot 10^{15} \left(T_*/100 \ \mathrm{GeV} \right) \left(g_{\mathrm{S}}/100 \right)^{1/3}$$

Characteristic amplitude of gravitational waves

GWs amplitude:

$$h_{\rm c}^2 = \int_{-\infty}^{\infty} h_{\rm c}^2(k)\,{\rm d}\ln k$$

$$\mathbf{h_{\rm c}^2(k)} = (a_*/a_0)k \int_{4\pi} \left(\left|\tilde{h}_+^{\rm phys}\right|^2 + \left|\tilde{h}_\times^{\rm phys}\right|^2 \right) k^2\,{\rm d}\Omega_k$$

Frequency:

$$f = H_*(a_*/a_0)(k/2\pi) \approx 1.6475 \cdot 10^{-5}(k/2\pi)\ {\rm Hz}$$

for $T_* = 100$ GeV, $g_{\rm S} \approx g_* = 100$.

Numerical results for decaying MHD turbulence[7]

Initial conditions[8]

- Fully helical stochastic magnetic field
- Batchelor spectrum, i.e., $E_{\rm M} \propto k^4$ for small k
- Kolmogorov spectrum for inertial range, i.e., $E_{\rm M} \propto k^{-5/3}$
- Total energy density at t_* is $\sim 10\%$ to the radiation energy density
- Spectral peak at $k_{\rm M} = 100 \cdot 2\pi$, normalized with $k_H = 1/(cH)$

Numerical parameters

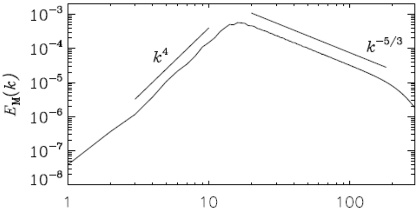
- 1152^3 mesh gridpoints
- 1152 processors
- Wall-clock time of runs is $\sim 1 - 5$ days

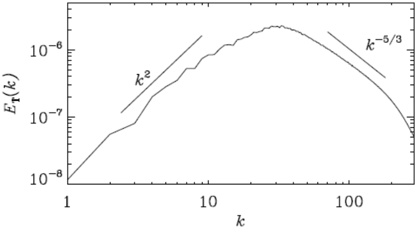
[7] A. Roper Pol, *et al.* arXiv:1903.08585

[8] A. Brandenburg, *et al.* *Phys. Rev. D* **96**, 123528 (2017)

Initial magnetic spectra

$k_{\mathrm{M}} = 15$





$$E_{\mathrm{M}}(k) = \frac{1}{2} \int_{4\pi} \left(\tilde{\mathbf{B}}(k) \cdot \tilde{\mathbf{B}}^*(k) \right) k^2 d\Omega$$

$$\Omega_{\mathrm{M}} = \int_0^\infty E_{\mathrm{M}}(k) dk$$

$$E_{\mathrm{T}}(k) = \int_{4\pi} \left(\tilde{\mathbf{T}}_{ij}(k) \tilde{\mathbf{T}}_{ij}^*(k) \right) k^2 d\Omega$$

$$E_{\mathrm{T}}(k) = \frac{1}{2} \int_{4\pi} \left(\tilde{\mathbf{B}}^2(k) \tilde{\mathbf{B}}^{2,*}(k) \right) k^2 d\Omega$$

$$\Omega_{\mathrm{T}} = \int_0^\infty E_{\mathrm{T}}(k) dk$$

Detectability with LISA

LISA

- Laser Interferometer Space Antenna (LISA) is a space–based GW detector
- LISA is planned for 2034
- LISA was approved in 2017 as one of the main research missions of ESA
- LISA is composed by three spacecrafts in a distance of 2.5M km

