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Theory of the kinetic helicity effect on turbulent diffusion of magnetic and scalar fields

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ABSTRACT

Kinetic helicity is a fundamental characteristics of astrophysical turbulent flows. It is not only responsible for the generation of large-scale magnetic fields in the Sun, stars, and spiral galaxies, but it also affects turbulent diffusion resulting in the dissipation of large-scale magnetic fields. Using the path integral approach for random helical velocity fields with a finite correlation time and large Reynolds numbers, we show that turbulent magnetic diffusion is reduced by the kinetic helicity, while the turbulent diffusivity of a passive scalar is enhanced by the helicity. The latter can explain the results of recent numerical simulations for forced helical turbulence. One of the crucial reasons for the difference between the kinetic helicity effect on magnetic and scalar fields is related to the helicity dependence of the correlation time of a turbulent velocity field.

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1. INTRODUCTION

The evolution of solar and Galactic large-scale magnetic fields can be understood in terms of mean-field dynamo theory applying various analytical methods (see, e.g., Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich et al. 1983; Ruzmaikin et al. 1988; Rüdiger et al. 2013; Moffatt & Dormy 2019; Rogachevskii 2021; Shukurov & Subramanian 2022). Helical motions emerge in inhomogeneous or density stratified turbulence, give rise to an α effect, and produce large-scale dynamo action in combination with a nonuniform (differential) rotation, while turbulent magnetic diffusion limits the growth rate of the field.

It has recently been shown using direct numerical simulations (DNS) (Brandenburg et al. 2017; Brandenburg et al. 2025) that helical turbulent motions of the plasma affect not only the α effect, but also the turbulent magnetic diffusion. In particular, the kinetic helicity $H_{\rm K}=\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle$ was found to lower the turbulent magnetic diffusion coefficient $\eta_{_{\rm t}}$, where $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are fluctuations of velocity and vorticity, and angular brackets denote ensemble averaging. On the other hand, DNS showed that the kinetic helicity increases the turbulent diffusion coefficient for passive scalars (Brandenburg et al. 2025).

Using the renormalization group approach in the limit of low magnetic Reynolds numbers, it has been recently shown by Mizerski (2023) that the decrease of the turbulent magnetic diffusion coefficient in comparison with that for a nonhelical random flow is of the order of ${\rm Rm}^2 (H_{\rm K}\tau_{\rm c})^2/\langle \boldsymbol{u}^2\rangle$, where ${\rm Rm}=\tau_{\rm c}\,\langle \boldsymbol{u}^2\rangle/\eta$ is the magnetic Reynolds number, η is the magnetic diffusion caused by an electrical conductivity of the plasma, and $\tau_{\rm c}$ is the turbulent correlation time. Early theoretical predictions by Nicklaus & Stix (1988) based on the cumulant expansion method demonstrated the opposite effect where the turbulent magnetic diffusion coefficient increases with kinetic helicity—in contradiction to the subsequent numerical results of Brandenburg et al. (2017).

By means of the Feynman diagram technique, it has been found that kinetic helicity increases the turbulent diffusion of a passive scalar (Dolginov & Silant'ev 1987). Later, the increase of passive scalar diffusion

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of up to 50% by kinetic helicity has been confirmed by Chkhetiani et al. (2006) applying the renormalization group approach. On the other hand, applying the renormalization-group theory it has been demonstrated that there are no effect of helicity on the effective eddy viscosity (Zhou 1990). Various helicity effects on different characteristics of turbulence are discussed in the recent review by Pouquet & Yokoi (2022).

In the present study, we apply the path-integral approach (see, e.g., Dittrich et al. 1984; Kleeorin et al. 2002; Elperin et al. 2000, 2001) for a random helical velocity field with a finite correlation time for large fluid and magnetic Reynolds numbers. We derive equations for the mean magnetic field and the mean scalar field (e.g., the mean particle number density). We have shown that the turbulent magnetic diffusion coefficient decreases because of the kinetic helicity. On the other hand, the kinetic helicity increases turbulent diffusion coefficient of the scalar field. Both effects are of the order of $(H_{\rm K}\tau_{\rm c})^2/\langle \boldsymbol{u}^2\rangle$.

To derive the mean-field equations for the magnetic and scalar fields, we use an exact solution of the governing equations (i.e., the induction equation for the magnetic field and the convection–diffusion equation for the scalar field) in the form of a functional integral for an arbitrary velocity field. The microscopic diffusion can be described by a Wiener random process, and the functional integral implies an averaging over the Wiener random process. The used form of the exact solution of the governing equations allows us to separate the averaging over the Wiener random process and a random velocity field. The derived mean-field equations for the magnetic and scalar fields are generally integro-differential equations. However, when the characteristic scale of variation of the mean fields is much larger than the correlation length of a random velocity field, second-order equations (in spatial variables) are recovered for the mean fields.

For the derivation of the mean-field equations, we consider a random helical velocity field with a small yet finite constant renewal time. Thus, we apply a model with two random processes: the Wiener random process which describes the microscopic diffusion and the random velocity field between the renewals. This model reproduces important features of some real turbulent flows. For instance, the interstellar turbulence which is driven by supernovae explosions, loses memory in the instants of explosions (see, e.g., Zeldovich et al. 1990; Lamburt et al. 2000). Between the renewals, the velocity field can be random with its intrinsic statistics. To obtain a statistically stationary random velocity field, we assume that the velocity fields between renewals have the same statistics.

This paper is organized as follows. In Section 2 we outline the governing equations and the procedure of the derivation of the equation for the mean magnetic field. In Section 3 we derive the equation for the turbulent magnetic diffusion coefficient. For comparison with the magnetic case, we derive the mean-field equation for the particle number density in Section 4 and obtain an expression for the turbulent diffusion coefficient. In Section 5 we compare the theoretical predictions with the results of the direct numerical simulations. Finally, we draw conclusions in Section 6.

