

CONVECTIVE INSTABILITY IN A COMPRESSIBLE ATMOSPHERE. II.

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ABSTRACT

The onset of steady convection in a polytropic atmosphere with constant viscosity is studied numerically.

Subject headings: convection — stars: atmospheres

I. INTRODUCTION

The object of this paper and of Paper I (Spiegel 1965) is to study the onset of convection in a highly compressible medium. As noted in Paper I, this work lies at the intersection of two sequences of research started by Rayleigh (1916) on the one hand and Lamb (1910) on the other. The former studied the influence of viscosity and conductivity on the onset of convection in a medium with very slight variations of density. The latter studied convective growth rates in a polytropic atmosphere without either of these dissipative effects.

From Rayleigh's work we know that neither viscosity nor conductivity alone can stabilize ordinary thermal convection. Hence we expect that both are needed to establish a criterion for the onset of convection in a medium spanning many scale heights, even though the viscosity may be small. Of course, the exact conditions for the onset of convection in stars are not of vital interest in the present state of stellar convection theory. Nevertheless, the nature of the critical condition for the onset of convection does enter into some theories of convective heat transfer. To see whether this condition is greatly affected by appreciable compressibility, it may suffice to consider a very simple equilibrium model such as a polytropic atmosphere.

Approximate treatments of the onset of steady convection in a polytropic atmosphere were given by Kato and Unno (1960), Unno, Kato, and Makita (1960), and in Paper I. Here we present careful numerical solutions of the linear equations treated with analytical techniques in Paper I. However, we should note that there exist other numerical treatments of this problem, by Vickers (1971) and Tadjbakhsh and Quarles (unpublished MS), and we shall refer to these below. Moreover, Graham (1975) has described two-dimensional numerical experiments on convection in a perfect gas with constant viscosity and thermal conductivity, whose static equilibrium state is a polytrope. His mildly nonlinear solutions, as well as our linear results, show that the preferred motion extends across the layer. These solutions provide no justification for assuming any preferred length scale other than that associated with the depth of the convecting layer.

II. EQUATIONS AND METHODS

For a plane-parallel atmosphere with constant thermal conductivity we have equilibrium solutions for the pressure, density, and temperature of the form

$$p_0 = Pz^{m+1}, \quad \rho_0 = \frac{P}{R_*\beta_0} z^m, \quad T_0 = \beta_0 z, \quad (2.1)$$

where P and β_0 are constants of integration, $m = g/(R_*\beta_0) - 1$, g is the acceleration of gravity, and R_* is the gas constant. The depth, z , is measured parallel to gravity, and a constant of integration has been adjusted so that p_0 and T_0 both vanish at the same height, as in the so-called radiative-zero solution of stellar structure theory (Schwarzschild 1958).

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The development of small perturbations to this equilibrium is governed by the coupled thermal and Navier-Stokes equations. On assuming constant viscosity and a dependence on horizontal coordinates x, y like $\exp[i(k_x x + k_y y)]$ we obtain a set of linear partial differential equations in time and z . From the studies of convection in nondissipative polytropes, we know that, as for atmospheres in general, there are two distinct kinds of modes: acoustic and gravity. Instability in the former takes the form of growing oscillations, and precise conditions for the onset of such instability (or overstability) have yet to be determined. The instability of the gravity modes appears to be in the form of exponential growth, and we conjecture that as in the inviscid case there is no crossover of acoustic and gravity modes. The condition for the onset of such instability is then governed by steady solutions of the linear equations, and so we have only to deal with ordinary differential equations.

As in Paper I we study a fluid extending from $z = z_0$ at the top to $z = z_0 + d$ at the bottom, and use the dimensionless coordinate $\zeta = z/d$ as independent variable. The quantity $Z \equiv \zeta_0^{-1} \equiv d/z_0$ is a measure of the depth of the layer and varies from zero in the Boussinesq limit to infinity for the polytrope with vanishing density at the surface. We introduce a Rayleigh number

$$\mathcal{R}(\zeta) = \frac{(g/T_0)\beta d^4}{(K/\rho_0 C_p)(\mu/\rho_0)}, \quad (2.2)$$

which varies as ζ^{2m-1} , where K is the thermal conductivity, μ is the shear viscosity, C_p is the specific heat at constant pressure, and $\beta = \beta_0 - g/C_p$.

