# CHAPTER 5

# RADIATIVE EXCHANGE BETWEEN GRAY, DIFFUSE SURFACES

## 5.1 INTRODUCTION

In this chapter we shall begin our analysis of radiative heat transfer rates within enclosures without a participating medium, making use of the view factors developed in the preceding chapter. We shall first deal with the simplest case of a black enclosure, that is, an enclosure where all surfaces are black.

Such simple analysis may often be sufficient, for example, for furnace applications with soot-covered walls. This will be followed by expanding the analysis to enclosures with gray, diffuse surfaces, whose radiative properties do not depend on wavelength, and which emit as well as reflect energy diffusely. Considerable experimental evidence demonstrates that most surfaces emit (and, therefore, absorb) diffusely except for grazing angles ($\theta > 60°$), which are unimportant for heat transfer calculations (for example, Fig. 3-1). Most surfaces tend to be fairly rough and, therefore, reflect in a relatively diffuse fashion. Finally, if the surface properties vary little across that part of the spectrum over which the blackbody emissive powers of the surfaces are appreciable, then the simplification of gray properties may be acceptable.

In both cases—black enclosures as well as enclosures with gray, diffuse surfaces—we shall first derive the governing integral equation for arbitrary enclosures, which is then reduced to a set of algebraic equations by applying it to idealized enclosures. At the end of the chapter solution methods to the general integral equations are briefly discussed.

## 5.2 RADIATIVE EXCHANGE BETWEEN BLACK SURFACES

Consider a black-walled enclosure of arbitrary geometry and with arbitrary temperature distribution as shown in Fig. 5-1. An energy balance for $dA$ yields, from equation (4.1),

$$q(\mathbf{r}) = E_b(\mathbf{r}) - H(\mathbf{r}), \tag{5.1}$$

where $H$ is the irradiation onto $dA$. From the definition of the view factor, the rate with which energy leaves $dA'$ and is intercepted by $dA$ is $(E_b(\mathbf{r}')\,dA')\,dF_{dA'-dA}$. Therefore, the total rate of

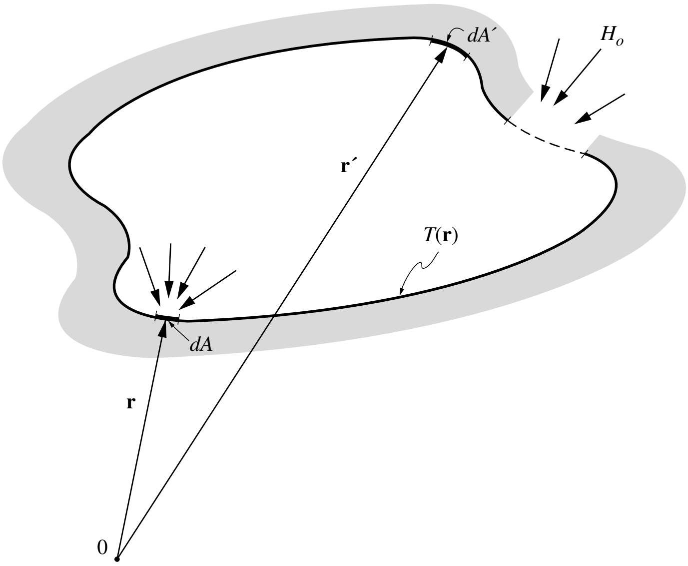



**FIGURE 5-1**
A black enclosure of arbitrary geometry.

incoming heat transfer onto $dA$ from the entire enclosure and from outside (for enclosures with some semitransparent surfaces and/or holes) is

$$H(\mathbf{r})\,dA = \int_A E_b(\mathbf{r}')\,dF_{dA'-dA}\,dA' + H_o(\mathbf{r})\,dA, \tag{5.2}$$

where $H_o(\mathbf{r})$ is the external contribution to the irradiation, i.e., any part not due to emission from the enclosure surface. Using reciprocity, this may be stated as

$$\begin{aligned} H(\mathbf{r}) &= \int_A E_b(\mathbf{r}')\,dF_{dA-dA'} + H_o(\mathbf{r}) \\ &= \int_A E_b(\mathbf{r}')\frac{\cos\theta\cos\theta'}{\pi S^2}(\mathbf{r},\mathbf{r}')\,dA' + H_o(\mathbf{r}), \end{aligned} \tag{5.3}$$

where $\theta$ and $\theta'$ are angles at the surface elements $dA$ and $dA'$, respectively, and $S$ is the distance between them, as defined in Section 4.2. For an enclosure with known surface temperature distribution, the local heat flux is readily calculated as[1]

$$\mathbf{q}(\mathbf{r}) = E_b(\mathbf{r}) - \int_A E_b(\mathbf{r}')\,dF_{dA-dA'} - H_o(\mathbf{r}). \tag{5.4}$$