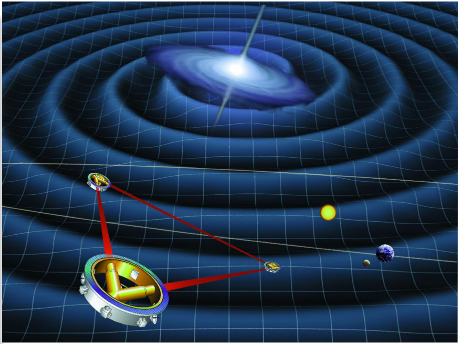
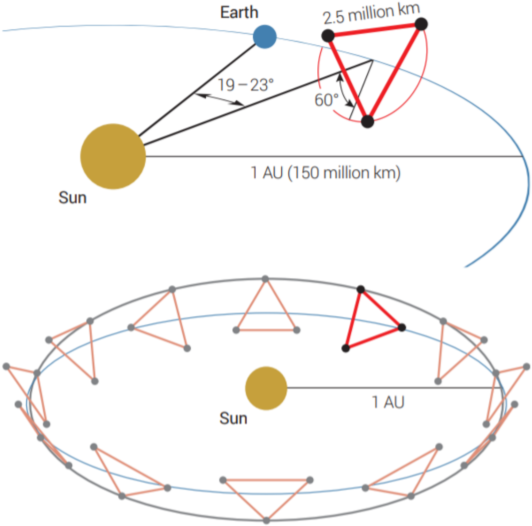
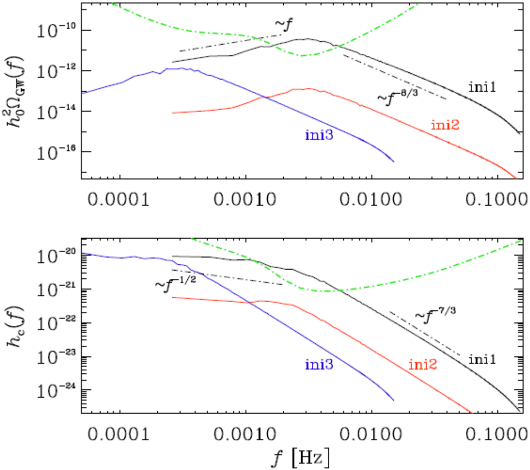


Figure: Artist's impression of LISA from Wikipedia

Orbit of LISA

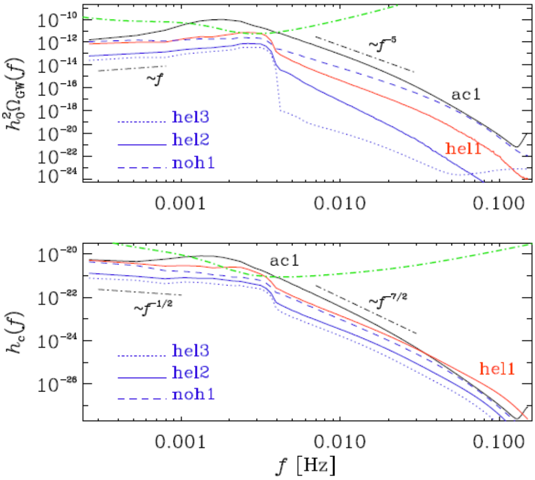


Numerical results for decaying MHD turbulence



Run	$\mathcal{E}_0, \mathcal{F}_0$	η	$\Omega_i^{\max}$	$\Omega_{\rm GW}^{\rm sat}$	i	hel	$t_{\max}$	N
ini1	—	5e-6	1.16e-01	2.05e-09	M	y	1.00	100
ini2	—	5e-8	7.62e-03	6.38e-12	M	y	1.00	100
ini3	—	5e-7	7.62e-03	6.36e-10	M	y	1.00	10

Forced turbulence (built-up primordial magnetic fields and hydrodynamic turbulence)



Run	$\mathcal{E}_0, \mathcal{F}_0$	η	$\Omega_i^{\max}$	$\Omega_{\rm GW}^{\rm sat}$	i	hel	$t_{\max}$	N
hel1	1.4e-3	5e-7	2.17e-02	4.43e-09	M	y	1.10	100
hel2	8.0e-4	5e-7	7.18e-03	4.67e-10	M	y	1.10	100
hel3	2.0e-3	5e-7	4.62e-03	2.09e-10	M	y	1.01	100
hel4	1.0e-4	2e-6	5.49e-03	1.10e-11	M	y	1.01	1000
noh1	1.4e-3	5e-7	1.44e-02	3.10e-09	M	n	1.10	100
noh2	8.0e-4	2e-6	4.86e-03	3.46e-10	M	n	1.10	100
ac1	3.0	2e-5	1.33e-02	5.66e-08	K	n	1.10	100
ac2	3.0	5e-5	1.00e-02	3.52e-08	K	n	1.10	100
ac3	1.0	5e-6	2.87e-03	2.75e-09	K	n	1.10	100

Conclusions

- We have implemented a module within the open-source Pencil Code that allows to obtain background stochastic GW spectra from primordial magnetic fields and hydrodynamic turbulence.
- For some of our simulations we obtain a detectable signal by future GW detector LISA.
- GW equation is normalized such that it can be easily scaled for different times within the radiation-dominated epoch.
- Bubble nucleation and magnetogenesis physics can be coupled to our equations for more realistic production analysis.

The End

Thank You!

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