The scalar, vector, and tensor modes in gravitational wave turbulence simulations

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Abstract. We study the gravitational wave (GW) signal sourced by primordial turbulence that is assumed to be present at cosmological phase transitions like the electroweak and quantum chromodynamics phase transitions. We consider various models of primordial turbulence, such as those with and without helicity, purely hydrodynamical turbulence induced by fluid motions, and magnetohydrodynamic turbulence whose energy can be dominated either by kinetic or magnetic energy, depending on the nature of the turbulence. We also study circularly polarized GWs generated by parity violating sources such as helical turbulence. Our ultimate goal is to determine the efficiency of GW production through different classes of turbulence. We find that the GW energy and strain tend to be large for acoustic or irrotational turbulence, even though its tensor mode amplitude is relatively small at most wave numbers. Only at very small wave numbers is the spectral tensor mode significant, which might explain the efficient GW production in that case.

Keywords: gravitational waves, magnetohydrodynamics, turbulence, cosmological phase transitions

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1. Introduction

Gravitational waves (GWs) can provide a window into the earliest epochs of the universe when it was opaque to electromagnetic radiation (see Ref. [1], Chapters: 21–22). Being massless and decoupled from other physical interactions (weak, strong, and electromagnetic) GWs propagate almost freely after their generation preserving their spectral characteristics and diluting only with the expansion of the universe. In particular, cosmological GW backgrounds produced by magnetohydrodynamic (MHD) turbulence conserve the characteristics of the source, i.e., the spectral peak and the energy density of the produced GW signal can be related to the characteristic scale and total energy density of the sourcing magnetic or velocity field at the time of generation (with a precision of about one order of magnitude for the energy density depending on the duration of the turbulence sourcing) [10].

If GWs are generated from the early universe at energy scales around the electroweak (EW) and quantum chromodynamics (QCD) phase transitions are detected, they will carry information about the fundamental physical processes at play when the Universe was only $10^{-15}$ and $10^{-5}$ seconds old, respectively. The Laser Interferometer Space Antenna (LISA), slated to launch in 2034, will be sensitive to the GWs generated around the 100 GeV – 1 TeV scale (see Refs. [11–13] for pioneering works and Ref. [14] for a review and reference therein). It is also conceivable that the GWs generated at the EW epoch can be detected in atomic interferometry experiments [15], and those originated at the QCD epoch (or around) at typical frequencies of $10^{-7}$–$10^{-9}$ Hz in pulsar timing array (PTA) experiments [15][18]. Indeed PTAs provide detection of GW signals in the nHz regime making the GW signal arising at QCD energy scale accessible. Recently, the NANOGrav collaboration has reported the detection of a signal, which could be compatible with a stochastic GW background [19]. If this detection is confirmed in the future, it could be a manifestation of the stochastic GW background originated from the epoch around the QCD energy scale [20–26]. Interestingly, the source of these GWs might be the presence of primordial magnetic fields at the QCD scale [21].

Stochastic relic GW backgrounds can be generated through different processes in the early universe, such as amplification of quantum-mechanical fluctuations during inflation, topological defects, cosmic strings, and cosmological phase transitions, see Ref. [29] for a review and references therein. In the case of cosmological phase transitions GWs are generated either from the collision of bubbles during first order phase transitions, like the EW or QCD phase transitions (see Refs. [12, 13] for pioneering works), or from the hydrodynamic and magnetohydrodynamic (MHD) turbulence produced in association with the generation of primordial magnetic fields (PMFs) and turbulence at those early times; see Refs. [9,30,31] for pioneering works, and for a recent review, see Sec. 8 of Ref [29] and references therein.

The stochastic magnetic fields and the turbulent velocity fields of the primordial plasma contribute to the anisotropic stress tensor that sources the primordial GWs. In addition there might be parity violating (or CP violating) processes in the early universe that lead to the generation of helical magnetic fields, or helical turbulent motions, and thus produce circularly polarized GWs [32]. In particular, the detection of circularly polarized GWs will shed light on phenomena of fundamental symmetry.

§ We discard the damping due to neutrino free streaming [2–8] or from anisotropic stresses [9].
∥ The commonly discussed sources of GWs at the nHz regime are mergers of supermassive black holes [19,27,28].
breaking in the early universe, such as parity violation, and potentially can serve as an explanation of the lepto- and baryogenesis problem (see Refs. [33–36] for pioneering works, and Ref. [37] for a recent review).

In this paper, we study the scalar, vector, and tensor mode decomposition of different turbulence sources with the objective to help understand their efficiency in producing GWs. More precisely we have considered different classes of turbulence that may be realized in the early universe. In particular we have studied kinetically and magnetically dominant helical and non-helical cases as well as two types of acoustic turbulence. The latter type consists of interacting sound waves. At large enough Mach numbers, the flow is dominated by shocks. Notably it was shown that the efficiency of sound waves as a source generating relic GWs is not only comparable but exceeds that from bubble collisions; see Ref. [38] for a pioneering work, and Ref. [39] for a recent study and references therein. However the generation mechanism related to sound waves requires a first order PT. The ultimate goal of the work presented here is to determine the physical conditions in the early universe that reach maximal efficiency of the GW signal production without being limited by the first order PT consideration. We also study helical vs non-helical sources behavior, and we confirm that, for distinguishing between parity conserving and parity violating sources, the measurement of polarization degrees in the low frequency tail of the spectrum could be used to identify the inverse cascade (which applies to the helical turbulence case) versus the slow inverse transfer.

This paper is arranged as follows. In Sec. 2 we briefly review the decomposition of the MHD stress tensor sourcing the GWs into the scalar, vector, and tensor components, and provide a background on the modeling of turbulent sources. This is followed by detailed numerical modeling for several types of turbulent hydrodynamic and/or MHD flows in Sec. 3. In Sec. 4 we describe in brief the degree of circular polarization of both the source and the generated GWs, and present numerical simulations of the degree of polarization of the source. Finally in Sec. 5 we present our conclusions. Throughout this work, we set $\hbar = c = k_B = 1$, and use the metric signature $(-, +, +, +)$. We also set the permeability of free space to unity, i.e., $\mu_0 = 1$, expressing the electromagnetic quantities in Lorentz-Heaviside units. Repeated indices denote a summation, unless otherwise specified. Greek indices run over spacetime coordinates, while Latin indices run over spatial coordinates.

2. Formalism

The stress tensor that sources GWs in the early universe receives contributions from PMFs [9], as well as the turbulent velocity field of the primordial plasma [40,41], and the dynamics of phase transitions [11,13]. In the following subsection, we discuss the decomposition of the stress tensor into its scalar, vector, and tensor (i.e., the transverse-traceless) components following the prescription by Lifshitz [42], and in the next subsection, we will present a brief overview of the different types of turbulent sources.

¶ This is not to be conflated with the transverse-traceless gauge in which the gravitational waves are computed far from the source.
2.1. The scalar-vector-tensor decomposition of the stress tensor

According to the Lifshitz prescription, any symmetric rank-2 tensor $\lambda_{ij}(x)$ in $d$-dimensions can be decomposed as \[44\]
\[\lambda_{ij} = L \delta_{ij} + \nabla_i \nabla_j \lambda + \nabla_i \lambda_j + \lambda_{ij},\]
where $\lambda_i$ is divergenceless, $\nabla_i \lambda_i = 0$, $\lambda_{ij}$ is divergenceless and traceless, $\nabla_i \lambda_{ij} = \nabla_j \lambda_{ij} = 0$, $\lambda_{ii} = 0$, and
\[\nabla_i \lambda_{jj} = \frac{1}{2} (\nabla_i \lambda_j + \nabla_j \lambda_i),\]
correspond to the removal of trace, and the symmetrization operators, respectively. This decomposition is commonly referred to as a scalar-vector-tensor (SVT) decomposition. Extracting the components in real space is algebraically involved, and is discussed in detail in Appendix A.