To simplify the problem it is customary to break up the enclosure into $N$ isothermal subsurfaces, as shown in Fig. 4-2*b*. Then equation (5.4) becomes

$$q_i(\mathbf{r}_i) = E_{bi} - \sum_{j=1}^{N} E_{bj}\int_{A_j} dF_{dA_i-dA_j} - H_{oi}(\mathbf{r}_i), \tag{5.5}$$

or, from equation (4.16),

$$q_i(\mathbf{r}_i) = E_{bi} - \sum_{j=1}^{N} E_{bj}F_{di-j}(\mathbf{r}_i) - H_{oi}(\mathbf{r}_i). \tag{5.6}$$

[1]When looking at equation (5.4) one is often tempted by intuition to replace $dF_{dA-dA'}$ by $dF_{dA'-dA}$. It should always be remembered that we have used reciprocity, since $dF_{dA'-dA}$ is per unit area at $\mathbf{r}'$, while equation (5.4) is per unit area at $\mathbf{r}$.

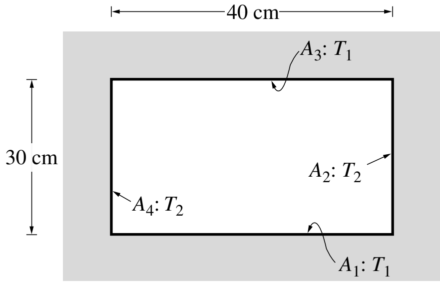



**FIGURE 5-2**
Two-dimensional black duct for Example 5.1.

Even though the temperature may be constant across $A_i$, the heat flux is usually not since (*i*) the local view factor $F_{di-j}$ nearly always varies across $A_i$, and (*ii*) the external irradiation $H_{oi}$ may not be uniform. We may calculate an *average heat flux* by averaging equation (5.6) over $A_i$. With $\int_{A_i} F_{di-j}\, dA_i = A_i F_{i-j}$ this leads to

$$q_i = \frac{1}{A_i}\int_{A_i} q_i(\mathbf{r}_i)\, dA_i = E_{bi} - \sum_{j=1}^{N} E_{bj} F_{i-j} - H_{oi}, \quad i = 1, 2, \ldots, N, \tag{5.7}$$

where $q_i$ and $H_{oi}$ are now understood to be average values.

Employing equation (4.18) we rewrite $E_{bi}$ as $\sum_{j=1}^{N} E_{bi} F_{i-j}$, or

$$q_i = \sum_{j=1}^{N} F_{i-j}(E_{bi} - E_{bj}) - H_{oi}, \quad i = 1, 2, \ldots, N. \tag{5.8}$$

In this equation the heat flux is expressed in terms of the net radiative energy exchange between surfaces $A_i$ and $A_j$,

$$Q_{i-j} = q_{i-j} A_i = A_i F_{i-j}(E_{bi} - E_{bj}) = -Q_{j-i}. \tag{5.9}$$

**Example 5.1.** Consider a very long duct as shown in Fig. 5-2. The duct is 30 cm × 40 cm in cross-section, and all surfaces are black. The top and bottom walls are at temperature $T_1 = 1000\,\text{K}$, while the side walls are at temperature $T_2 = 600\,\text{K}$. Determine the net radiative heat transfer rate (per unit duct length) on each surface.

***Solution***

We may use either equation (5.7) or (5.8). We shall use the latter here since it takes better advantage of the symmetry of the problem (i.e., it uses the fact that the net radiative exchange between two surfaces at the same temperature must be zero). Thus, with no external irradiation, and using symmetry (e.g., $E_{b1} = E_{b3}$, $F_{1-2} = F_{1-4}$, etc.),

$$\begin{aligned} q_1 &= F_{1-2}(E_{b1} - E_{b2}) + F_{1-3}(E_{b1} - E_{b3}) + F_{1-4}(E_{b1} - E_{b4}) \\ &= 2F_{1-2}(E_{b1} - E_{b2}) = q_3, \\ q_2 &= q_4 = 2F_{2-1}(E_{b2} - E_{b1}). \end{aligned}$$