2. GOVERNING EQUATIONS

The magnetic field $\boldsymbol{B}(t, \boldsymbol{r})$ is determined by the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \eta \Delta \boldsymbol{B}, \tag{1}$$

where $\boldsymbol{u}$ is a random velocity field. For simplicity, we consider an incompressible velocity field. Below we derive the equation for the mean magnetic field in a random helical velocity field with a finite correlation time for large fluid and magnetic Reynolds numbers.

Following a previously developed method (Dittrich et al. 1984; Kleeorin et al. 2002), we use an exact solution of Equation (1) with an initial condition $\boldsymbol{B}(t=s, \boldsymbol{x}) = \boldsymbol{B}(s, \boldsymbol{x})$ in the form of the Feynman-Kac formula:

$$B_i(t, \boldsymbol{x}) = \Big\langle G_{ij}(t, s, \boldsymbol{\xi}(t,s))\, B_j(s, \boldsymbol{\xi}(t,s)) \Big\rangle_{\boldsymbol{\xi}}, \tag{2}$$

where the function $G_{ij}(t, s, \boldsymbol{\xi})$ is determined by

$$\frac{dG_{ij}(t, s, \boldsymbol{\xi})}{ds} = N_{ik}\, G_{kj}(t, s, \boldsymbol{\xi}), \tag{3}$$

$N_{ij} = \nabla_j u_i$ is the velocity gradient matrix, $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} - \boldsymbol{x}$, and $\langle ... \rangle_{\boldsymbol{\xi}}$ denotes averaging over the Wiener paths

$$\boldsymbol{\xi}(t,s) = \boldsymbol{x} - \int_0^{t-s} \boldsymbol{u}[t-\mu, \boldsymbol{\xi}(t,\mu)]\, d\mu + \sqrt{2\eta}\, \mathbf{w}(t-s). \tag{4}$$

Here $\mathbf{w}(t)$ is a Wiener random process defined by the properties $\langle \mathbf{w}(t) \rangle_{\mathbf{w}} = 0$, and $\langle {\rm w}_i(t+\tau) {\rm w}_j(t) \rangle_{\mathbf{w}} = \tau \delta_{ij}$, and $\langle \ldots \rangle_{\mathbf{w}}$ denotes averaging over the statistics of the Wiener process. We use the Fourier transform defined as

$$\boldsymbol{B}(t, \boldsymbol{\xi}) = \int \exp(i \boldsymbol{\xi} \cdot \boldsymbol{q}) \boldsymbol{B}(s, \boldsymbol{q})\, d\boldsymbol{q}. \tag{5}$$

Substituting Equation (5) into Equation (2), we obtain

$$B_i(s,\boldsymbol{x}) = \int \Big\langle G_{ij}(t,s,\boldsymbol{\xi}(t,s)) \, \exp[i\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{q}] \, B_j(s,\boldsymbol{q}) \Big\rangle_{\boldsymbol{\xi}} \times \exp(i\boldsymbol{q}\cdot\boldsymbol{x}) \, d\boldsymbol{q} \, . \quad (6)$$

In Equation (6) we expand the function $\exp[i\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{q}]$ in a Taylor series at $\boldsymbol{q}=0$, i.e., $\exp[i\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{q}] = \sum_{k=0}^{\infty}(1/k!)(i\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{q})^k$. Using the identity $(i\boldsymbol{q})^k \exp[i\boldsymbol{x}\cdot\boldsymbol{q}] = \boldsymbol{\nabla}^k \exp[i\boldsymbol{x}\cdot\boldsymbol{q}]$ and Equation (6), we arrive at the expression

$$B_i(t,\boldsymbol{x}) = \Big\langle G_{ij}(t,s,\boldsymbol{\xi}) \Big[\sum_{k=0}^{\infty} \frac{(\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla})^k}{k!}\Big] \Big\rangle_{\boldsymbol{\xi}} \times \int B_j(s,\boldsymbol{q}) \exp(i\boldsymbol{q}\cdot\boldsymbol{x}) \, d\boldsymbol{q}. \quad (7)$$

The inverse Fourier transform implies that $B_j(s,\boldsymbol{x}) = \int B_j(s,\boldsymbol{q}) \exp(i\boldsymbol{q}\cdot\boldsymbol{x}) \, d\boldsymbol{q}$, so that Equation (7) can be rewritten as

$$B_i(t,\boldsymbol{x}) = \Big\langle G_{ij}(t,\boldsymbol{\xi}) \, \exp(\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla}) \Big\rangle_{\boldsymbol{\xi}} B_j(s,\boldsymbol{x}). \quad (8)$$

Equation (5) can be formally regarded as an inverse Fourier transform of the function $B_i(t,\boldsymbol{\xi})$. However, $\boldsymbol{\xi}$ is the Wiener path which is not a standard spatial variable. On the other hand, Equation (8) was also derived in Appendix A of Kleeorin et al. (2002) applying a more rigorous method; see also Dittrich et al. (1984). In this derivation the Cameron-Martin-Girsanov theorem was used.

3. MEAN-FIELD EQUATIONS FOR THE MAGNETIC FIELD

In this section we derive mean-field equation for a magnetic field using a random helical velocity field with a small yet finite constant renewal time. These results can be also generalized for a random renewal time (see, e.g., Lamburt et al. 2000; Kleeorin et al. 2002; Elperin et al. 2001). Assume that in the intervals $\ldots(-\tau,0];(0,\tau];(\tau,2\tau];\ldots$ the velocity fields are statistically independent and have the same statistics. This implies that the velocity field looses memory at the prescribed instants $t=k\tau$, where $k=0,\pm1,\pm2,\ldots$. This velocity field cannot be considered as a stationary (in statistical sense) field for small times $\sim\tau$, however, it behaves like a stationary field for $t\gg\tau$.