The process, however, is less cumbersome in momentum space where the analog of Eq. (1) can be written as
\[\tilde{\lambda}_{ij} = \tilde{L} \delta_{ij} - k_i k_j \tilde{\lambda}^S + i k_i \tilde{\lambda}^V_j + \tilde{\lambda}^T_{ij},\]
where tildes denote quantities in Fourier space, $k_i$ is the $i$-th component of $k$, and the superscripts $S$, $V$, and $T$ denote the scalar, vector, and tensor contributions respectively. Analogous to the real space expressions, we have $k_i \tilde{\lambda}^S = 0, k_i \tilde{\lambda}^T_j = k_j \tilde{\lambda}^T_{ij} = 0$, and $\tilde{\lambda}^T_{ii} = 0$, $k_i k_j \equiv k_i k_j - k^2 \delta_{ij}/d$ with $k = |k|$, and the parentheses in the indices denote a symmetrization as in Eq. (3). $\tilde{L}$ is proportional to the trace, i.e., $\tilde{L} = \tilde{\lambda}_{ii}/d$. The other scalar part is extracted out as
\[\tilde{\lambda}^S = - \frac{d}{d-1} \frac{k_i k_j \tilde{\lambda}^S_{ij}}{k^2} = - \frac{d}{d-1} \frac{k_i k_j}{k^2} \left( \tilde{\lambda}_{ij} - \frac{1}{d} \delta_{ij} \tilde{\lambda}_{ii} \right),\]
where $\tilde{k}_i \equiv k_i/k$, and the vector contribution turns out to be
\[\tilde{\lambda}^V_j = \left( \frac{2i}{k^2} \left( k_j \tilde{\lambda}^S_{(ij)} + \frac{d}{d} - \frac{1}{d} k_i k^2 \tilde{\lambda}^S \right) \right)\]
\[= \frac{2i}{k^2} \left( k_j \tilde{\lambda}^S_{(ij)} - k_i \tilde{k}_p \tilde{k}_q \tilde{\lambda}^S_{(pq)} \right).\]
Finally, the tensor part can thus be extracted from Eq. (5) as
\[\tilde{\lambda}^T_{ij} = \tilde{\lambda}^S_{(ij)} - \frac{d}{d-1} \tilde{k}_i \tilde{k}_j \tilde{k}_p \tilde{k}_q \tilde{\lambda}^S_{(pq)} - \left[ \left\{ \tilde{k}_i \tilde{k}_j \tilde{k}_p \tilde{\lambda}^S_{(jp)} + (i \leftrightarrow j) \right\} - 2 \tilde{k}_i \tilde{k}_j \tilde{k}_r \tilde{k}_s \tilde{\lambda}^S_{(rs)} \right].\]

The more conventional way to extract the transverse-traceless part of the stress tensor is through the four-point projection operator $P_{ijlm}(k)$, defined as $P_{ijlm}(k) \equiv \left[ P_{ij} P_{lm} - \frac{1}{d} P_{ij} P_{lm} \right] (k)$ with $P_{ij}(k) = \delta_{ij} - k_i k_j$, as is done, for example, in Ref. [45] (see their Eq. (22)). It can be shown in a straightforward way that the result of this operation gives the same expression as in Eq. (8). We now apply this formalism to the energy-momentum tensor for the magnetic and the plasma velocity fields.

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4 We also adopt the notational convention that a function $f(r)$ of real space coordinates $r$ is expressed in terms of its Fourier transform $f(k)$ as $f(r) = (2\pi)^{-3} \int d^3 k e^{-i k r} f(k)$, and the inverse transform is defined accordingly.
2.2. SVT Decomposition of Magnetic Stress

For a magnetic field $\mathbf{B}$, the stress tensor can be written as

$$\tau_{ij}(x) = B_i(x) B_j(x) - \frac{1}{2} \delta_{ij} B^2(x),$$

(9)

where $B_i$ are the spatial components of the three-dimensional vector $\mathbf{B}$. The magnetic field $\mathbf{B}$ is solenoidal, i.e., $\nabla \cdot \mathbf{B} = 0$. This tensor can be decomposed into its SVT contributions using Eq. (1).

$$\tau_{ij} = T \delta_{ij} + \nabla (\nabla \cdot \tau) \tau + \nabla (\nabla \cdot \tilde{\tau}) \tilde{\tau},$$

(10)

where $\nabla \cdot \tilde{\tau}_i = 0$, $\nabla \cdot \tilde{\tau}_{ij} = \nabla \nabla \cdot \tilde{\tau}_{ij} = 0$, and $\tilde{\tau}_{ii} = 0$. An example is given in Appendix B for a one-dimensional Beltrami field. The term proportional to the trace is $T = -B_i B_i/6 \equiv -B^2/6$, and the other scalar component is (see Appendix A for the derivation of these expressions for general $\lambda_{ij}$)

$$\tau = -\frac{1}{2} \int dV' \int dV'' \frac{\nabla^{ij} [B^{ij}(x'')] - 3 \nabla_j B_i(x'') \nabla_i B_j(x'')} {16 \pi^2 |x - x'| |x' - x''|},$$

(11)

and the vector contribution is

$$\tilde{\tau}_j = \int dV' \frac{\nabla_j \{B^{ij}(x') + 2 \nabla^{ij} \tau(x')\} - 3 B_i(x') \nabla_j B_i(x')} {6 \pi |x - x'|}.$$ 

(12)

Note that the solenoidality of the magnetic field has been used to obtain Eqs. (11) and (12). The tensor mode is then easily extracted by subtracting the scalar and vector contributions,

$$\tilde{\tau}_{ij} = B_i B_j - \nabla (\nabla \cdot \tilde{\tau}) \tilde{\tau} - \nabla (\nabla \cdot \tilde{\tau}) \tilde{\tau},$$

(13)

where, as before, $B_i B_j \equiv (B_i B_j - \delta_{ij} B^2/3)$. In Fourier space, the stress tensor in Eq. (9) becomes a convolution,

$$\tilde{\tau}_{ij}(k) = \frac{1}{(2\pi)^3} \left( \tilde{B}_i \times \tilde{B}_j \right)(k) \equiv \frac{1}{(2\pi)^3} \int d^3p \tilde{B}_i(p) \tilde{B}_j(k - p),$$

(14)

where \( \tilde{B}_i(k) \) is the Fourier transform of $B_i(x)$, and we have omitted the isotropic term in Eq. (9). For the stress-tensor in Eq. (11), the expression analogous to Eq. (14) is,

$$\tilde{\tau}_{ij} = \tilde{T} \delta_{ij} - k_i k_j \tilde{\tau}^S + i k_i \tilde{\tau}^Y + \tilde{\tau}^T_{ij},$$

(15)

with $k_i \tilde{\tau}^S = 0$, $k_i \tilde{\tau}^T = k_j \tilde{\tau}^T_{ij} = 0$, and $\tilde{\tau}^S_{ii} = 0$. The term proportional to the trace is $\tilde{T} = (2\pi)^{-3/2} (\tilde{B}_i \times \tilde{B}_j)(k)/3$, while

$$\tilde{\tau}^S(k) = -\frac{1}{(2\pi)^3} \frac{3 k_i k_j}{k^2} \tilde{B}_i(k) \tilde{B}_j(k),$$

(16)

where, continuing the spirit of using angle bracket notations, $\tilde{B}_i(k) \equiv (\tilde{B}_i \times \tilde{B}_j)(k) - \delta_{ij} (\tilde{B}_s \times \tilde{B}_s)(k)/3$. For clarity, we suppress the functional dependence on $k$ henceforth. The vector contribution turns out to be

$$\tilde{\tau}^V = -\frac{2i k_j}{k^2} \left[ \frac{1}{(2\pi)^3} \tilde{B}_i \times \tilde{B}_j \right] + k_i k_j \tilde{\tau}^S,$$

(17)

and finally, the tensor part can thus be extracted as

$$\tilde{\tau}^T_{ij} = \frac{1}{(2\pi)^3} \tilde{B}_i \times \tilde{B}_j + k_i k_j \tilde{\tau}^S - i k_i \tilde{\tau}^V.$$ 