Only the view factors $F_{1-2}$ and $F_{2-1}$ are required, which are readily determined from the crossed-strings method as

$$\begin{aligned} F_{1-2} &= \frac{30 + 40 - (\sqrt{30^2 + 40^2} + 0)}{2 \times 40} = \frac{1}{4}, \\ F_{2-1} &= \frac{A_1}{A_2} F_{1-2} = \frac{40}{30} \times \frac{1}{4} = \frac{1}{3}. \end{aligned}$$

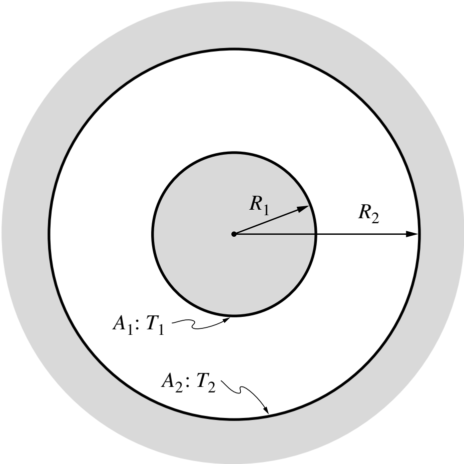



**FIGURE 5-3**
Concentric black spheres for Example 5.2.

Therefore (using a prime to indicate "per unit duct length"),

$$\begin{aligned} Q_1' &= Q_3' = 2A_1'F_{1-2}\sigma(T_1^4 - T_2^4) \\ &= 2\times 0.4\,\text{m}\times 0.25\times 5.670\times 10^{-8}\,\frac{\text{W}}{\text{m}^2\,\text{K}^4}(1000^4 - 600^4)\,\text{K}^4 = 9870\,\text{W/m} \\ Q_2' &= Q_4' = 2A_2'F_{2-1}\sigma(T_2^4 - T_1^4) = -9870\,\text{W/m} \end{aligned}$$

It is apparent from this example that the sum of all surface heat transfer rates must vanish. This follows immediately from *conservation of energy*: The total heat transfer rate into the enclosure (i.e., the heat transfer rates summed over all surfaces) must be equal to the rate of change of radiative energy within the enclosure. Since radiation travels at the speed of light, steady state is reached almost instantaneously, so that the rate of change of radiative energy may nearly always be neglected. Mathematically, we may multiply equation (5.7) by $A_i$ and sum over all areas:

$$\sum_{i=1}^{N}(Q_i + A_iH_{oi}) = \sum_{i=1}^{N} A_iE_{bi} - \sum_{i=1}^{N} A_i \sum_{j=1}^{N} E_{bj}F_{i-j} = \sum_{i=1}^{N} A_iE_{bi} - \sum_{j=1}^{N} A_jE_{bj} \sum_{i=1}^{N} F_{j-i} = 0. \qquad (5.10)$$

This relationship is most useful to check the correctness of one's calculations, or their accuracy (for computer calculations).

**Example 5.2.** Consider two concentric, isothermal, black spheres with radii $R_1$ and $R_2$, and temperatures $T_1$ and $T_2$, respectively, as shown in Fig. 5-3. Show how the temperature of the inner sphere can be deduced, if temperature and heat flux of the outer sphere are measured.

***Solution***
We have only two surfaces, and equation (5.8) becomes

$$q_1 = F_{1-2}(E_{b1} - E_{b2}); \quad q_2 = F_{2-1}(E_{b2} - E_{b1}).$$

Since all radiation from Sphere 1 travels to 2, we have $F_{1-2} = 1$ and, by reciprocity, $F_{2-1} = A_1/A_2$. Thus,

$$Q_1 = -Q_2 = A_1\sigma(T_1^4 - T_2^4).$$

Solving this for $T_1$ we get, with $A_i = 4\pi R_i^2$,

$$T_1^4 = T_2^4 - \left(\frac{R_2}{R_1}\right)^2 \frac{q_2}{\sigma}.$$

Whenever $T_1$ is larger than $T_2$, $q_2$ is negative, and vice versa.

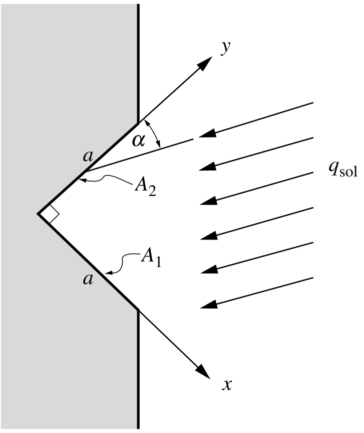



**FIGURE 5-4**
Right-angled groove exposed to solar irradiation, Example 5.3.

**Example 5.3.** A right-angled groove, consisting of two long black surfaces of width $a$, is exposed to solar radiation $q_{sol}$ (Fig. 5-4). The entire groove surface is kept isothermal at temperature $T$. Determine the net radiative heat transfer rate from the groove.