The perturbation equations for the vertical velocity W and the temperature fluctuation Θ are (cf. Paper I)

$$(D^2 - a^2)\Theta = A\zeta^m W, \quad (2.3)$$

$$\mathcal{L}W = -A^{-1}Z^{2m-1}\mathcal{R}_0 a^2 \zeta^m \Theta, \quad (2.4)$$

where

$$a^2 = (k_x^2 + k_y^2)d^2, \quad A = (m+1)\beta PC_p d^{m+2}/(gK), \quad D \equiv d/d\zeta, \\ \mathcal{L} \equiv \zeta(D^2 - a^2)^2 + \zeta(D^2 - a^2)D \frac{m}{\zeta} - (m+1)(D^2 - a^2)\left(D + \frac{m}{\zeta}\right) + \frac{m(m+1)}{3\zeta} a^2, \quad (2.5)$$

and $\mathcal{R}_0 \equiv \mathcal{R}(\zeta_0)$ is the Rayleigh number evaluated at the top of the layer. These equations can be combined to yield

$$\zeta^{-m}(D^2 - a^2)\zeta^{-m}\mathcal{L}W = -Z^{2m-1}\mathcal{R}_0 a^2 W. \quad (2.6)$$

We retain the idealized boundary conditions that were applied in Paper I: the medium is assumed to be bounded at $\zeta = \zeta_0$ and $\zeta_0 + 1$ by rigid, completely slippery, infinitely conducting walls at fixed temperatures. Then

$$\left. \begin{aligned} \zeta^m W &= 0 \\ D^2 W + \frac{m}{\zeta} DW - \frac{m}{\zeta^2} W &= 0 \\ \Theta &= 0 \end{aligned} \right\} \quad \text{at } \zeta = \zeta_0, \zeta_0 + 1. \quad (2.7)$$

These conditions express the vanishing of the mass flux, the tangential viscous stress, and the temperature perturbations on the boundaries, and have been written in a more general form than in Paper I to allow for the possibility that $\zeta_0 = 0$.

Equations (2.3)–(2.5) subject to conditions (2.7) define an eigenvalue problem for \mathcal{R}_0 at fixed m, Z , and a . This was solved by finite differences with two independently written programs. The first used a shooting method (Keller 1968) with fourth-order predictor-corrector or Runge-Kutta integration and a uniform mesh spacing in $\log \zeta$; the second used Newton-Raphson iteration on centered second-order accuracy difference equations (cf. Baker, Moore, and Spiegel 1971) with mesh points distributed according to a first-derivative stretching procedure (Gough, Spiegel, and Toomre 1975). Results from the two programs were compared over the range of Z discussed in this paper; the greatest difference, which occurred for the largest value of Z , was less than 0.1 percent when 300 mesh points were used. The first program, which was faster, was used to obtain the results quoted here.

In all cases the differential equations were cast into six first-order equations before differencing. When reducing the equations to such a form it is convenient to work with variables such as the vorticity and its derivative rather than, say, $D^2 W$ and $D^3 W$. If we did not do this, we would have to subtract large numbers in applying the boundary conditions when ζ_0 is small. We used the variables $W, X \equiv (D^2 + m\zeta^{-1}D - m\zeta^{-2})W$ and Θ , together with their first derivatives.

When $\zeta_0 = 0$ ($Z = \infty$), the series expansion of the solution about $\zeta = 0$ (Ince 1926) was matched onto a numerical solution for $\zeta > \zeta'$ (where ζ' was typically 10^{-2}). The six independent solutions of the equations in the neighborhood of $\zeta = 0$ have the forms $\zeta^{-m}, \zeta, \zeta^2, \zeta^{m+3}, \zeta^{m+4}$, and $\zeta^{m+4} \ln(1/\zeta)$. The vanishing of the mass flux at $\zeta = 0$ eliminates the first of these, the stress condition eliminates ζ^2 , and the temperature condition eliminates ζ^{m+3} . There remain three solutions which must be integrated to the other boundary. Incidentally, if instead of having a rigid lid on the