(18)
2.3. SVT Decomposition of Kinetic Stress

For a fluid with velocity $\mathbf{u}$, the stress tensor analogous to Eq. (9) can be written as

$$\sigma_{ij}(\mathbf{x}) = w\gamma^2 u_i u_j - p\delta_{ij}, \quad (19)$$

where $w = p + \rho$ is the enthalpy, $p$ and $\rho$ are the pressure and density of the fluid respectively, $\gamma = (1 - u^2)^{-1/2}$ is the Lorentz factor, and the spacetime dependence of all the fields are suppressed. The SVT decomposition of this tensor, analogous to Eq. (10), can be written as

$$\sigma_{ij} = S\delta_{ij} + \nabla\langle i \nabla j \rangle \sigma + \nabla\langle i \sigma j \rangle + \bar{\sigma}_{ij}, \quad (20)$$

and as before, $\nabla\bar{\sigma}_i = 0, \nabla\bar{\sigma}_{ij} = \nabla j \bar{\sigma}_{ij} = 0,$ and $\bar{\sigma}_{ij} = 0$. The mathematical details are more involved compared to the case of magnetic fields, since the velocity fields are allowed to have a longitudinal component, as opposed to the case with magnetic fields. In addition, we consider non-relativistic fluid velocities ($\gamma \sim 1$) for simplicity of the expressions in the analytical results. However, this term is recovered in the numerical treatment. We omit the general expressions for the $S$, $V$, and $T$ components of the stress tensor in real space, which are provided in Appendix C.

The stress tensor can be written in $k$-space in a manner analogous to Eq. (13) as

$$\tilde{\sigma}_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^6} \left((\tilde{p} + \tilde{\rho}) \ast \tilde{u}_i \ast \tilde{u}_j\right)(\mathbf{k}) - \tilde{p}(\mathbf{k})\delta_{ij}$$

$$= \frac{1}{(2\pi)^6} \int d^3 p \int d^3 q \left[(\tilde{p} + \tilde{\rho})(\mathbf{p})\tilde{u}_i(\mathbf{q})\tilde{u}_j(\mathbf{k} - \mathbf{p} - \mathbf{q})\right] - \tilde{p}(\mathbf{k})\delta_{ij}. \quad (21)$$

This is again decomposed as,

$$\tilde{\sigma}_{ij} = \tilde{S}\delta_{ij} - k_{ij}\tilde{k}_j\tilde{\sigma}^S + ik_{ij}\tilde{\sigma}^V + \tilde{\sigma}_{ij}^T, \quad (22)$$

where $k_{ij}\tilde{\sigma}^V = 0, k_{ij}\tilde{\sigma}_{ij}^T = 0,$ and $\tilde{\sigma}_{ij}^T = 0$. $\tilde{S}$ is again proportional to the trace, $\tilde{S} = -\tilde{p}(\mathbf{k}) + (2\pi)^{-6}((\tilde{p} + \tilde{\rho}) \ast \tilde{u}_i \ast \tilde{u}_j)(\mathbf{k})/3$, and the other scalar part is

$$\tilde{\sigma}^S(\mathbf{k}) = -\frac{1}{(2\pi)^6} \frac{3}{2} \frac{k_{ij}}{k^2} (\tilde{p} + \tilde{\rho}) \ast \tilde{u}_i \ast \tilde{u}_j(\mathbf{k}), \quad (23)$$

and as in Eqs. (17) and (18), we have

$$\tilde{\sigma}^V_i = -\frac{2ik_{ij}}{k^2} \left[\frac{1}{(2\pi)^6} (\tilde{p} + \tilde{\rho}) \ast \tilde{u}_i \ast \tilde{u}_j + k_{(ij)}\tilde{\sigma}^S\right], \quad (24)$$

and,

$$\tilde{\sigma}^T_{ij} = \frac{1}{(2\pi)^6} (\tilde{p} + \tilde{\rho}) \ast \tilde{u}_i \ast \tilde{u}_j + k_{(ij)}\tilde{\sigma}^S - ik_{(ij)}\tilde{\sigma}^V. \quad (25)$$

* In a magnetized plasma, an important quantity associated with the magnetic field is the Alfvén velocity $v_A$ defined as $v_A = \mathbf{B}/\sqrt{\rho}$.  
* If we have a solenoidal flow, i.e., $\nabla \cdot \mathbf{u} = 0$, then the anisotropic stress tensor for the fluid velocity $\gamma u_i$ is analogous to that of magnetic fields.
2.4. Modeling of Turbulent Sources

As mentioned above, the primordial plasma that generates the GWs is highly turbulent, and contains magnetic fields. It is therefore essential to study the MHD evolution of the plasma and understand the various kinds of turbulent initial conditions. The purpose of this subsection is to provide an elementary background of MHD turbulence, in preparation for the numerical results discussed in Sec. 3.

Turbulent flows are characterized by the magnetic and fluid Reynolds numbers, \( \text{Re}_M \) and \( \text{Re} \) respectively, defined as
\[
\text{Re}_M = \frac{vL}{\eta}, \quad \text{Re} = \frac{vL}{\nu},
\]
for a typical fluid velocity \( v \) on the length scale \( L \), and \( \eta \) and \( \nu \) being the magnetic diffusivity and kinematic viscosity respectively. \( \text{Re}_M \) characterizes the relative importance of the diffusion term over the induction term, while \( \text{Re} \) tells us about the importance of the nonlinear velocity term relative to the viscous dissipation term.

In MHD simulations, one generally studies the evolution of the energy spectra \( E_i(k) \) of the velocity and magnetic fields (for \( i = K, M \) respectively). In a turbulent medium, the energy spectra have a range of scales where the spectral shape is given by the Kolmogorov spectrum, i.e., \( E_i(k) \propto k^{-5/3} \) – this is called the inertial subrange; see [46] for a review. Scales smaller than the inertial range are dominated by viscous eddies, while larger scales have energy carrying eddies. Turbulence involves nonlinear dynamics, which leads to an energy cascade whereby energy is transferred from larger to smaller scales (direct cascade) or vice versa (inverse cascade). The latter, for example, is interesting in the context of evolution of primordial magnetic fields, since it can explain the presence of magnetic fields correlated at sufficiently large scales in the universe after they evolve through MHD evolution of the plasma.

The physics governing the MHD evolution of the magnetic and velocity fields of the plasma affects the instantaneous decay exponents of the mean energies. For example, if we have helical magnetic fields in a highly conducting plasma, magnetic helicity is conserved throughout the evolution; in that case, magnetic energy decays with time \( t \) as \( \mathcal{E}_M(t) \propto t^{-2/3} \). On the other hand, for non-helical fields, one has \( \mathcal{E}_M(t) \propto t^{-1} \). The correlation length of the turbulence, \( \xi_M \), which is defined as a weighted average over the spectrum as \( \xi_M = \mathcal{E}_i^{-1} \int dk k^{-1} E_i(k) \), tends to increase in a corresponding fashion. The increase is fastest for helical turbulence, where \( \xi_M \propto t^{2/3} \), and slower for nonhelical turbulence, where \( \xi_M \propto t^{1/2} \). This can have a noticeable effect on the resulting GW production [10].