The velocity fields before and after renewal are assumed to be statistically independent. We use this assumption to decouple averaging into averaging over two time intervals. In particular, the function $G_{ij}(t,\boldsymbol{\xi})$ in Equation (8) is determined by the velocity field after the renewal, while the magnetic field $B_j(s,\boldsymbol{x})$ is determined by the velocity field before renewal.

In Equation (8) we specify instants $t=(m+1)\tau$ and $s=m\tau$, and average it over random velocity field, which yield the equation for the mean magnetic field $\overline{\boldsymbol{B}}$ as

$$\overline{B}_i[(m+1)\tau,\boldsymbol{x}] = P_{ij}(\tau,\boldsymbol{x},i\boldsymbol{\nabla}) \, \overline{B}_j(m\tau,\boldsymbol{x}), \quad (9)$$

where $\overline{B}_i[(m+1)\tau,\boldsymbol{x}] = \langle B_i((m+1)\tau,\boldsymbol{x})\rangle_{\boldsymbol{u}}$, $\overline{B}_j(m\tau,\boldsymbol{x}) = \langle B_j(m\tau,\boldsymbol{x})\rangle_{\boldsymbol{u}}$, and

$$P_{ij}(\tau,\boldsymbol{x},i\boldsymbol{\nabla}) = \langle\langle G_{ij}(\tau,\boldsymbol{\xi}) \, \exp[\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla}]\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}}. \quad (10)$$

Here the time $s=m\tau$ is the last renewal time before $t=(m+1)\tau$ and $t-s=\tau$. Averaging of the functions $G_{ij}(\tau,\boldsymbol{\xi}) \, \exp[\tilde{\boldsymbol{\xi}}(\tau)\cdot\boldsymbol{\nabla}]$ and $B_j(m\tau,\boldsymbol{x})$ over random velocity field $\langle\ldots\rangle_{\boldsymbol{u}}$ can be decoupled into the product of averages since $B_j(m\tau,\boldsymbol{x})$ and $G_{ij}(\tau,\boldsymbol{\xi}) \, \exp[\tilde{\boldsymbol{\xi}}(\tau)\cdot\boldsymbol{\nabla}]$ are statistically independent. Indeed, the field $B_j(m\tau,\boldsymbol{x})$ is determined in the time interval $(-\infty,m\tau]$, whereas the function $G_{ij}(\tau,\boldsymbol{\xi}) \, \exp[\tilde{\boldsymbol{\xi}}(\tau)\cdot\boldsymbol{\nabla}]$ is defined on the interval $(m\tau,(m+1)\tau]$. Due to a renewal, the velocity field as well as its functionals $B_j(m\tau,\boldsymbol{x})$ and $G_{ij}(\tau,\boldsymbol{\xi}) \, \exp[\tilde{\boldsymbol{\xi}}(\tau)\cdot\boldsymbol{\nabla}]$ in these two time intervals are statistically independent (see Dittrich et al. 1984; Kleeorin et al. 2002, for details).

Considering a very small renewal time and expanding into Taylor series the functions $G_{ij}(\mu,\boldsymbol{\xi})$ and $\exp[\tilde{\boldsymbol{\xi}}(\mu)\cdot\boldsymbol{\nabla}]$ entering in $P_{ij}(\mu,\boldsymbol{x},i\boldsymbol{\nabla})$ (see Equation (10)), we obtain

$$P_{ij}(\mu,\boldsymbol{x},i\boldsymbol{\nabla}) = \Big\langle\Big\langle\Big(\delta_{ij} + \mu N_{ij} + \frac{\mu^2}{4} N_{ik}N_{kj} + \ldots\Big) \times\Big(1 + \tilde{\xi}_m\nabla_m + \frac{1}{2}\tilde{\xi}_m\tilde{\xi}_n\nabla_m\nabla_n + \ldots\Big)\Big\rangle_{\boldsymbol{\xi}}\Big\rangle_{\boldsymbol{u}}. \quad (11)$$

Here we take into account that the solution of Equation (3) can be written as

$$G_{ij}(\mu) = \delta_{ij} + \int_0^{\mu} N_{ij}(\mu')\, d\mu' + \frac{1}{2}\int_0^{\mu} N_{ik}(\mu')\, d\mu' \int_0^{\mu'} N_{kj}(\mu'')\, d\mu'' + \ldots \quad (12)$$

We consider a random incompressible velocity field with a Gaussian statistics. We also consider a homogeneous turbulence with the large fluid and magnetic Reynolds numbers. Therefore, the operator $P_{ij}(\mu,\boldsymbol{x},i\boldsymbol{\nabla})$ is given by

$$P_{ij}(\mu,\boldsymbol{x},i\boldsymbol{\nabla}) = \delta_{ij} + \mu\langle\langle\tilde{\xi}_m N_{ij}\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}}\nabla_m + \Big\{\frac{1}{2}\delta_{ij}\langle\langle\tilde{\xi}_m\tilde{\xi}_n\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}} + \frac{\mu^2}{8}\Big[\langle\langle\tilde{\xi}_m N_{ik}\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}}\langle\langle\tilde{\xi}_n N_{kj}\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}} + \langle\langle\tilde{\xi}_m N_{kj}\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}} \, \langle\langle\tilde{\xi}_n N_{ik}\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}}\Big]\Big\}\nabla_m\nabla_n + \ldots, \quad (13)$$

where we keep only nonzero correlation functions. Now we determine the correlation function $\langle\langle\tilde{\xi}_m\tilde{\xi}_n\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}}$ for small μ as

$$\langle\langle\tilde{\xi}_m\tilde{\xi}_n\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}} = \mu^2\langle u_m u_n\rangle + \frac{\mu^4}{4}\Big[\langle u_s \nabla_p u_n\rangle\,\langle u_p \nabla_s u_m\rangle + \langle u_s u_p\rangle\,\langle(\nabla_p u_n)\,(\nabla_s u_m)\rangle\Big] + 2\eta\langle \mathrm{w}_m \mathrm{w}_n\rangle_{\mathbf{W}}, \qquad (14)$$

where we neglected terms $\sim \mathrm{O}(\mu^5)$ and hereafter we denote $\langle ...\rangle$ as the averaging over statistics of random velocity field.