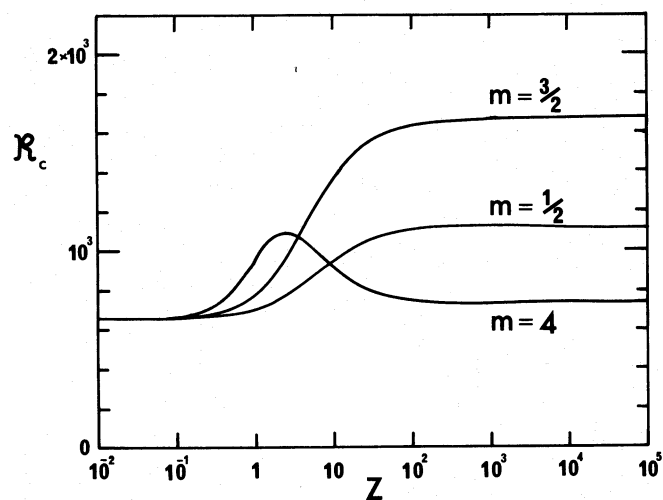


FIG. 1a

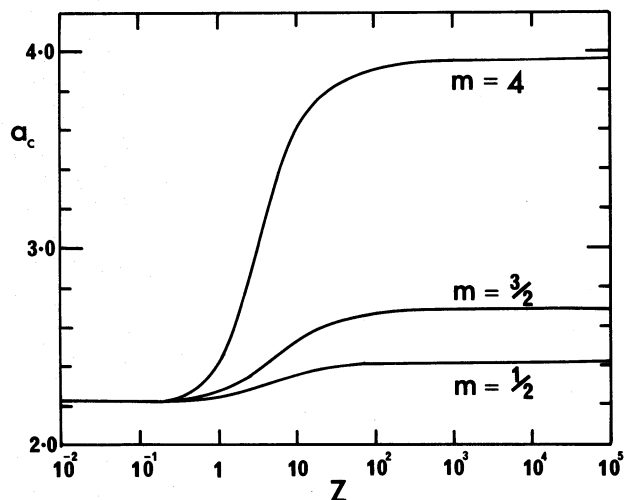


FIG. 1b

FIG. 1a.—Critical Rayleigh number \mathcal{R}_c , evaluated at the middle of the layer, as a function of the layer depth parameter Z , for three values of the polytropic index m of the static atmosphere.

FIG. 1b.—Dependence of the critical wavenumber a_c on Z for various m .

atmosphere in the case $\zeta_0 = 0$ one has a vacuum, and imposes the condition that all components of the stress vanish on the perturbed surface, the same three solutions are selected by the boundary conditions.

III. RESULTS

Eigenvalues of \mathcal{R}_0 for fixed a , m , and Z were obtained from the system described in § II. For given m and Z the values of \mathcal{R}_0 tend to infinity as a tends either to zero or infinity, just as in Boussinesq convection. There is a discrete set of solutions, each with a different number of nodes of W and Θ . Since \mathcal{R}_0 increases with the number of nodes,

TABLE 1

CRITICAL RAYLEIGH NUMBER \mathcal{R}_c (upper figure) AND CRITICAL WAVENUMBER a_c (lower figure) FOR DIFFERENT VALUES OF Z AND m .

m	Z								
	0.1	0.5	1	5	10	10^2	10^3	10^5	∞
0.5	658 2.22	669 2.22	690 2.24	838 2.31	930 2.35	1110 2.41	1144 2.42
1.0	659 2.22	681 2.23	725 2.25	1021 2.36	1189 2.42	1458 2.52	1499 2.53
1.5	660 2.22	694 2.24	762 2.26	1183 2.44	1381 2.53	1638 2.67	1668 2.69	1671 2.69	1671 2.69
2.0	660 2.22	708 2.25	800 2.28	1290 2.54	1462 2.68	1631 2.87	1647 2.89	1649 2.89	1649 2.89
2.5	661 2.22	722 2.25	837 2.30	1322 2.68	1426 2.87	1480 3.10	1481 3.13
3.0	662 2.22	737 2.26	871 2.33	1279 2.86	1299 3.09	1248 3.36	1237 3.40
3.5	663 2.22	752 2.27	902 2.37	1175 3.06	1118 3.33	993 3.63	974 3.68
4.0	664 2.22	767 2.28	925 2.41	1033 3.27	918 3.58	754 3.91	736 3.95	733 3.96	733 3.96
4.5	665 2.22	782 2.29	941 2.46	874 3.50	726 3.84	552 4.19	530 4.24
5.0	666 2.23	795 2.31	947 2.52	716 3.74	556 4.08	391 4.48	370 4.52

NOTE.—When $Z = 0$ the problem is Boussinesq and $\mathcal{R}_c = 658$, $a_c = 2.22$, irrespective of m .