3.1. Numerical modeling setup

In the following, we present the results of a numerical SVT decomposition of the stress tensor. We consider four distinct types of isotropic hydrodynamic and MHD turbulence similar to those considered in recent numerical studies of GW production [10, 48]. These flows are distinguished by different types of forcing; a summary is given in Table 1. The forcing energy is injected at wave numbers around \( k_* \), which we arrange to be six times larger than the smallest wavenumber \( k_1 = 2\pi/L \) in our

\[\int dk E_i(k)\]
The SVT modes of GW turbulence simulations

triply-periodic computational domain of side length $L$. For most of the cases, we choose $k_1 = 100 H_\star$, and in two cases, we use $k_1 = 0.3 H_\star$. Energy is dissipated through a Laplacian diffusion operator with coefficients $\nu$ and $\eta$ for the viscosity and the magnetic diffusivity, which act on the smallest resolved length scales. We use conformal time normalized to the Hubble time at the time of source initiation $H_\star^{-1}$, which corresponds to $t = 1$ in our units. Following the recent work of Ref. [10], we apply a forcing term (in the momentum equation or in the induction equation for kinetic or magnetic turbulence, respectively) for one Hubble time until $t = 2$. The forcing has an initial stage $1 < t < 2$ when the forcing term is present, an intermediate stage $2 < t < 3$ when it decreases linearly, and a final stage $t > 3$ when it is absent. The forcing is random and applied at each time step. This allows to sustain stationary turbulence for a finite time during the intermediate stage, which leads to freely decaying turbulence in the final stage. For Run A, we apply forcing in the form of spherical expansion waves through the gradients of randomly placed Gaussians with amplitude $f_0$ [49], which are irrotational, i.e., curl-free. This case was also studied in Ref. [48], and is similar to their Run ac1, except that in their case the forcing was only applied until $t = 1.1$, and the intermediate stage is not considered. In Run B, we also apply irrotational forcing, but in the form of plane waves, again with amplitude $f_0$. The remaining four runs also have plane wave forcings, but are all vortical, calculated as a curl of a randomly oriented vector with amplitude $f_0$ and can have helicity, which is quantified by the parameter $\sigma \in [0,1]$. Run C has nonhelical forcing, while Run D has a helical forcing. Runs E and F have magnetic fields and the forcing is applied in the induction equation, again first without helicity (Run E) and with helicity (Run F). In Runs G and H, forcing is applied at small $k$ ($k_\star = 2$) during a time interval $1 \leq t \leq 2$, using either vortical (Run G) or irrotational (Run H) forcing. In all cases, we present the results of the SVT decomposition in the form of power spectra per linear wave number interval for the projected part of the stress tensor. The efficiency of GW production will be related to the contribution by the tensor mode with respect to the scalar and vector modes of the turbulent source. The energy injection wave numbers are $k = k_\star \equiv 600$ (Runs A–F) and 2 (Runs G and H). The resulting spectra of the stress peak at $k = 2k_\star \approx 1200$ and 4, respectively. Note that we have used normalized wave numbers $k$ such that $k = 1$ corresponds to the Hubble scale $H_\star^{-1}$ at the time of generation.

3.2. Comparison for Acoustic and Vortical Turbulence

In Table I we list the maximum kinetic or magnetic energies $E_i^{\max}$, where $i = Ka$ and $Kv$ refer to kinetic energy densities for acoustic and vortical turbulence, and $Mv$ for magnetic energy densities, where the turbulence is also vortical. These values are normalized by the radiation energy, just like the resulting GW energy, $E_{GW}^{\text{sat}}$, which refers to the GW energy averaged over oscillations around the saturated stationary value. In earlier work of Ref. [48], the saturation GW energy $E_{GW}^{\text{sat}}$ was found to be proportional to the square of the ratio $E_i^{\max}/k_\star$. To characterize the efficiency in GW energy conversion, we quote in Table I the factor $q = k_\star (E_{GW}^{\text{sat}})^{1/2}/E_i^{\max}$. In Ref. [48], the values of $q$ were found to be between one and ten, but here we find even larger values.

We begin with the case of acoustic turbulence in Runs A and B. This type of turbulence is unusual in that it processes very little vorticity. In Fig. I we show the power spectra of the projected stress tensor for acoustic turbulence. Those flows have
always zero helicity. The spectrum has a subinertial range proportional to $k^{3/2}$ and an inertial range which is proportional to $k^{-2}$ for Run A and proportional to $k^{-1.8}$ for Run B, similar to what is expected for acoustic turbulence [50]. The scalar and vector components dominate over the tensor components at almost all wave numbers, although the scalar component becomes smaller than the tensor component at small wave numbers ($k < 400$). In addition, at small wave numbers, the vector components dominate over the scalar component.

The situation is quite different for vortical turbulence. We consider here cases without kinetic helicity (Run C) and with kinetic helicity (Run D). These two runs are similar in that they have a much shallower inertial range spectrum proportional to $k^{-1}$. In addition, we also consider two cases with magnetic fields, which are driven by an electromotive force. The functional form is the same as for the forcing in the momentum equation, but the amplitude factor $f_0$ tends to be much smaller while still leading to comparable stresses. This is a consequence of this forcing occurring underneath the curl of the induction equation. Run E has null magnetic helicity and Run F has maximum magnetic helicity; see Fig. 3. In these two cases, we find a somewhat steeper inertial range spectrum proportional to $k^{-5/3}$, but for Run F there is a clear uprise of the spectrum at low wavenumber, which is likely a consequence of an inverse cascade; see also [40] for hydromagnetic decay simulations showing inverse cascading. Those cases are rather similar in that now the tensor component is

<table>
<thead>
<tr>
<th>Run</th>
<th>$\xi_i^\text{max}$</th>
<th>$\sigma$</th>
<th>wave</th>
<th>$E_{GW}^{\text{sat}}$</th>
<th>$\nu (= \eta)$</th>
<th>$f_0$</th>
<th>$q$</th>
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<tbody>
<tr>
<td>A</td>
<td>Ka</td>
<td>600</td>
<td>0</td>
<td>expan</td>
<td>$1.5 \times 10^{-5}$</td>
<td>$10^{-6}$</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>Ka</td>
<td>600</td>
<td>0</td>
<td>plane</td>
<td>$1.6 \times 10^{-3}$</td>
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<td>$2 \times 10^{-3}$</td>
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<td>$1.1 \times 10^{-3}$</td>
<td>$1 \times 10^{-2}$</td>
<td>$4 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Figure 1. SVT decomposition for Runs A (left) and B (right) for irrotational turbulence produced by spherical expansion waves and plane wave forcing, respectively.
dominant; see Fig. 3.

As in the case of acoustic turbulence, the vector mode dominates over the scalar mode at the largest length scales (k < 400), with Run F being the exception (where the vector and scalar modes are comparable, and both much smaller than the tensor mode). For vortical hydrodynamic turbulence, the tensor and vector modes are comparable in the inertial range, while for magnetic turbulence the tensor mode dominates over this regime. Hence, in the magnetic turbulence case, the spectrum of the total unprojected stress tensor agrees surprisingly well with that of the tensor mode (for all k in the helical case, and at large k for the non-helical case), so the computational effort for doing such a projection could potentially be ignored in a first approximation, as was done in Refs. [51,52]. This is, however, not the case for acoustic turbulence, where the projection is essential.

It is worth noting that all spectra of Figs. 1–3 show a subinertial range proportional to k or k^{3/2}, which is shallower than the generic k^2 spectrum. In the present case, the subinertial range is not long enough to make strong claims, but it is useful to recall that deviations from a k^2 spectrum of the stress have been found when there are departures from Gaussianity of the underlying magnetic or velocity fields [53]. Although the departures from Gaussianity may not yet be very strong in the present simulations, it is worth noting that real astrophysical magnetic fields have much larger magnetic Reynolds numbers and may well lead to much more significant departures.
3.3. On the Relative GW Strength for Acoustic and Vortical Turbulence

In this connection, it is interesting to note that the apparent strength of GWs for acoustic turbulence cannot straightforwardly be explained by the SVT decomposition. If there was such a connection, one would have expected the GW energy to be less for acoustic turbulence than for vortical turbulence, because the tensor mode (i.e., the actual source for the GWs) is weaker there. However, this is not the case [48]. We must therefore conclude that this difference between acoustic and vortical turbulence is explained by other factors, most likely the detailed aspects of the temporal variation of the stress. In fact, to drive GWs, the temporal stress frequency must be close to the frequency of light at each spatial Fourier mode and it seems plausible that this is more easily satisfied for acoustic turbulence. It can be argued that only the parallel component of the perturbations in magnetic fields contributes to the energy spectrum of the stress tensor of $B^2$ sourcing the GWs. Thus the acoustic modes of turbulence are more efficient in transferring energy to the GWs than the vortical modes, and thus GWs sourced by acoustic turbulence carry more energy compared to those by vortical turbulence.