***Solution***
Again, we may employ either equation (5.7) or (5.8). However, this time the enclosure is not closed; and we must close it artificially. We note that any radiation leaving the cavity will not come back (barring any reflection from other surfaces nearby). Thus, our artificial surface should be black. We also assume that, with the exception of the (parallel) solar irradiation, no external radiation enters the cavity. Since the solar irradiation is best treated separately through the external irradiation term $H_o$, our artificial surface is nonemitting. Both criteria are satisfied by covering the groove with a black surface at 0 K. Even though we now have three surfaces, the last one does not really appear in equation (5.7) (since $E_{b3} = 0$), but it does appear in equation (5.8). Using equation (5.7) we find

$$\begin{aligned} q_1 &= E_{b1} - F_{1-2}E_{b2} - H_{o1} = \sigma T^4(1 - F_{1-2}) - q_{sol}\cos\alpha, \\ q_2 &= E_{b2} - F_{2-1}E_{b1} - H_{o2} = \sigma T^4(1 - F_{2-1}) - q_{sol}\sin\alpha. \end{aligned}$$

From Configuration 33 in Appendix D we find, with $H = 1$,

$$F_{1-2} = \tfrac{1}{2}\left(2 - \sqrt{2}\right) = 0.293 = F_{2-1},$$

and

$$Q' = a(q_1 + q_2) = a\left[\sqrt{2}\sigma T^4 - q_{sol}(\cos\alpha + \sin\alpha)\right].$$

These examples demonstrate that equation (5.8) is generally more convenient to use for closed configurations, since it takes advantage of the fact that the net exchange between two surfaces at the same temperature (or with itself) is zero. Equation (5.7), on the other hand, is more convenient for open configurations, since the hypothetical surfaces employed to close the configuration do not contribute (because of their zero emissive power): With this equation the hypothetical closing surfaces may be completely ignored!

Equation (5.7) may be written in a third form that is most convenient for computer calculations. Using *Kronecker's delta function,* defined as

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{5.11}$$

we find $\sum_{j=1}^{N} \delta_{ij} = 1$ and $\sum_{j=1}^{N} E_{bj}\delta_{ij} = E_{bi}$. Thus,

$$q_i = \sum_{j=1}^{N} (\delta_{ij} - F_{i-j})E_{bj} - H_{oi}, \quad i = 1, 2, \ldots, N. \tag{5.12}$$

Let us suppose that for surfaces $i = 1, 2, \ldots, n$ the heat fluxes are prescribed (and temperatures are unknown), while for surfaces $i = n+1, \ldots, N$ the temperatures are prescribed (heat fluxes unknown). Unlike for the heat fluxes, no explicit relations for the unknown temperatures exist. Placing all unknown temperatures on one side of equation (5.12), we may write

$$\sum_{j=1}^{n}(\delta_{ij} - F_{i-j})E_{bj} = q_i + H_{oi} + \sum_{j=n+1}^{N} F_{i-j}E_{bj}, \quad i = 1, 2, \ldots, n, \tag{5.13}$$

where everything on the right-hand side of the equation is known. In matrix form this is written[2] as

$$\mathbf{A} \cdot \mathbf{e_b} = \mathbf{b}, \tag{5.14}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 - F_{1-1} & -F_{1-2} & \cdots & -F_{1-n} \\ -F_{2-1} & 1 - F_{2-2} & \cdots & -F_{2-n} \\ \vdots & & \ddots & \vdots \\ -F_{n-1} & -F_{n-2} & \cdots & 1 - F_{n-n} \end{pmatrix}, \tag{5.15}$$

$$\mathbf{e_b} = \begin{pmatrix} E_{b1} \\ E_{b2} \\ \vdots \\ E_{bn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} q_1 + H_{o1} + \sum_{j=n+1}^{N} F_{1-j}\, E_{bj} \\ q_2 + H_{o2} + \sum_{j=n+1}^{N} F_{2-j}\, E_{bj} \\ \vdots \\ q_n + H_{on} + \sum_{j=n+1}^{N} F_{n-j}\, E_{bj} \end{pmatrix}. \tag{5.16}$$