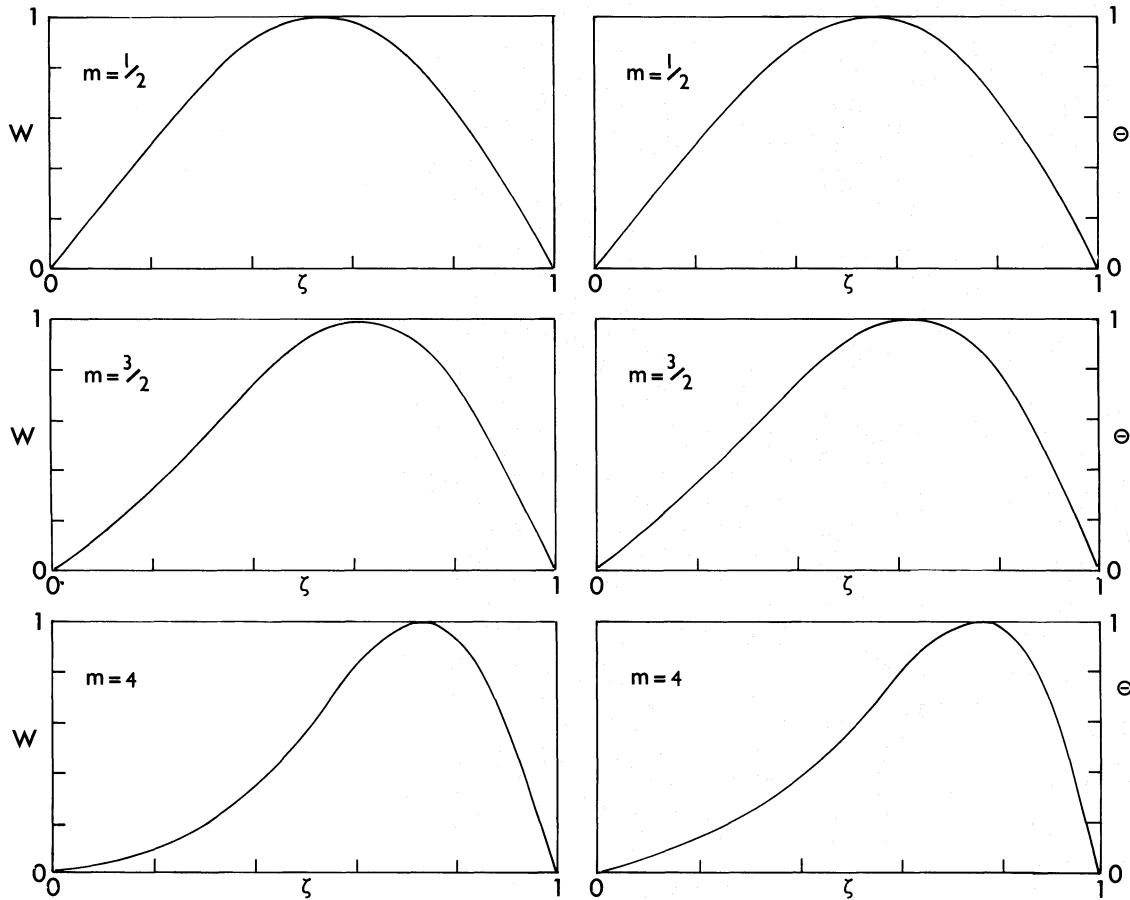


FIG. 2.—Form of the eigenfunctions of vertical velocity W and temperature fluctuation Θ for $Z = \infty$ at various values of m

only the fundamental solutions with no internal nodes have been studied in detail. For these the function $\mathcal{R}_0(a)$ always has a single minimum \mathcal{R}_c at a particular value a_c and, as usual, we infer that for $\mathcal{R}_0 > \mathcal{R}_c$ convection will occur. What is of interest here is the way \mathcal{R}_c and a_c vary with layer depth, measured by Z . In Figure 1 we illustrate this variation for several values of m . The quantity actually plotted in Figure 1a is \mathcal{R}_c , where $\mathcal{R} \equiv \mathcal{R}(\zeta_0 + \frac{1}{2})$ is the value of the Rayleigh number at the middle of the layer. We have studied only cases with $m > 0.5$ and for all these, as $Z \rightarrow \infty$, \mathcal{R}_c tends to a finite limit which is equal to its value at $Z = \infty$. Values of \mathcal{R}_c and a_c are also presented in Table 1 for various values of m and Z .