In Fig. 4 we show the positions of Runs A–F in a $\varepsilon_{GW}^{sat}$ versus $\varepsilon_{M,K}^{max}$ diagram. For orientation, we also show the old data points from the Ref. [48]. We see that the new acoustic or irrotational turbulence simulations are now even slightly more efficient in driving GWs: Run B has not only the smallest kinetic energy of all the runs, but it also has a relatively large GW energy.

Larger GW energies can be obtained by decreasing $k_\ast$. In Fig. 5 we show the results for $k_\ast = 2$ and compare vortical and irrotational hydrodynamic forcings. Similarly to the cases with larger $k_\ast$, we see, again, that the tensor mode of acoustic turbulence is weaker (second panel of Fig. 4), but this is only true of the initial range, which contributes little to the total GW energy production. Most of the GW energy comes from wave numbers $k$ below the peak, and there the tensor mode is, similarly to the cases with larger $k_\ast$, the largest (left panel of Fig. 5 around the peak, and right panel for low $k$) or, at least comparable to the vector mode, which is the largest (left panel of Fig. 5 for low $k$).
The SVT modes of GW turbulence simulations

4. Relation Between Stress and Strain Spectra

4.1. GW Equation for the Radiation Dominated Epoch

In this section, we connect the observable quantities related to GWs to properties of the source obtained in the previous section. GWs are considered to be small tensor perturbations over the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, and the dynamics of these GWs is given by the linearized Einstein equations. The perturbed metric is taken to be,

\[ ds^2 = -dt_P^2 + a^2(t_P)[\delta_{ij} + h_{ij,P}(x,t)]dx_idx_j, \]

where \( x \) are the comoving coordinates, \( t_P \) is the physical time, \( a(t_P) \) is the scale factor of the universe, and \( h_{ij,P} \) are the physical strains. The equation governing the evolution of these GWs is given by [43, 44],

\[ \frac{\partial^2}{\partial t_P^2} + 3\frac{H(t_P)}{a(t_P)} \frac{\partial}{\partial t_P} - \nabla^2 a^2 h_{ij,P}(x,t_P) = 16\pi G T^{TT}_{ij,P}(x,t_P), \]

where \( G \) is Newton’s constant, \( H(t_P) = da/(a dt_P) \) is the Hubble rate, \( T^{TT}_{ij,P} \) is the physical stress tensor, the superscript \( TT \) denotes the the transverse and traceless (TT) projection, and \( \nabla \) signifies derivatives with respect to the comoving coordinates \( x \).

The energy density in the magnetic fields and plasma velocity dilutes with the expansion of the universe, and to scale out the effects of this expansion, we need to express the GW equation in terms of comoving stress tensor \( T_{ij} = a^4 T_{ij,P} \), the scaled strain \( h_{ij} = a h_{ij,P} \), and the conformal time \( t \) defined as \( dt_P = a dt \). Also in the radiation dominated epoch, the scale factor is linear in conformal time, and the damping term in the above equation vanishes. Eq. (28) can then be expressed as

\[ h''_{ij} - \nabla^2 h_{ij} = \frac{16\pi G}{a} T^{TT}_{ij}(x,t), \]

where the primes denote derivatives with respect to the conformal time \( t \). The analytic solution of the GW equation is discussed in several references; see, for example, Refs. [32, 51]. The full numerical solution for a turbulent source is also discussed in Refs. [48, 54].

The four-dimensional power spectrum \( H_{ijlm}(k, t, \tau) \) of the energy density tensor is defined as

\[ \frac{1}{(2\pi)^6} \left< \tilde{T}_{ij}(k, t) \tilde{T}_{lm}(k', t + \tau) \right> = (2\pi)^3 \delta^3(k - k') H_{ijlm}(k, t, \tau), \]
where \( \hat{T}_{ij}(k, t) \) is the Fourier transform of \( T_{ij}(x, t) \), we have used the reality condition \( \hat{T}_{ij}(-k, t) = \hat{T}_{ij}^*(k, t) \). As mentioned previously, \( w \equiv \rho + p \) is the enthalpy density of the primordial plasma, where \( \rho \) and \( p \) being the energy density and the pressure of the primordial plasma, and all quantities are comoving.

In the standard framework there are two degrees of freedom for \( h_{ij} \) (a \( 3 \times 3 \) tensor has 9 components, the symmetry condition \( h_{ij} = h_{ji} \) leaves 6 degrees of freedom, the transverse nature of GWs, i.e., \( k_i h^{ij} = 0 \), reduces that to 3, while the traceless condition \( h_{ii} = 0 \) further reduces the total number of the degrees of freedom to 2) that correspond to two possible states of GWs polarization. In the case of a parity even source, these two states are equal to each other, while a parity odd sources generates two states that are perpendicular to each other. The basis is defined as

\[
\hat{e}_R^{ij}(\hat{k}) = \frac{1}{2} (\hat{e}_1 \pm i \hat{e}_2)_j (\hat{e}_1 \pm i \hat{e}_2)_i.
\]

It is easy to check that \((e^{ij}_R)^* = e^{ij}_L\), and vice versa; and also that

\[
\hat{e}^{ij}_R(\hat{k}) \hat{e}^{ij}_R(\hat{k}) = 1,
\]

\[
\hat{e}^{ij}_R(\hat{k}) \hat{e}^{ij}_L(\hat{k}) = \hat{k}_i \hat{e}^{R,L}_i = \delta_{ij} \hat{e}^{R.L}_j = 0.
\]

The Fourier space stress tensor can now be decomposed into the left and right handed components,

\[
\hat{T}_{ij}(k, t) = \hat{T}_{L}(k, t) \hat{e}^{ij}_L(\hat{k}) + \hat{T}_{R}(k, t) \hat{e}^{ij}_R(\hat{k}).
\]

We can also construct symmetric and antisymmetric projectors \([55]\),

\[
S_{ijlm}(\hat{k}) \equiv \frac{1}{2} \left[ \hat{e}^{ij}_R \hat{e}^{lm}_R + \hat{e}^{ij}_L \hat{e}^{lm}_L \right],
\]

\[
A_{ijlm}(\hat{k}) \equiv \frac{1}{2i} \left[ \hat{e}^{ij}_R \hat{e}^{lm}_L - \hat{e}^{ij}_L \hat{e}^{lm}_R \right].
\]

These projection operators satisfy the properties that \( S_{ijlm}(\hat{k})S_{ijlm}(\hat{k}) = A_{ijlm}(\hat{k})A_{ijlm}(\hat{k}) = 2 \), and \( S_{ijlm}(\hat{k})A_{ijlm}(\hat{k}) = 0 \), and pick out respectively the symmetric and antisymmetric components of a fourth-order tensor.

The degree of polarization of the source can then be expressed as

\[
\mathcal{P}_T(k, t) = \frac{T^A(k, t)}{T^S(k, t)},
\]

where

\[
T^\alpha(k, t) = \frac{1}{2u^2} \int_{\Omega_k} d\Omega_k k^2 \int d^3 k' \alpha_{ijlm}(k) \left\langle \hat{T}_{ij}^\alpha(k, t) \hat{T}_{lm}(k', t) \right\rangle,
\]

with \( \alpha \) denoting either \( S \) or \( A \).

In terms of the polarization components of Eq. \([44]\), we can write

\[
T^S(k) = \frac{1}{2u^2} \int_{\Omega_k} d\Omega_k k^2 \int d^3 k' \left\langle \hat{T}_R^L(k) \hat{T}_L(k') + \hat{T}_L^L(k) \hat{T}_L(k') \right\rangle,
\]

\[
T^A(k) = \frac{1}{2u^2} \int_{\Omega_k} d\Omega_k k^2 \int d^3 k' \left\langle \hat{T}_R^R(k) \hat{T}_R(k') - \hat{T}_L^R(k) \hat{T}_L(k') \right\rangle.
\]
where the explicit $t$-dependence of the quantities in Eq. (39) is dropped for brevity. $\mathcal{P}_T(k,t)$ thus captures the relative inequality of power in the left and right handed components of the stress tensor. One can alternatively work in the linear polarization basis, where the physical meaning of the polarization degree of the source is not so straightforward. The definition of the basis, as well as the expressions analogous to those above are presented in Appendix D. For more details about the linear and circular polarization bases, and the decomposition of the four-dimensional power spectrum into its symmetric and antisymmetric parts, see Ref. [55], where the stress tensor is sourced by magnetic fields. Notably, the analytical estimates for the polarization degree differ significantly from the numerical simulations due to the simplified treatment of the time-decorrelation functions and the neglect of different scaling laws for the energy density and helicity decays scalings; see Ref. [10] for more discussion.