To determine the correlation function $\langle u_i \nabla_p u_j\rangle$, we use a model for the second moment $\langle u_i(\boldsymbol{k})u_j(-\boldsymbol{k})\rangle$ of homogeneous incompressible and helical turbulence in Fourier space in the following form:

$$\langle u_i(\boldsymbol{k})u_j(-\boldsymbol{k})\rangle = \frac{E_u(k)}{8\pi k^2}\bigg[\Big(\delta_{ij} - \frac{k_i\,k_j}{k^2}\Big)\langle \boldsymbol{u}^2\rangle - \frac{\mathrm{i}}{k^2}\,\varepsilon_{ijp}\,k_p\,\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle\bigg], \qquad (15)$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity, δ_{ij} is the Kronecker fully symmetric unit tensor, ε_{ijp} is the Levy-Civita fully antisymmetric unit tensor, $\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle$ is the kinetic helicity, the energy spectrum function is $E_u(k) = (2/3)\,k_0^{-1}\,(k/k_0)^{-5/3}$ in the inertial range of turbulence $k_0 \le k \le k_\nu$, the wave number $k_0 = 1/\ell_0$, the length ℓ_0 is the integral scale of turbulence, the wave number $k_\nu = \ell_\nu^{-1}$, the length $\ell_\nu = \ell_0 \mathrm{Re}^{-3/4}$ is the Kolmogorov (viscous) scale. After integration in the Fourier space we obtain that the correlation function $\langle u_i u_j\rangle$ in the physical space is $\langle u_i u_j\rangle = \langle \boldsymbol{u}^2\rangle\,\delta_{ij}/3$. Using Equation (15), after integration in the Fourier space, we arrive at the following expression:

$$\langle u_i \nabla_p u_j\rangle = -\frac{1}{6}\varepsilon_{ijp}\,\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle. \qquad (16)$$

Using Equations (14) and (16), we obtain that the correlation function $\langle\langle\tilde{\xi}_m\tilde{\xi}_n\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}}$ is given by

$$\langle\langle\tilde{\xi}_m\tilde{\xi}_n\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}} = \mu\delta_{mn}\bigg\{2\eta + \frac{\mu}{3}\bigg[\langle \boldsymbol{u}^2\rangle - \frac{\mu^2}{24}\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle^2\bigg]\bigg\}. \qquad (17)$$

Here we have neglected a small contribution ($\sim \mu^4$) caused by non-helical part of turbulence. In a similar way, we obtain that the correlation function $\langle\langle\tilde{\xi}_i N_{jp}\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}}$ is given by

$$\langle\langle\tilde{\xi}_i N_{jp}\rangle_{\boldsymbol{\xi}}\rangle_{\boldsymbol{u}} = -\mu\langle u_i \nabla_p u_j\rangle = \frac{\mu}{6}\varepsilon_{ijp}\,\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle. \qquad (18)$$

Since $\partial\overline{\boldsymbol{B}}/\partial t = \lim\limits_{\mu\to 0}[\overline{B}_i((m+1)\mu,\boldsymbol{x}) - \overline{B}_i(m\mu,\boldsymbol{x})]/\mu$, Equations (13)–(14) and (16)–(18) yield the mean-field equation:

$$\frac{\partial\overline{\boldsymbol{B}}(t,\boldsymbol{x})}{\partial t} = \alpha\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} + (\eta+\eta_{\mathrm{t}})\Delta\overline{\boldsymbol{B}}, \qquad (19)$$

where the turbulent magnetic diffusion coefficient is given by

$$\eta_{\mathrm{t}} = \frac{\tau_{\mathrm{c}}}{3}\,\bigg[\langle \boldsymbol{u}^2\rangle - \frac{\tau_{\mathrm{c}}^2}{3}\,\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle^2\bigg], \qquad (20)$$

and the α effect is $\alpha = -\tau_{\mathrm{c}}\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle/3$. In the derivation of Equations (19)–(20), we take into account that $\lim\limits_{\mu\to 0}(\mu\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle) = 2\tau_{\mathrm{c}}\,\langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle$ and $\lim\limits_{\mu\to 0}(\mu\langle \boldsymbol{u}^2\rangle) = 2\tau_{\mathrm{c}}\,\langle \boldsymbol{u}^2\rangle$. Here we also use that $\mathrm{div}\overline{\boldsymbol{B}} = 0$ and $\varepsilon_{ijp}\,\varepsilon_{mnp} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$. The coefficient $a_* < 1$ takes into account the fact that the correlation time is scale-dependent inside the inertial range of turbulence. It also takes into account possible anisotropies of turbulence.

It has been demonstrated by DNS (Brandenburg et al. 2025), that the correlation time of turbulent velocity field depends on the kinetic helicity. It follows from Equation (20) that

$$\frac{\eta_{\mathrm{t}}(H_{\mathrm{K}})}{\eta_{\mathrm{t}}(0)} = \frac{\tau_{\mathrm{c}}(H_{\mathrm{K}})}{\tau_0}\,\bigg[1 - \frac{\tau_{\mathrm{c}}^2(H_{\mathrm{K}})}{3}\,\frac{H_{\mathrm{K}}^2}{\langle \boldsymbol{u}^2\rangle}\bigg], \qquad (21)$$

where $H_{\mathrm{K}} = \langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle$ is the kinetic helicity density, $\eta_{\mathrm{t}}(0) = \eta_{\mathrm{t}}(H_{\mathrm{K}}=0)$ is the turbulent magnetic diffusivity at zero kinetic helicity, and $\tau_0 = \tau_{\mathrm{c}}(H_{\mathrm{K}}=0)$ is the correlation time.