That the critical Rayleigh number has a finite value when $Z = \infty$ is true regardless of the depth ζ_1 at which the Rayleigh number is evaluated, provided $\zeta_1 \neq 0$. But the actual limiting value does depend on ζ_0 . This dependence of the critical Rayleigh number on the depth at which it is evaluated makes it difficult to draw an unambiguous conclusion about the effect of density variation on stability. Yet the qualitative indication of these results is that a compressible medium is generally more stable than a nearly incompressible one of the same depth.

The basis of this remark lies partly in the behaviour of the eigenfunctions illustrated in Figure 2. This shows $W(\zeta)$ and the corresponding value of $\Theta(\zeta)$, the temperature fluctuation. In thick layers the eigenfunctions have small amplitudes in the upper portion of the layer. As $\zeta_0 \rightarrow 0$ ($Z \rightarrow \infty$), the effect becomes very pronounced. At the same time, the eigenfunctions in the lower part of the layer look rather like those of Boussinesq convection. Thus it is only in the lower portion of the layer, where $1 + \zeta_0 > \zeta > \zeta^*$, say, that the amplitude of convection is significant in linear theory, and the effective Rayleigh number for this part of the layer, which varies as $(1 + \zeta_0 - \zeta^*)^4$, is reduced. Hence there is effective stabilization. More specifically we may choose ζ^* as the level at which the dependent variable X is zero [which is near the point of inflection in $W(\zeta)$] and evaluate the effective Rayleigh number at a point midway between ζ^* and the base of the layer:

$$\mathcal{R}_{\text{eff}} \equiv \left[\frac{\zeta^* + \frac{1}{2}(1 + \zeta_0 - \zeta^*)}{\zeta_0 + \frac{1}{2}} \right]^{2m-1} (1 + \zeta_0 - \zeta^*)^4 \mathcal{R}_c. \quad (3.1)$$

At marginal stability the dependence of \mathcal{R}_{eff} on Z and m is much weaker than that of \mathcal{R}_c , especially when m is large

and the eigenfunctions are strongly concentrated near the base of the layer. Moreover, the effective critical wave-number $a_{\text{eff}} \equiv (1 + \zeta_0 - \zeta^*)a_c$ varies less than a_c . The fact that \mathfrak{R}_c itself decreases with m at large Z , as shown in Figure 1 and Table 1, is somewhat misleading, for, when $m \gtrsim 2.5$, \mathfrak{R} is evaluated in the inert upper part of the layer. The Rayleigh number $\mathcal{R}(\zeta_0 + 1)$, evaluated at the base of the layer, increases monotonically with both m and Z .

We may mention here that our results have been confirmed by Graham (1975). Another numerical study of this problem by Tadjbakhsh and Quarles (unpublished MS) gives only values of \mathfrak{R}_c for various m and Z , and for a variety of boundary conditions. Our results agree to within (or disagree by) about 20 percent in values of \mathfrak{R}_c .

Vickers (1971) likewise presents numerical solutions of the equations studied here and these disagree markedly with our results. Vickers's values of the critical Rayleigh number (evaluated at $\zeta = \zeta_0 + \frac{1}{2}$) differ from ours by more than a factor of 2 in some cases. The source of the difference seems to be a reversal of the sign of Θ associated with a spike which appears at the upper surface. This reversal in Θ , which seriously modifies the eigenvalues, is possible only if W also changes sign. Consider equation (2.4) relating Θ , the temperature fluctuation, and W , the vertical velocity. Since Θ must vanish at the endpoints ζ_0 and $\zeta_0 + 1$, the solution of equation (2.4) for Θ in terms of W is readily seen to be

$$\Theta(\zeta) = A \int_{\zeta_0}^{\zeta_0+1} G(\zeta, \xi) \xi^m W(\xi) d\xi, \quad (3.2)$$

$$\begin{aligned} G(\zeta, \xi) &= -\frac{1}{a \sinh a} \sinh [a(1 + \zeta_0 - \xi)] \sinh [a(\zeta - \zeta_0)] \quad (\zeta < \xi) \\ &= -\frac{1}{a \sinh a} \sinh [a(1 + \zeta_0 - \zeta)] \sinh [a(\xi - \zeta_0)] \quad (\zeta > \xi). \end{aligned} \quad (3.3)$$