The $n \times n$ matrix $\mathbf{A}$ is readily inverted on a computer (generally with the aid of a software library subroutine), and the unknown temperatures are calculated as

$$\mathbf{e_b} = \mathbf{A}^{-1} \cdot \mathbf{b}. \tag{5.17}$$

## 5.3 RADIATIVE EXCHANGE BETWEEN GRAY, DIFFUSE SURFACES

We shall now assume that all surfaces are gray, that they are diffuse emitters, absorbers, and reflectors. Under these conditions $\epsilon = \epsilon'_\lambda = \alpha'_\lambda = \alpha = 1 - \rho$. The total heat flux leaving a surface at location $\mathbf{r}$ is, from Fig. 4-1,

$$J(\mathbf{r}) = \epsilon(\mathbf{r})E_b(\mathbf{r}) + \rho(\mathbf{r})H(\mathbf{r}), \tag{5.18}$$

which is called the *surface radiosity* $J$ at location $\mathbf{r}$. Since both emission and reflection are diffuse, so is the resulting intensity leaving the surface:

$$I(\mathbf{r}, \hat{\mathbf{s}}) = I(\mathbf{r}) = J(\mathbf{r})/\pi. \tag{5.19}$$

Therefore, an observer at a different location is unable to distinguish emitted and reflected

[2]For easy readability of matrix manipulations we shall follow here the convention that a two-dimensional matrix is denoted by a bold capitalized letter, while a vector is written as a bold lowercase letter.

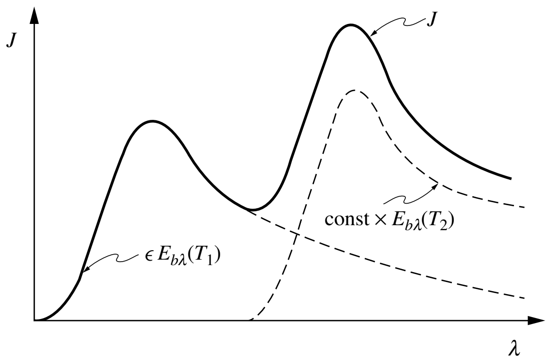



**FIGURE 5-5**
Qualitative spectral behavior of radiosity for irradiation from an isothermal source.

radiation on the basis of *directional* behavior. However, the observer may be able to distinguish the two as a result of their different *spectral* behavior. Consider Example 5.2 for the case of a black outer sphere but a gray, diffuse inner sphere. On the inner sphere the emitted radiation has the spectral distribution of a blackbody at temperature $T_1$, while the reflected radiation—which was originally emitted at the outer sphere—has the spectral distribution of a blackbody at temperature $T_2$. Thus, the spectral radiosity will behave as shown qualitatively in Fig. 5-5. An observer will be able to distinguish between emitted and reflected radiation if he has the ability to distinguish between radiation at different wavelengths. A gray surface does not have this ability, since it behaves in the same fashion toward *all* incoming radiation at *any* wavelength, i.e., it is "color blind." Consequently, a gray surface does not "know" whether its irradiation comes from a gray, diffuse surface or from a black surface with an effective emissive power $J$. This fact simplifies the analysis considerably since it allows us to calculate radiative heat transfer rates between surfaces by balancing the net outgoing radiation (i.e., emission and reflection) traveling directly from surface to surface (as opposed to emitted radiation traveling to another surface directly or after any number of reflections). For this reason the following analysis is often referred to as the *net radiation method.*

Making an energy balance on a surface $dA$ in the enclosure shown in Fig. 5-6 we obtain from equation (4.2)

$$q(\mathbf{r}) = \epsilon(\mathbf{r})E_b(\mathbf{r}) - \alpha(\mathbf{r})H(\mathbf{r}) = J(\mathbf{r}) - H(\mathbf{r}). \tag{5.20}$$

The irradiation $H(\mathbf{r})$ is again found by determining the contribution from a differential area $dA'(\mathbf{r}')$, followed by integrating over the entire surface. From the definition of the view factor the heat transfer rate leaving $dA'$ intercepted by $dA$ is $(J(\mathbf{r}')\,dA')\,dF_{dA'-dA}$. Thus, similar to the black-surfaces case,

$$H(\mathbf{r})\,dA = \int_A J(\mathbf{r}')\,dF_{dA'-dA}\,dA' + H_o(\mathbf{r})\,dA, \tag{5.21}$$

where $H_o(\mathbf{r})$ is again any external radiation arriving at $dA$. Using reciprocity this equation reduces to

$$H(\mathbf{r}) = \int_A J(\mathbf{r}')\,dF_{dA-dA'} + H_o(\mathbf{r}). \tag{5.22}$$

Substitution into equation (5.20) yields

$$q(\mathbf{r}) = \epsilon(\mathbf{r})E_b(\mathbf{r}) - \alpha(\mathbf{r})\left[\int_A J(\mathbf{r}')\,dF_{dA-dA'} + H_o(\mathbf{r})\right]. \tag{5.23}$$