We assume that

$$\tau_{\mathrm{c}}(H_{\mathrm{K}}) = \tau_0\,(1 + C_\tau \epsilon_{\mathrm{f}}^{\zeta}), \qquad (22)$$

where $\epsilon_{\mathrm{f}} = \langle \boldsymbol{u}\cdot\boldsymbol{\omega}\rangle\ell_0/\langle \boldsymbol{u}^2\rangle$ is the normalized kinetic helicity. Equation (22) has recently been supported by the direct numerical simulations of forced turbulence (Brandenburg et al. 2025), where $\zeta = 4$ and $C_\tau = 0.5$ for $\mathrm{Re} \approx 14$. This numerical result has been obtained using two independent methods based on the noninstantaneous correlation functions and the rate of energy dissipation. Equation (22) has also been confirmed for $\mathrm{Re} = 120$; see Figure 1, where we show the dependence of τ_{c} on ϵ_{f}. Here, the simulations had a forcing wavenumber $k_{\mathrm{f}} = 5.1\,k_1$, where k_1 is the lowest wavenumber in the domain. In this case, the results are well approximated by $\zeta = 4$ and $C_\tau = 0.37$, where we have assumed $\ell_0 = 1/k_{\mathrm{f}}$.

Therefore, the turbulent magnetic diffusion coefficient is

$$\eta_{\mathrm{t}} = \eta_{\mathrm{t}}(0)\,\Big[1 + C_\tau\epsilon_{\mathrm{f}}^4 - \tfrac{1}{3}\,\left(1 + C_\tau\epsilon_{\mathrm{f}}^4\right)^3\,\epsilon_{\mathrm{f}}^2\Big]. \qquad (23)$$

It follows from Equation (23) that the turbulent magnetic diffusion coefficient is reduced by the kinetic helicity (see Section 5).

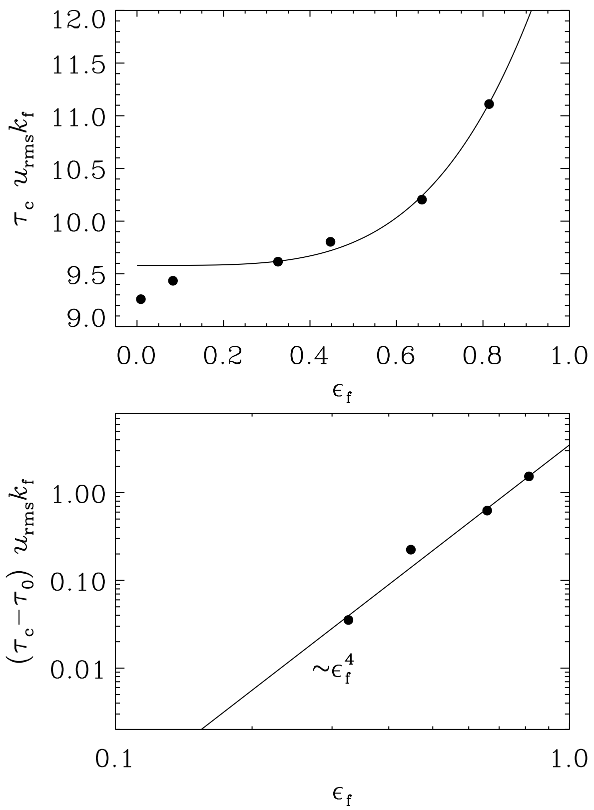

Figure 1. Dependence of τ_c on ϵ_f and $\mathrm{Re} \approx 120$. The solid line gives the fit with $\zeta = 4$ and $C_\tau = 0.37$. In the second panel, we used $\tau_0 u_{\rm rms} k_{\rm f} = 9.6$.

4. MEAN-FIELD EQUATION FOR PARTICLE NUMBER DENSITY

The evolution of the number density $n(t, \boldsymbol{r})$ of small particles advected by a random incompressible fluid flow is determined by the following convection–diffusion equation

$$\frac{\partial n}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} n = \kappa \Delta n, \tag{24}$$

where $\boldsymbol{u}$ is a random velocity field of the particles which they acquire in a random fluid velocity field and κ is the coefficient of molecular (Brownian) diffusion. Following to the method described in Sections 2–3 (see also Elperin et al. 2000, 2001), we derive the mean-field equation for the particle number density. We use an exact solution of Equation (24) with an initial condition $n(t = s, \boldsymbol{x}) = n(s, \boldsymbol{x})$ in the form of the Feynman-Kac formula:

$$n(t, \boldsymbol{x}) = \Big\langle n(s, \boldsymbol{\xi}(t,s)) \Big\rangle_{\boldsymbol{\xi}}, \tag{25}$$

where $\langle ... \rangle_{\boldsymbol{\xi}}$ implies the averaging over the Wiener paths:

$$\boldsymbol{\xi}(t,s) = \boldsymbol{x} - \int_0^{t-s} \boldsymbol{u}[t-\mu, \boldsymbol{\xi}(t,\mu)]\, d\mu + \sqrt{2\kappa}\, \mathbf{w}(t-s).$$

(26)

We assume that

$$n(t, \boldsymbol{\xi}) = \int \exp(i \boldsymbol{\xi} \cdot \boldsymbol{q}) n(s, \boldsymbol{q})\, d\boldsymbol{q}. \tag{27}$$