Since GWs are sourced by the TT projected stress tensor, i.e., the tensor component $\lambda_{ij}$ of the stress tensor [cf. Eq. (1)], we built the four-dimensional power spectrum $H_{ijlm}(k,t,\tau)$ from this TT component and decomposed it into parity odd and even parts. It is of interest to see whether it is possible to ascribe a degree of polarization analogous to that in Eq. (37) to the scalar and vector components of the stress tensor. We defer a discussion of that to Appendix E.

### 4.2. Shell-integrated Spectra

In a manner analogous to Eq. (30), we can form a four-dimensional power spectrum $Q_{ijlm}(k,t,\tau)$ from the time derivative of the strains in Fourier space

$$\frac{1}{32\pi G} \left\langle \dot{h}_{ij,\nu}(k,t) \dot{h}_{lm,\nu}(k',t+\tau) \right\rangle = (2\pi)^3 \delta^3(k-k')Q_{ijlm}(k,t,\tau).$$

(41)

For more details on the fourth order correlation functions and their Fourier transforms, see Ref. [56].

The polarization degree of GWs is written as

$$\mathcal{P}_{GW}(k,t) = \frac{H_{GW}(k,t)}{E_{GW}(k,t)},$$

(42)

where $E_{GW}(k,t)$ and $H_{GW}(k,t)$ are respectively the spectral GW energy density and the spectral GW helical energy density, obtained respectively by the application of the projection operators of Eq. (35) on the LHS of Eq. (41), in a way similar to Eq. (38).

In the circular polarization basis, we get (dropping the explicit $t$-dependence)

$$E_{GW}(k) = \frac{1}{(2\pi)^6} \frac{1}{32\pi G} \int_{\Omega_k} d\Omega_k k^2 \int d^3k' \times \left\langle \dot{h}_R^*(k)\dot{h}_R(k') + \dot{h}_L^*(k)\dot{h}_L(k') \right\rangle,$$

(43)

$$H_{GW}(k) = \frac{1}{(2\pi)^6} \frac{1}{32\pi G} \int_{\Omega_k} d\Omega_k k^2 \int d^3k' \times \left\langle \dot{h}_R^*(k)\dot{h}_R(k') - \dot{h}_L^*(k)\dot{h}_L(k') \right\rangle,$$

(44)

where $\dot{h}_{ij,\nu}(k,t)$ is decomposed in the circular polarization basis,

$$\dot{h}_{ij}(k,t) = \dot{h}_L(k,t) e^L_{ij}(k) + \dot{h}_R(k,t) e^R_{ij}(k).$$

(45)

As expected, $\mathcal{P}_{GW}(k,t)$ captures the relative inequality of power in the left and right handed components of the GW perturbations.
4.3. Numerical Results for the Strain and Stress Spectra

We now solve for the resulting GWs numerically using the approach described in Refs. [48, 54]. Here we focus on a comparison of the results with irrotational plane wave (Run B) and vortical plane wave forcings without (Run C) and with (Run D) magnetic field, and with helicity (Run F). In each of the cases, we show the spectra of both the TT-projected stress of the magnetic (kinetic) fields $\bar{\tau}_{ij}$ ($\bar{\sigma}_{ij}$), and the strain $h$ scaled with a $k^2$ factor. Here we denote the latter simply as $Sp(k^2h)$, but mean by this all of the components of the strain tensor. Likewise, we denote the stress spectrum by $Sp(\tau)$, and mean by this the TT-projected stress tensor with the $16\pi G$ factor included, so that it would be equal to $Sp(k^2h)$, if the second conformal time derivative of the strain in Eq. (29) was absent. The result is shown in Fig. 6 for Runs B, C, E, and F.

We see from Fig. 6(d) that in the magnetic case with helicity, the spectra of $Sp(k^2h)$ and $Sp(\tau)$ are in almost perfect agreement with each other at all wave numbers, except near $2k_*$. The situation changes when there is no helicity and $Sp(k^2h)$ is now slightly above $Sp(\tau)$; see Fig. 6(c). The difference increases further when we compare with purely hydrodynamic cases (Run C); see Fig. 6(b). Particularly striking is a difference for irrotational turbulence, where we see significant differences between the two spectra at all wave numbers (Run B); see Fig. 6(a).

Our results suggest that, although there is a correspondence between $Sp(k^2h)$ and $Sp(\tau)$, there can also be a significant departure and even a shift between them. Thus, even though the TT-projection of the stress has resulted in a significant decrease in the projected stress (see Fig. 4), the stress in the irrotational hydrodynamic cases have now other qualities that make it more efficient for driving GWs. As already discussed...
above, this could be caused by larger power at larger temporal frequencies. This is supported by our findings that in the cases with the smallest efficiency (Run D), the two spectra are almost equal, i.e., \( \text{Sp}(\tau) \approx \text{Sp}(k^2 h) \).

4.4. Polarization Spectra of Stress and Strain

It is common to define the degree of polarization in terms of the strains themselves, as opposed to our definition in terms of the time derivatives of the strains (see, e.g., Eq. (6) of Ref. [32]). It can be argued that these definitions are equivalent for GWs sourced by stationary turbulence, where one can apply the aeroacoustic approximation. For stationary turbulence, one can take the time derivatives out of the time averaging operation on the four-point correlation function of the stress tensor (see Eq. (12) of Ref. [51]). The same reasoning applies to the four-point correlation function of the GW strain, since the strains are expected to have the same statistical properties as the turbulent source. This means

\[
\left\langle h_{ij,p}^{TT}(x,t)h_{lm,p}^{TT}(x + \xi, t + \tau) \right\rangle_t = -\partial_t^2 \left\langle h_{ij,p}^{TT}(x,t)h_{lm,p}^{TT}(x + \xi, t + \tau) \right\rangle_t,
\tag{46}
\]

where the subscript \( t \) is used to denote that the angle brackets now also include time averaging, in addition to the ensemble averaging. In frequency space, the time derivatives become factors of frequency, and cancel out when calculating the degree of polarization (see Eq. (12)).

Let us now discuss the degree of polarization of the source, \( \mathcal{P}_T(k) \), and of the strain, \( \mathcal{P}_h(k) \), from numerical simulations. In Fig. 8(a) and (b) we show \( \mathcal{P}_T(k) \) for the source for the runs of Ref. [10] for kinetic and magnetic driving. The run names for the kinetically driven cases are K0, K01, K03, K05, and K1 for \( \sigma = 0, 0.1, 0.3, 0.5 \), and 1, respectively. The magnetically driven runs have analogous names M0, M01, M03, M05, and M1.

Remarkably, in the kinetically driven case, the polarization of the source never reaches 100\%, and yet, the polarization of \( h \) does reach nearly 100\%. This was already found in Ref. [10], and their results are convenient comparison also reproduced in Fig. 8(c) and (d). Furthermore, in the magnetically driven case, the source reaches nearly 100\%, but not at the same wave number where the GWs become 100\% polarized. Instead, maximum polarization happens here at the smallest scale.
5. Conclusions

We have computed the SVT decomposition for six different flows: kinetically and magnetically driven flows, with and without helicity, as well as two types of acoustically driven flows. In addition, we have considered two cases with significantly smaller forcing wave numbers. All our flows had a turbulent inertial range spectrum, as well as a subinertial range spectrum. However, significant differences can be seen in the relative strengths of scalar, vector, and tensor modes of the turbulent source. The largest contribution to the tensor mode comes from the flows in Runs E and F. Both are magnetically driven, although only one of them had helicity. In the presence of magnetic helicity (Run F), an inverse cascade emerges, which leads to an additional contribution to the tensor mode at small wave numbers. The scalar and vector modes of the turbulence stress are small and of comparable strengths in the helically driven case. Without helicity, however, the vector mode exceeds the scalar mode at small and intermediate length scales. For helically driven flows, there is no significant inverse cascade, and the tensor mode is no longer so dominant. Instead, tensor and vector modes are now of comparable strength. At smaller wave numbers, the vector mode does even become dominant.