Thus, the unknown heat flux (or temperature) could be calculated if the radiosity field had been known. A governing integral equation for radiosity is readily established by solving

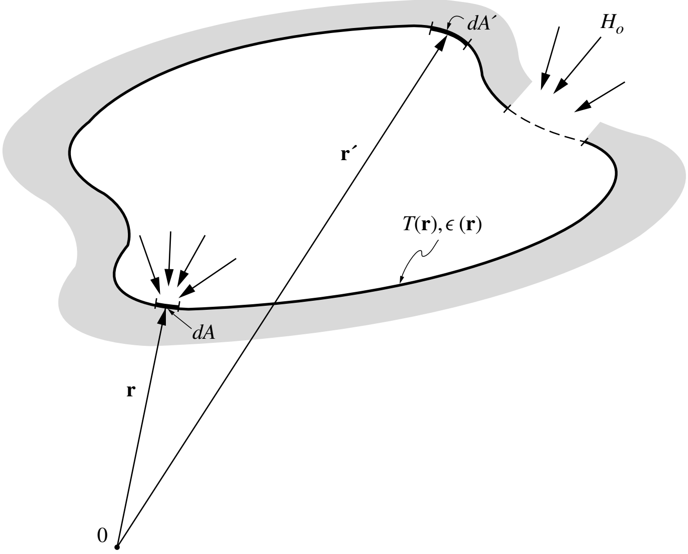



**FIGURE 5-6**
Radiative exchange in a gray, diffuse enclosure.

equation (5.20) for $J$:

$$J(\mathbf{r}) = \epsilon(\mathbf{r})E_b(\mathbf{r}) + \rho(\mathbf{r})\left[\int_A J(\mathbf{r}')\,dF_{dA-dA'} + H_o(\mathbf{r})\right], \tag{5.24}$$

for those surface locations where the temperature is known, or

$$J(\mathbf{r}) = q(\mathbf{r}) + \int_A J(\mathbf{r}')\,dF_{dA-dA'} + H_o(\mathbf{r}), \tag{5.25}$$

for those parts of the surface where the local heat flux is specified. However, in problems without participating media there is rarely a need to determine radiosity, and it is usually best to eliminate radiosity from equation (5.23). Expressing radiosity in terms of local temperature and heat flux and eliminating irradiation $H$ from equation (5.20) we have

$$q - \alpha q = (\epsilon E_b - \alpha H) - \alpha(J - H) = \epsilon E_b - \alpha J.$$

Up to this point we have differentiated between emittance and absorptance, to keep the relations as general as possible (i.e., to accommodate nongray surface properties if necessary). We shall now invoke the assumption of gray, diffuse surfaces, or $\alpha = \epsilon$. Then

$$q(\mathbf{r}) = \frac{\epsilon(\mathbf{r})}{1-\epsilon(\mathbf{r})}[E_b(\mathbf{r}) - J(\mathbf{r})]. \tag{5.26}$$

Solving for radiosity, we get

$$J(\mathbf{r}) = E_b(\mathbf{r}) - \left(\frac{1}{\epsilon(\mathbf{r})} - 1\right)q(\mathbf{r}). \tag{5.27}$$

Substituting this into equation (5.23), we obtain an integral equation relating temperature $T$ and heat flux $q$:

$$\frac{q(\mathbf{r})}{\epsilon(\mathbf{r})} - \int_A \left(\frac{1}{\epsilon(\mathbf{r}')} - 1\right) q(\mathbf{r}')\,dF_{dA-dA'} + H_o(\mathbf{r}) = E_b(\mathbf{r}) - \int_A E_b(\mathbf{r}')\,dF_{dA-dA'}. \tag{5.28}$$

Note that equation (5.28) reduces to equation (5.4) for a black enclosure. However, for a black enclosure with known temperature field the local heat flux can be determined with a simple integration over emissive power. For a gray enclosure an *integral equation* must be solved, i.e., an equation where the unknown dependent variable $q(\mathbf{r})$ appears inside an integral. This requirement makes the solution considerably more difficult.