Substituting Equation (27) into Equation (25), we obtain

$$n(s, \boldsymbol{x}) = \int \Big\langle \exp[i \tilde{\boldsymbol{\xi}} \cdot \boldsymbol{q}]\, n(s, \boldsymbol{q}) \Big\rangle_{\boldsymbol{\xi}} \exp(i \boldsymbol{q} \cdot \boldsymbol{x})\, d\boldsymbol{q}. \tag{28}$$

In Equation (28) we expand the function $\exp[i \tilde{\boldsymbol{\xi}} \cdot \boldsymbol{q}]$ in Taylor series at $\boldsymbol{q} = 0$ and use the identity $(i\boldsymbol{q})^k \exp[i\boldsymbol{x} \cdot \boldsymbol{q}] = \boldsymbol{\nabla}^k \exp[i \boldsymbol{x} \cdot \boldsymbol{q}]$, which yields

$$n(t, \boldsymbol{x}) = \Big\langle \Big[\sum_{k=0}^{\infty} \frac{(\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla})^k}{k!} \Big] \Big\rangle_{\boldsymbol{\xi}} \int n(s, \boldsymbol{q}) \exp(i \boldsymbol{q} \cdot \boldsymbol{x})\, d\boldsymbol{q}. \tag{29}$$

Applying the inverse Fourier transform $n(s, \boldsymbol{x}) = \int n(s, \boldsymbol{q}) \exp(i \boldsymbol{q} \cdot \boldsymbol{x})\, d\boldsymbol{q}$, we obtain

$$n(t, \boldsymbol{x}) = \Big\langle \exp(\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla}) \Big\rangle_{\boldsymbol{\xi}} n(s, \boldsymbol{x}). \tag{30}$$

Equation (30) has been also derived applying a more rigorous method in Appendix A of Elperin et al. (2000). In this derivation the Cameron-Martin-Girsanov theorem is applied.

To derive mean-field equation for a particle number density, we consider a random velocity field with a finite constant renewal time. In Equation (30) we specify instants $t = (m+1)\tau$ and $s = m\tau$, and average this equation over a random velocity field. This yields the mean-field equation for the particle number density as

$$\overline{n}[(m+1)\tau, \boldsymbol{x}] = P(\tau, \boldsymbol{x}, i\boldsymbol{\nabla})\, \overline{n}(m\tau, \boldsymbol{x}), \tag{31}$$

where $\overline{n}[(m+1)\tau, \boldsymbol{x}] = \langle n((m+1)\tau, \boldsymbol{x}) \rangle_{\boldsymbol{u}}$, $\overline{n}(m\tau, \boldsymbol{x}) = \langle n(m\tau, \boldsymbol{x}) \rangle_{\boldsymbol{u}}$, and

$$P(\tau, \boldsymbol{x}, i\boldsymbol{\nabla}) = \langle \langle \exp[\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla}] \rangle_{\boldsymbol{\xi}} \rangle_{\boldsymbol{u}}. \tag{32}$$

We consider a random velocity field with a Gaussian statistics and with large fluid Reynolds numbers and large Peclet numbers. For a small renewal time, expanding the function $\exp[\tilde{\boldsymbol{\xi}}(\tau) \cdot \boldsymbol{\nabla}]$ into Taylor series, we obtain

$$P(\mu, \boldsymbol{x}, i\boldsymbol{\nabla}) = 1 + \frac{1}{2} \langle \langle \tilde{\xi}_m \tilde{\xi}_n \rangle_{\boldsymbol{\xi}} \rangle_{\boldsymbol{u}} \nabla_m \nabla_n + ..., \tag{33}$$

where

$$\langle \langle \tilde{\xi}_m \tilde{\xi}_n \rangle_{\boldsymbol{\xi}} \rangle_{\boldsymbol{u}} = \mu \delta_{mn} \Big\{ 2\kappa + \frac{\mu}{3} \Big[\langle \boldsymbol{u}^2 \rangle - \frac{\mu^2}{24} \langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle^2 \Big] \Big\}. \tag{34}$$

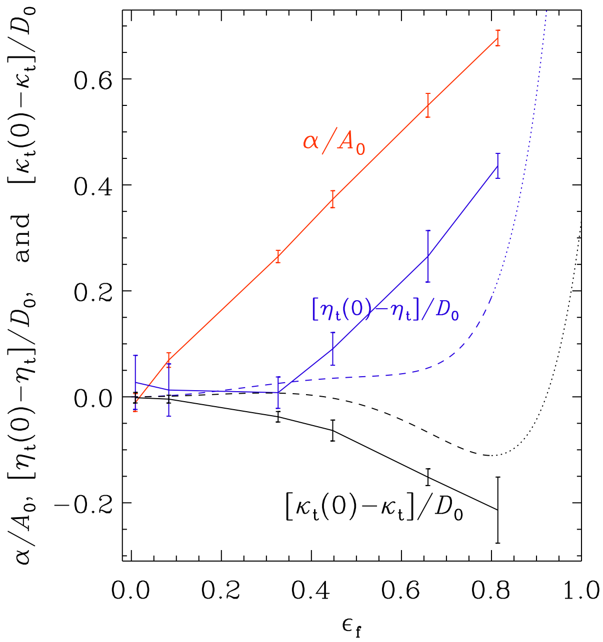

Figure 2. Dependencies of α (red solid line), $\eta_t(0)-\eta_t$ (blue solid line) and $\kappa_t(0)-\kappa_t$ (black solid line) on the fraction ϵ_f of the kinetic helicity for Re $\approx$ 120. The theoretical dependencies for $0<\epsilon_f<0.8$ [see Equations (23) and (38)] are shown as dashed lines. The theoretical results for $\epsilon_f \gtrsim 0.8$ are shown as dotted lines, because they may not be reliable.