The Green function G is negative in the open interval $(\zeta_0, \zeta_0 + 1)$. Hence for the fundamental solution with no internal zeros of W , Θ and W have opposite signs everywhere in $(\zeta_0, \zeta_0 + 1)$. The numerical solution for Θ illustrated in Vickers's Figure 1 does change sign in $(\zeta_0, \zeta_0 + 1)$ while the graph of W shows no perceptible sign change. In that case his solution could not satisfy the governing differential equations.

Vickers has recently informed us that though it is not evident from his Figure 1, W vanishes at $\zeta - \zeta_0 \simeq 10^{-4}$ (and again at $\zeta = \zeta_0$) in the numerical solution illustrated there. In the domain where W is positive, W remains very small, according to Vickers, though the precise value of the maximum is no longer available. His Figure 1a suggests that $W \ll 0.05$; but if we set $W = 0.05$ throughout the interval in question, we overestimate the maximum possible effect of this sign reversal in W on the form of Θ . We can then use equation (3.2) to show that although Θ has a zero in this case, it occurs close to $\zeta = \zeta_0$ which is qualitatively different from the location of the zero of Θ in Vickers's Figure 1b.

IV. DISCUSSION

Our principal aim in the present calculations is to shed light on the qualitative influence of strong density stratification on convection. This is a feature which is not taken account of in current theories of stellar convection other than by a crude approximation. The question that we would like to confront with these results is whether density stratification is stabilizing or not. In fact it is not evident that this can be resolved unambiguously, since the governing parameter, the Rayleigh number, is highly variable in a stratified layer. If we evaluate \mathcal{R} at the middle of the layer we obtain exact agreement with the Boussinesq case when $Z \rightarrow 0$. As we saw in Paper I, this agreement is good to $O(Z)$ for very small Z . The thin layer expansion of Paper I can be extended, and we find for \mathfrak{R}_c , the critical Rayleigh number evaluated at $\zeta = \zeta_0 + \frac{1}{2} = Z^{-1} + \frac{1}{2}$,

$$\mathfrak{R}_c = \frac{27\pi^4}{4} (1 + \mathfrak{R}_2 Z^2 + \dots), \quad (4.1)$$

where the precise expression for \mathfrak{R}_2 , which is extremely complicated, may be approximated to an accuracy of 10^{-6} by

$$\mathfrak{R}_2 = \frac{1}{621} \left(1 + \frac{583}{6210} m \right) \left(1 + \frac{7411}{60} m \right). \quad (4.2)$$

Thus, as Z increases, so does the critical value of \mathfrak{R} , and this we interpret as enhanced stability. Our conclusion is that a layer with strong density stratification is more stable than a layer of the same thickness with weak density stratification. This is apparent from equations (4.1) and (4.2) since increasing m implies increasing density stratification.

The origin of this enhanced stability is understandable, as we implied in § III, in terms of the behavior of $W(\zeta)$. The amplitude of W is quite small in the upper part of the layer when the layer thickness Z is large. The WKB analysis of Paper I predicts a turning point in the solutions for W in deep layers above which W decreases exponentially. This behavior is mirrored by a point of inflection in the numerical solutions. Its origin seems clear. Near the

top of the layer the kinematic viscosity and thermal diffusivity both become very large and provide great local stabilization for $m > 0.5$. The point seems evident, but is often ignored in discussions of stellar fluid dynamics.

It has often been argued that motion in stars will conserve the momentum density ρv and hence produce large velocities near their surfaces. A discussion centering on this point has arisen, for example, in the analogous problem of the behavior of meridional currents near a stellar surface (Mestel 1966; Smith 1970). Our results raise another possibility: at least in this problem, the enormous damping at low density is a powerful effect, and superficial speeds may be smaller than those at greater depths. Although molecular viscosity may be insufficient to reduce the tangential velocity gradient, turbulent viscosity might force the meridional velocity to be almost uniform near the surface (see also Osaki 1972).