For acoustically or irrotationally driven flows, the situation is drastically different in that the tensor mode is now comparatively weak, while scalar and vector modes
are dominant and of similar strengths. Furthermore, there is very little difference between two fundamentally different realizations of acoustic turbulence: plane wave and spherical expansion wave forcings. Both show a very strong “bottleneck” effect, i.e., a shallower spectrum at high wave numbers, which was already found in Ref. [49]. Bottleneck effects are known to exist also for vortical turbulence [57], but they are not as strong [59]. The planar wave forcing was thought to be more similar to the vortical plane wave forcing, but this is not the case. Even though the acoustically driven flows have comparatively weak tensor modes, they are most effective in driving GWs. The reason for this higher efficiency of acoustic flows in generating GWs is not very well understood; but it might be interesting to note that at least in a gravitationally stratified fluid, maximum acoustic wave power output was found for flows with significant longitudinal modes compared to transverse ones [58]. The planar wave-driven acoustic flows are even slightly more effective (Run B, \( q = 32 \)), than spherical expansion wave-driven flows (Run A, \( q = 19 \)).

Given that the presence of helicity had no significant effect on the results of the SVT decomposition, it was of interest to look for differences in the degree of polarization of the source and the resulting GWs. The difference is indeed immense, but it results almost entirely from the inverse cascade, which is only possible in the magnetically driven case. In these cases, the degree of polarization of the source is virtually independent of the fractional helicity. This is probably because, even for weak magnetic helicity, inverse cascading is possible. At the driving scale, however, the degree of polarization is never close to 100\% – neither in the magnetically driven cases, nor in the kinetically driven ones. In the latter, however, fractional helicity of the velocity does lead to a smaller fractional helicity of the stress. By contrast, in the magnetically driven cases, all spectra of the polarization degree are similar and they seem to be controlled by some kind of forward cascade from the largest scale downward to all smaller scales. This includes even the driving scale of the system, whose degree of polarization seems to be controlled by the stress on the largest scale of the domain.

Clearly, spectral power on the scale of the domain must be interpreted with care, because the intention of periodic box simulations is to represent an infinite and unbounded domain. The results will therefore be different if one considered a larger domain. It is conceivable, then, that inverse cascading also works in an infinitely extend domain, but, again, only on the scale of the domain will the polarization degree be close to maximum.

The work presented here will be important for identifying the nature of the turbulent motions in the early universe once a GW signal from turbulent sources will be detected. More specifically, it can help to disentangle the turbulent conditions in the early universe regarding the dependence on irrotational (longitudinal) and vortical (solenoidal) flows.

Data availability—The source code used for the simulations of this study, the Pencil Code, is freely available [60]. The simulation setups and the corresponding data are freely available from Refs. [61, 62].

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Appendix A. Helmholtz Theorem and the real space SVT decomposition

The scalar-vector-tensor (SVT) decomposition is the most generalized way to decompose linearized perturbations of a symmetric rank-2 tensor based on how these components transform under rotations. In this formulation, the evolution equations for the scalar, vector, and tensor components of the metric perturbations are decoupled from each other at the linear level. It is a more generalized version of the Helmholtz decomposition by which a vector is decomposed into divergence-free and curl-free parts.

In Sec. 2.1, we have provided the expressions for the SVT decomposition of a general symmetric rank-2 tensor in momentum space. In this Appendix, we show the corresponding decomposition in real space.

The Helmholtz theorem states that any 3-vector $b_i$ can be decomposed into a scalar part $b$ and a vector part $\bar{b}_i$,

$$b_i = \nabla_i b + \bar{b}_i.$$  \hfill (A.1)

The first term on the RHS is curl-free and the second term is divergence-free, $\nabla_i \bar{b}_i = 0$. To extract this latter part, we need to solve for $b$; we get the following condition on $b$,

$$\nabla^2 b = \nabla_i b_i(x).$$  \hfill (A.2)

This corresponds to a Poisson’s equation, whose general solution in three dimensions is

$$b = -\int dV' \frac{\nabla'_i b_i(x')}{{4\pi|x - x'|}},$$  \hfill (A.3)

where the prime on the $\nabla$ indicates that the derivative is with respect to the components of $x'$; we thus get,

$$\bar{b}_i = b_i + \nabla_i \int dV' \frac{\nabla'_j b_j(x')}{{4\pi|x - x'|}}.$$  \hfill (A.4)

Analogous to the case for vectors, we have SVT decomposition for tensors – any symmetric rank-2 tensor $\lambda_{ij}(x)$ can be decomposed as in Eq. (1). Taking the trace of Eq. (1), we get,

$$L = \frac{1}{d}\lambda_{ii},$$  \hfill (A.5)
which allows us to rewrite Eq. (1), using the condensed notation \( \lambda_{ij} \equiv \lambda_{ij} - \delta_{ij} \lambda_{ss}/d, \)

\[
\lambda_{ij} = \nabla_i \nabla_j \lambda + \nabla_i \bar{\lambda}_j + \bar{\lambda}_{ij}.
\]

Acting on this with \( \nabla_i \) and then with \( \nabla_j \), we get,

\[
\nabla_i \nabla_j \lambda_{ij} = \frac{d-1}{d} \nabla^2 (\nabla^2 \lambda),
\]

and thus, we can solve for \( \lambda \), by solving the Poisson’s Eq. (A.7) first for \( \nabla^2 \lambda \), and then for \( \lambda \) in \( d \) dimensions:

\[
\lambda = \frac{d}{d-1} \frac{\Gamma^2 (\frac{d}{2} - 1)}{16 \pi^d} \int dV' \int dV'' \frac{\nabla'' \nabla'' \lambda_{ij}(x'')}{|x - x'|^{d-2} |x' - x''|^{d-2}}.
\]

To extract \( \bar{\lambda}_i \), we insert this value of \( \lambda \) into Eq. (A.7) and act on it with \( \nabla_j \),

\[
\nabla^2 \bar{\lambda}_i = 2 \nabla_j \lambda_{ij} - \frac{2(d-1)}{d} \nabla_i (\nabla^2 \lambda),
\]

where \( \lambda \) is understood to be known from Eq. (A.8). Solving this,

\[
\bar{\lambda}_i = \frac{\Gamma (\frac{d}{2} - 1)}{2 \pi^\frac{d}{2}} \int dV' \frac{\nabla' \lambda_{ij}(x') - (d-1) \nabla' \nabla^2 \lambda(x')/d}{|x - x'|^{d-2}}.
\]

Finally, the traceless, divergenceless part of the energy-momentum tensor is,

\[
\bar{\lambda}_{ij} = \lambda_{(ij)} - \frac{d}{d-1} \frac{\Gamma^2 (\frac{d}{2} - 1)}{16 \pi^d} \nabla_i \nabla_j \int dV' \int dV'' \frac{\nabla'' \nabla'' \lambda_{(mn)}(x'')}{|x - x'|^{d-2} |x' - x''|^{d-2}}
\]

\[
+ \frac{\Gamma (\frac{d}{2} - 1)}{4 \pi^\frac{d}{2}} \int dV' \left[ \nabla' \lambda_{(im)}(x') + \frac{\Gamma^2 (\frac{d}{2} - 1)}{16 \pi^d} \nabla' \nabla^2 \int dV'' \int dV''' \frac{\nabla''' \nabla''' \lambda_{(mn)}(x''')}{|x - x'|^{d-2} |x' - x''|^{d-2} |x'' - x'''|^{d-2}}\right]
\]

\[
+ (i \leftrightarrow j)\right] .
\]