Since $\partial\overline{n}/\partial t = \lim_{\mu\to 0}[\overline{n}((m+1)\mu,\boldsymbol{x})-\overline{n}(m\mu,\boldsymbol{x})]/\mu$, Equations (31), (33) and (34) yield the mean-field equation for the particle number density $\overline{n}(t,\boldsymbol{x})$ as

$$\frac{\partial\overline{n}(t,\boldsymbol{x})}{\partial t} = (\kappa+\kappa_t)\Delta\overline{n}, \tag{35}$$

where the turbulent diffusion coefficient is given by

$$\kappa_t = \frac{\tau_c}{3}\left[\langle\boldsymbol{u}^2\rangle - \frac{\tau_c^2}{6}\,\langle\boldsymbol{u}\cdot\boldsymbol{\omega}\rangle^2\right]. \tag{36}$$

It follows from Equation (36) that

$$\frac{\kappa_t(H_u)}{\kappa_t(0)} = \frac{\tau_c(H_K)}{\tau_0}\left[1-\frac{\tau_c^2(H_K)}{6}\,\frac{H_K^2}{\langle\boldsymbol{u}^2\rangle}\right], \tag{37}$$

where $\kappa_t(0)=\kappa_t(H_K=0)$ and $\tau_0=\tau_c(H_K=0)$. Since $\tau_c(H_K)/\tau_0 = 1+C_\tau\epsilon_f^4$ [see Equation (22)], the turbulent diffusion coefficient is

$$\kappa_t = \kappa_t(0)\,\left[1+C_\tau\epsilon_f^4-\tfrac{1}{6}\,\left(1+C_\tau\epsilon_f^4\right)^3\epsilon_f^2\right]. \tag{38}$$

Using Equation (38) we will show in Section 5 that the turbulent diffusion coefficient for the scalar field is enhanced by the kinetic helicity.

5. COMPARISONS WITH NUMERICAL RESULTS

As in Brandenburg et al. (2025), we compute a turbulent velocity field by solving the fully compressible momentum equation with an isothermal equation of state. In Brandenburg et al. (2017), only fully helical cases were compared with nonhelical ones. Brandenburg et al. (2025) did consider runs with intermediate helicity of the forcing, but only for Re $\approx$ 14. Here, we also compute such cases with Re $\approx$ 120. This is accomplished by adding a fraction $\sigma i k_p \epsilon_{ijp} f_j$ to the nonhelical forcing function f_i.

We compute the turbulent transport coefficient α, η_t, and κ_t in Equations (19) and (35) from the turbulent velocity field discussed above. We use the test-field method (Schrinner et al. 2005, 2007; Brandenburg 2005; Brandenburg et al. 2008), where we solve numerically the equations for the fluctuating magnetic and passive scalar fields. These are nonlinear inhomogeneous equations, in which the product of the mean magnetic and passive scalar fields act as an inhomogeneous source term. Thus, the test-field equations are different from the original evolution equations, which are homogeneous. Moreover, the mean magnetic and passive scalar fields are not solutions to these equations, but consist of a set of mutually orthogonal fields that are called test fields. They are constructed such that we can compute the desired transport coefficient exactly and not as a fit or by some regression method (Brandenburg & Sokoloff 2002; Simard et al. 2016; Bendre et al. 2024).

The resulting turbulent transport coefficients depend on time and one or two space coordinates (here only on z, in addition to t). We are usually interested in their averaged values. To determine error bars, we also compute averages for any one third of the full time series. The results for Re $\approx$ 14 and $\approx$ 120 are given in Table 1 for different values of σ. For $\sigma<0.7$, $\epsilon_f=\langle\boldsymbol{u}\cdot\boldsymbol{\omega}\rangle\ell_0/\langle\boldsymbol{u}^2\rangle$ is well approximated by $2\sigma/(1+\sigma^2)$. For larger values, ϵ_f stays somewhat below this estimate. This departure contributes to the steep power-law scaling with $\zeta=4$.

The values α, η_t, and κ_t for Re $=120$ are plotted in Figure 2. As in Brandenburg et al. (2025), we present them in normalized form and divide α by $A_0=u_{\rm rms}/3$ and η_t and κ_t by $D_0=u_{\rm rms}/3k_f$. Note that $\eta_t(0)=\kappa_t(0)=D_0$. We see that α increases approximately linearly with ϵ_f. For η_t and κ_t, it is convenient to plot the differences from the nonhelical values, $\eta_t(0)$ and $\kappa_t(0)$, respectively. We see that for both functions, the differences are small when $\epsilon_f \lesssim 0.4$, and then depart from zero in opposite directions. This is also predicted by the theory. For $\epsilon_f \gtrsim 0.8$, however, there are major departures between our theory and the simulations. Note that the simulations (Brandenburg et al. 2025) predict similar results both for passive scalars using the test-field

Table 1. Values of $\kappa_{\rm t}$ and $\eta_{\rm t}$ normalized by $D_0 \equiv u_{\rm rms}/3k_{\rm f}$, as well as α normalized by $A_0 \equiv u_{\rm rms}/3$ for Re ≈ 14 and ≈ 120 for different values of σ. The values of Ma, $\omega_{\rm rms}/u_{\rm rms}k_{\rm f}$ and $\tau_{\rm c}u_{\rm rms}k_{\rm f}$ are also given.