As for a black enclosure it is customary to break up a gray enclosure into $N$ subsurfaces, over each of which the *radiosity* is assumed constant. Then equation (5.23) becomes

$$\frac{q_i(\mathbf{r}_i)}{\epsilon_i(\mathbf{r}_i)} = E_{bi}(\mathbf{r}_i) - \sum_{j=1}^{N} J_j F_{di-j}(\mathbf{r}_i) - H_{oi}(\mathbf{r}_i), \quad i = 1, 2, \ldots, N, \tag{5.29}$$

and, taking an average over subsurface $A_i$,

$$\frac{q_i}{\epsilon_i} = E_{bi} - \sum_{j=1}^{N} J_j F_{i-j} - H_{oi}, \quad i = 1, 2, \ldots, N. \tag{5.30}$$

Taking a similar average for equation (5.26) gives

$$q_i = \frac{\epsilon_i}{1-\epsilon_i}\left[E_{bi} - J_i\right]. \tag{5.31}$$

Solving for $J$ and substituting into equation (5.30) then leads to

$$\frac{q_i}{\epsilon_i} - \sum_{j=1}^{N}\left(\frac{1}{\epsilon_j} - 1\right) F_{i-j} q_j + H_{oi} = E_{bi} - \sum_{j=1}^{N} F_{i-j} E_{bj}, \quad i = 1, 2, \ldots, N. \tag{5.32}$$

This relation also follows directly from equation (5.28) if both $(1/\epsilon - 1)q$ and $E_b$ (the components of $J$) are assumed constant across the subsurfaces. Recalling the summation rule, $\sum_{j=1}^{N} F_{i-j} = 1$, we may also write equation (5.32) as an interchange between surfaces,

$$\frac{q_i}{\epsilon_i} - \sum_{j=1}^{N}\left(\frac{1}{\epsilon_j} - 1\right) F_{i-j}\, q_j + H_{oi} = \sum_{j=1}^{N} F_{i-j}(E_{bi} - E_{bj}), \quad i = 1, 2, \ldots, N. \tag{5.33}$$

Either one of these equations, of course, reduces to equation (5.8) for a black enclosure. Equation (5.32) is preferred for open configurations, since it allows one to ignore hypothetical closing surfaces; and equation (5.33) is preferred for closed enclosures, because it eliminates transfer between surfaces at the same temperature.

Sometimes one wishes to determine the radiosity of a surface, for example, in the field of pyrometry (relating surface temperature to radiative intensity leaving a surface). Depending on which of the two is unknown, elimination of $q_i$ or $E_{bi}$ from equation (5.30) with the help of equation (5.31) leads to

$$J_i = \epsilon_i E_{bi} + (1-\epsilon_i)\left(\sum_{j=1}^{N} J_j F_{i-j} + H_{oi}\right) \tag{5.34a}$$

$$= q_i + \sum_{j=1}^{N} J_j F_{i-j} + H_{oi}, \quad i = 1, 2, \ldots, N. \tag{5.34b}$$

These two relations simply repeat the definition of radiosity, the first stating that radiosity consists of emitted and reflected heat fluxes and the second that radiosity, or outgoing heat flux, is equal to net heat flux (with negative $q_{\rm in}$) plus the absolute value of $q_{\rm in}$.

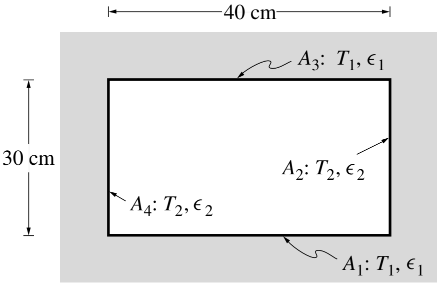



**FIGURE 5-7**
Two-dimensional gray, diffuse duct for Example 5.4.

**Example 5.4.** Reconsider Example 5.1 for a gray, diffuse surface material. Top and bottom walls are at $T_1 = T_3 = 1000\,\mathrm{K}$ with $\epsilon_1 = \epsilon_3 = 0.3$, while the side walls are at $T_2 = T_4 = 600\,\mathrm{K}$ with $\epsilon_2 = \epsilon_4 = 0.8$ as shown in Fig. 5-7. Determine the net radiative heat transfer rates for each surface.

***Solution***
Using equation (5.33) for $i = 1$ and $i = 2$, and recalling that $F_{1-2} = F_{1-4}$ and $F_{2-1} = F_{2-3}$,

$$i = 1: \quad \frac{q_1}{\epsilon_1} - 2\left(\frac{1}{\epsilon_2} - 1\right) F_{1-2}\, q_2 - \left(\frac{1}{\epsilon_1} - 1\right) F_{1-3}\, q_1 = 2F_{1-2}(E_{b1} - E_{b2}),$$
$$i = 2: \quad \frac{q_2}{\epsilon_2} - 2\left(\frac{1}{\epsilon_1} - 1\right) F_{2-1}\, q_1 - \left(\frac{1}{\epsilon_2} - 1\right) F_{2-4}\, q_2 = 2F_{2-1}(E_{b2} - E_{b1}).$$