### Appendix B. An example

It is instructive to consider an example for the formalism of Sec. 2.2. For a Beltrami field, \( B = (0, \sin kx, \cos kx) \), we have

\[
\tau_{ij} = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 0 \\
0 & -\cos 2kx & \sin 2kx \\
0 & \sin 2kx & \cos 2kx
\end{pmatrix}.
\]

One can easily verify that

\[
\bar{\tau}_{ij} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & -\cos 2kx & \sin 2kx \\
0 & \sin 2kx & \cos 2kx
\end{pmatrix}
\]

is traceless and transverse. Using \( \tau = (y^2 + z^2)/4 \), we have \( \nabla_i \nabla_j \tau = \text{diag}(-2, 1, 1)/6 \). Next, using \( T = -1/6 \), we see that \( T \delta_{ij} + \nabla_i \nabla_j \tau = \text{diag}(-1/2, 0, 0) \). Thus, following Eq. (10), we can write \( \tau_{ij} \) as \( T \delta_{ij} + \nabla_i \nabla_j \tau + \nabla_i \bar{\tau}_{j} + \bar{\tau}_{ij} \) with \( \bar{\tau}_j = 0 \). The Beltrami field has therefore only a spatially periodic tensor mode, a spatially (unbounded) nonperiodic scalar mode, and no vector mode.

\[\dagger\] For notes on deriving the Green’s function of \( \nabla^2 \) in \( d \) dimensions see, for example, the discussion in https://tinyurl.com/knns7bbc
Appendix C. The SVT decomposition of the fluid stress tensor

In this section, we provide the expressions for the scalar, vector, and tensor components of the plasma velocity stress tensor of Eq. (19). As before, one gets the trace-term to be

\[ S = -p + \frac{1}{3} w \gamma^2 u^2, \]  

(C.1)

and we can then solve for \( \sigma \),

\[ \sigma = \frac{3}{32\pi^2} \int dV' \int dV'' \frac{\nabla_i'' \nabla_j'' \{(p + \rho)(x'')(\gamma^2(x'')u_{i}(u_j)(x'')\} )}{|x - x'| |x' - x''|}. \]  

(C.2)

The vector part is also extracted in a straightforward manner,

\[ \bar{\sigma}_i = -\frac{1}{2\pi} \int dV' \nabla_j' \{(p + \rho)(x')\gamma^2(x')u_{i}(u_j)(x')\} - 2\nabla'_i \nabla^2 \sigma(x')/3 \]  

(C.3)

The tensor part finally comes out by subtracting the scalar and vector parts from the full tensor,

\[ \tilde{\sigma}_{ij} = w \gamma^2 u_{i}(u_j) - \nabla_{(i} \nabla_{j)} \sigma - \nabla_{(i} \bar{\sigma}_{j)}. \]  

(C.4)

Appendix D. The linear polarization basis

In a manner similar to the circular polarization basis, one can also define the linear polarization basis \[55]\:

\[ e_{ij}(\hat{k}) = e_{ij}^1 - e_{ij}^2, \quad e_{ij}^\times(\hat{k}) = e_{ij}^1 e_{ij}^2, \]

(D.1)

which is orthogonal, but not normalized, and has the properties

\[ e_{ij}^1(\hat{k}) e_{ij}^2(\hat{k}) = e_{ij}^2(\hat{k}) e_{ij}^1(\hat{k}) = 2, \quad e_{ij}^1(\hat{k}) e_{ij}^\times(\hat{k}) = 0. \]  

(D.2)

In addition, the fact that \( \hat{e}^1, \hat{e}^2, \) and \( \hat{k} \) form a right-handed orthonormal basis leads to the relations

\[ -\epsilon_{ipq} k_p e_{qj}^\times(\hat{k}) = e_{ij}^+(\hat{k}), \quad \epsilon_{ipq} k_p e_{qj}^+(\hat{k}) = e_{ij}^\times(\hat{k}). \]  

(D.3)

As in Eq. (D.1), we can decompose the stress tensor into + and \( \times \) polarization modes,

\[ \tilde{T}_{ij}(k, t) = \tilde{T}_+(k, t) e_{ij}^+(\hat{k}) + \tilde{T}_\times(k, t) e_{ij}^\times(\hat{k}). \]  

(D.4)

The projection operators of Eq. (D.3) can be written in this basis as

\[ S_{ijlm}(\hat{k}) = \frac{1}{2} \left[ e_{ij}^+ e_{lm}^+ + e_{ij}^\times e_{lm}^\times(\hat{k}) \right], \]  

(D.5)

\[ A_{ijlm}(\hat{k}) = \frac{1}{2} \left[ e_{ij}^\times e_{lm}^+ - e_{ij}^+ e_{lm}^\times(\hat{k}) \right], \]  

(D.6)

and the degree of polarization in of the source can be expressed with Eq. (37), where in terms of the components in the linear polarization basis,

\[ T^S(k) = \frac{1}{w^2} \int \Omega_k d^3k' \int d^3k' \left\langle \tilde{T}_+(k)\tilde{T}_+(k') + \tilde{T}_\times(k)\tilde{T}_\times(k') \right\rangle, \]  

(D.7)

\[ T^A(k) = \frac{1}{w^2} \int \Omega_k d^3k' \int d^3k' \left\langle \tilde{T}_\times(k)\tilde{T}_+(k') - \tilde{T}_+(k)\tilde{T}_\times(k') \right\rangle. \]  

(D.8)
where the explicit $t$-dependence is dropped as in Eq. (39). Similarly, the spectral energy densities of GWs is written in this basis as (again dropping explicit $t$-dependence)

\[ E_{GW}(k) = \frac{1}{16\pi^6} \int d\Omega_k k^2 \int d^3k' \times \left\langle \hat{h}_+(k)\hat{h}_+(k') + \hat{h}_x(k)\hat{h}_x(k') \right\rangle, \]

\[ H_{GW}(k) = \frac{1}{16\pi^6} \int d\Omega_k k^2 \int d^3k' \times \left\langle \hat{h}_x(k)\hat{h}_+(k') - \hat{h}_+(k)\hat{h}_x(k') \right\rangle. \]

(Appendix E. Polarization of the $S$ and $V$ components of the stress tensor)

One can define a four-dimensional power spectrum analogous to that in Eq. (30) from the scalar and vector components of the general stress tensor $\lambda_{ij}$ of Eq. (1). Indeed using Eq. (5), we can write

\[ \frac{1}{w^2} \left\langle k_i(k_j)\hat{\lambda}^S(k,t) \cdot k'_i k'_j \hat{\lambda}^S(k',t+\tau) \right\rangle \equiv (2\pi)^3 \delta^3(k - k') H_{ijlm}^S(k, t, \tau), \]

(E.1)

and

\[ \frac{1}{w^2} \left\langle k_i(k_j)\hat{\lambda}^V(k,t) \cdot k'_i k'_j \hat{\lambda}^V(k',t+\tau) \right\rangle \equiv (2\pi)^3 \delta^3(k - k') H_{ijlm}^V(k, t, \tau). \]

(E.2)

One can define antisymmetric analogs of these,

\[ \mathcal{H}_{ijlm}^{S,V}(k,t,\tau) \equiv -\epsilon_{ipq} k_j \hat{\lambda}_{ijlm}^{S,V}(k,t), \]

(E.3)

with respective “degrees of polarization” defined as

\[ P_T^{S,V}(k, t) = \frac{\int d\Omega k^2 \mathcal{H}_{ijlm}^{S,V}(k, t, 0)}{\int d\Omega k^2 \mathcal{H}_{ijlm}^{S,V}(k, t, 0)}. \]

(E.4)

It follows in a straightforward manner from Eq. (E.1) that $H_{ijlm}^S(k, t, 0) = 0$, and there is no polarization associated with the scalar part of the stress tensor, as would be intuitively expected. However, $\mathcal{H}_{ijlm}^V(k, t, 0)$ does not necessarily vanish, and one can associate a degree of polarization with the vector component of the source.

References

The SVT modes of GW turbulence simulations


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