Run	Re	σ	$2\sigma/(1+\sigma^2)$	$\epsilon_{\rm f}$	$\kappa_{\rm t}/D_0$	$\eta_{\rm t}/D_0$	α/A_0	Ma	$\omega_{\rm rms}/u_{\rm rms}k_{\rm f}$	$\tau_{\rm c}u_{\rm rms}k_{\rm f}$
A	13.8	0.10	0.20	0.20	2.40 ± 0.00	2.00 ± 0.00	-0.99 ± 0.01	0.099	6.1	0.107
B	14.0	0.20	0.38	0.38	2.42 ± 0.00	1.94 ± 0.01	-1.92 ± 0.00	0.100	6.0	0.104
C	14.2	0.30	0.55	0.54	2.48 ± 0.00	1.86 ± 0.00	-2.74 ± 0.01	0.101	5.9	0.099
D	14.5	0.40	0.69	0.67	2.53 ± 0.00	1.73 ± 0.01	-3.39 ± 0.00	0.103	5.8	0.093
E	14.8	0.50	0.80	0.76	2.59 ± 0.00	1.61 ± 0.01	-3.82 ± 0.02	0.106	5.7	0.087
F	15.4	0.70	0.94	0.87	2.81 ± 0.01	1.44 ± 0.00	-4.28 ± 0.04	0.110	5.5	0.076
G	15.8	1.00	1.00	0.91	2.88 ± 0.01	1.37 ± 0.02	-4.44 ± 0.05	0.113	5.3	0.071
H	120.6	0.00	0.00	0.01	2.27 ± 0.01	1.73 ± 0.05	0.05 ± 0.09	0.123	13.0	0.054
I	121.1	0.05	0.10	0.08	2.27 ± 0.01	1.75 ± 0.05	-0.35 ± 0.07	0.124	12.9	0.053
J	121.5	0.20	0.38	0.33	2.30 ± 0.01	1.75 ± 0.03	-1.35 ± 0.06	0.124	12.8	0.052
K	121.5	0.30	0.55	0.45	2.33 ± 0.02	1.67 ± 0.03	-1.90 ± 0.08	0.124	12.7	0.051
L	124.0	0.50	0.80	0.66	2.42 ± 0.02	1.49 ± 0.05	-2.81 ± 0.11	0.126	12.5	0.049
M	127.6	1.00	1.00	0.81	2.48 ± 0.06	1.32 ± 0.02	-3.45 ± 0.07	0.130	12.2	0.045

method and for active scalars based on the decay of an initial entropy perturbation.

The strong dependence of the theoretical results from Equations (23) and (38) involving high powers of $\epsilon_{\rm f}$ is related to the following reasons. The main contributions to the difference in turbulent diffusion coefficients for helical and nonhelical turbulence come from the fourth-order moments of a random velocity field. The second reason for the high powers of $\epsilon_{\rm f}$ in turbulent diffusion coefficients is related to the strong dependence of the correlation time of a random velocity field on $\epsilon_{\rm f}$ found in simulations.

The difference between the theoretical predictions and the simulations for $0.8 < \epsilon_{\rm f} \leq 1$ is related to the theory being based on the following assumptions: (i) the contributions of higher than fourth-order moments of a random velocity field are neglected; (ii) it is assumed that the velocity field has Gaussian statistics; and (iii) we use a model of a random velocity field with renovations.

A recent theoretical study by Kishore & Singh (2025), where a method based on the Furutsu-Novikov theorem (Furutsu 1963; Novikov 1965) has been applied, shows that the turbulent diffusivities of both the mean magnetic and passive scalar fields are suppressed by kinetic helicity. In that paper, however, the kinetic helicity dependence of the correlation time has not been taken into account (Kishore & Singh 2025). This may explain the discrepancy with the numerical results related to the helicity effect on turbulent diffusion of the scalar field (Brandenburg et al. 2025).

6. DISCUSSION

One of the main effects of astrophysical turbulent flows is a strong increase of the diffusion of the large-scale magnetic and scalar fields, which can be characterized in terms of the effective (turbulent) diffusion coefficients. The latter effect decreases the growth rates of the mean-field dynamo instability and various clustering instabilities related to scalar fields.

In the present study, we have developed a theory which qualitatively explains the nontrivial behavior of turbulent diffusion coefficients of the large-scale magnetic and scalar fields as functions of the kinetic helicity. These effects have been recently discovered by direct numerical simulations (Brandenburg et al. 2017; Brandenburg et al. 2025), which show that turbulent magnetic diffusion decreases with increasing kinetic helicity while turbulent diffusion of passive scalars increases with the helicity.

The main contribution to these effects comes from the fourth-order correlation function of the turbulent velocity field. This is the reason why widely used methods like the quasi-linear approach [the first-order smoothing approximation (FOSA) or the second-order correlation approximation (SOCA)] as well as the various τ approaches and the direct interaction approximation (DIA) cannot describe these effects. For instance, Gruzinov & Diamond (1995) use the quasi-linear approach to determine the turbulent transport coefficients (the α effect and turbulent magnetic diffusivity). The main assumption of the quasi-linear approach is that fluctuations are much smaller than the mean fields, so the fourth-order moments have been neglected by Gruzinov & Diamond (1995). All studies of the kinetic helicity effect on turbulent diffusivity discussed here apply various perturbation approaches which take into account the fourth-order moments of random Gaussian velocity fields with small yet finite correlation time.

The main goal of the present paper is to explain the results of the numerical simulations by Brandenburg et al. (2017) and Brandenburg et al. (2025), where we also take into account the kinetic helicity dependence of the correlation time of the random velocity field that has been found in DNS. We have applied the path-integral approach for random flows with a finite correlation time and for large Reynolds and Péclet numbers. We have assumed that the velocity field has Gaussian statistics, which allows us to represent the fourth-order moments of the turbulent velocity field as a product of second-order moments. A crucial role in the understanding of these effects is played by the kinetic helicity effect on the turbulent correlation time, which increases with increasing helicity. The results of the theory developed here are in a qualitative agreement with the numerical results of Brandenburg et al. (2017) and Brandenburg et al. (2025).

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Software and Data Availability. The source code used for the simulations of this study, the Pencil Code (Pencil Code Collaboration et al. 2021), is available on https://github.com/pencil-code/. The simulation setups and corresponding secondary data are available on http://doi.org/10.5281/zenodo.15084461.

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