This reduction in speeds at the surface increases as the horizontal scale of motion decreases and is evident, for this linear problem, even when the horizontal scale is noticeably in excess of the layer thickness, if the latter is itself considerable. This may affect the scales of convective motion observed near the solar surface. Linear theory suggests that the observed spectrum of horizontal length scales for convection should have a high-frequency cutoff, compared with that at greater depths. The turning point in the WKB approximation coincides with the level at which $\mathcal{R}a^{-4}$, a local Rayleigh number based on the horizontal wavenumber, becomes less than unity. At any given level there is a minimum wavelength λ_c such that all cells with $\lambda < \lambda_c$ have velocities that decay exponentially upward. It appears that for this reason molecular viscosity should suppress motion with horizontal scales below about 200 km at less than unit optical depth in the solar photosphere (Spiegel 1966). This coincides with the smallest scales observed (Harvey and Schwarzschild 1975), though the latter may be limited by observational resolution.

A similar argument might apply to the influence of turbulent viscosity on even larger scales of convection. If so, the spectrum of horizontal scales in the body of the convective zone would not resemble that observed at the solar surface. The kinematic eddy viscosity based on granular convection is around $10^{12} \text{ cm}^2 \text{ s}^{-1}$, which is about 10^9 times greater than the corresponding molecular viscosity, and the critical wavelength $\lambda_c \propto \mu^{1/4}$. Thus the fine structure of supergranulation might be filtered out owing to turning points in the large solutions caused by eddy viscosity. This effect is consistent with the observed cutoff at about 20,000 km (Leighton 1963).

Finally we should like to comment on the usefulness of WKB approximations in the light of the present numerical results. Numerical solutions are, if anything, easier to produce and far more accurate. However, approximate analytical approaches often lead more easily to physical insight. The turning point in the WKB solution is a clear case of this, as are the associated notions of local and global instability arising from it (cf. Paper I). However, the accuracy of the WKB solutions is limited, at least in leading order.

If we examine the WKB results for \mathcal{R}_c (cf. Paper I), we see that as $Z \rightarrow \infty$, \mathcal{R}_c tends to a constant, which is a correct qualitative result. But the \mathcal{R}_c so obtained is less than the value for $Z = 0$ while we know from the numerical results that it should be larger. To see this clearly, we have only to expand the \mathcal{R}_c given by WKB for small Z . Though we get the form (4.1), we find for $m = 2$ that $\mathcal{R}_2 = 3/32$ while the value obtained for \mathcal{R}_2 from equation (4.2) is 0.47. This is therefore a rather crude approximation.

For large Z the situation becomes somewhat worse, and the occurrence of a turning point in the layer forces the introduction of appropriate procedures such as the uniformly valid asymptotic approximation of Paper I. As we have just noted, this is not accurate as $Z \rightarrow \infty$. The reason for this failure is that the approximation as developed in Paper I is strictly valid only for $a \gg \max(1, Z)$. To obtain \mathcal{R}_c we need to extend the results to $a = O(1)$, and this produces inaccuracy both in the solution obtained and in the representation of the boundary conditions to leading order. However for $a \gg (1, Z)$, the approximation is good. For example, it is easy to see from the analysis of Paper I that the value of the Rayleigh number, evaluated at the bottom of the layer, for marginal stability is a^4 . We have checked that this agrees well with our numerical results. (The numerical results given by Vickers 1971 pertaining, to the accuracy of the WKB solution seem to be in error, as can be seen by detailed examination of the case $m = \frac{1}{2}$, which is analytically simple in WKB approximation.) This result is of qualitative interest since it bears out our interpretation of the form of the eigenfunctions. For large a and Z , the motion is confined to a relatively thin layer near the bottom of the zone, so the result should be similar to that in Boussinesq theory.

To improve the WKB results, we need to go to higher order terms, but they are unlikely to add qualitative information whereas for quantitative information numerical methods are superior. There is in Paper I the beginning of an extension of the WKB results to higher order, which actually causes the eigenfunctions to have amplitudes near the top which are too large, according to the numerical results. The fault is not in the WKB approximation but in the failure to extend the boundary conditions to higher order. We consider that in a problem such as this the leading order WKB approximation provides valuable insight, especially as it is obtained fairly readily. But any quantitative results obtained by extending the approximation beyond its strict limits of validity must be treated with caution.

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