We have already evaluated $F_{1-2} = \frac{1}{4}$ and $F_{2-1} = \frac{1}{3}$ in Example 5.1. From the summation rule $F_{1-3} = 1 - 2F_{1-2} = \frac{1}{2}$ and $F_{2-4} = 1 - 2F_{2-1} = \frac{1}{3}$. Substituting these, as well as emittance values, into the relations reduces them to the simpler form of

$$\left[\frac{1}{0.3} - \left(\frac{1}{0.3} - 1\right)\frac{1}{2}\right] q_1 - 2\left(\frac{1}{0.8} - 1\right)\frac{1}{4} q_2 = 2 \times \tfrac{1}{4}(E_{b1} - E_{b2}),$$
$$-2\left(\frac{1}{0.3} - 1\right)\frac{1}{3} q_1 + \left[\frac{1}{0.8} - \left(\frac{1}{0.8} - 1\right)\frac{1}{3}\right] q_2 = 2 \times \tfrac{1}{3}(E_{b2} - E_{b1}),$$

or

$$\frac{13}{6} q_1 - \frac{1}{8} q_2 = \frac{1}{2}(E_{b1} - E_{b2}),$$
$$-\frac{14}{9} q_1 + \frac{7}{6} q_2 = -\frac{2}{3}(E_{b1} - E_{b2}).$$

Thus,

$$\left(\frac{13}{6} \times \frac{7}{6} - \frac{14}{9} \times \frac{1}{8}\right) q_1 = \left(\frac{1}{2} \times \frac{7}{6} - \frac{2}{3} \times \frac{1}{8}\right)(E_{b1} - E_{b2}),$$
$$q_1 = \frac{3}{7} \times \frac{1}{2}(E_{b1} - E_{b2}) = \frac{3}{14}\sigma(T_1^4 - T_2^4),$$

and

$$\left(-\frac{1}{8} \times \frac{14}{9} + \frac{7}{6} \times \frac{13}{6}\right) q_2 = \left(\frac{1}{2} \times \frac{14}{9} - \frac{2}{3} \times \frac{13}{6}\right)(E_{b1} - E_{b2}),$$
$$q_2 = -\frac{3}{7} \times \frac{2}{3}(E_{b1} - E_{b2}) = -\frac{2}{7}\sigma(T_1^4 - T_2^4).$$

Finally, substituting values for temperatures,

$$Q_1' = 0.4\,\mathrm{m} \times \tfrac{3}{14} \times 5.670 \times 10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2\,K^4}}(1000^4 - 600^4)\,\mathrm{K}^4 = 4230\,\mathrm{W/m},$$
$$Q_2' = -0.3\,\mathrm{m} \times \tfrac{2}{7} \times 5.670 \times 10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2\,K^4}}(1000^4 - 600^4)\,\mathrm{K}^4 = -4230\,\mathrm{W/m}.$$

Of course, both heat transfer rates must again add up to zero. We observe that these rates are less than half the ones for the black duct.

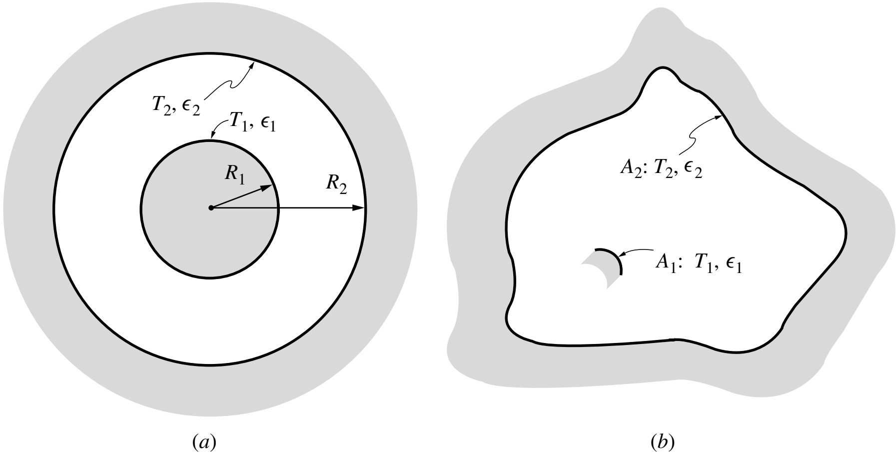



**FIGURE 5-8**
Radiative transfer between (*a*) two concentric spheres, (*b*) a convex surface and a large isothermal enclosure.

**Example 5.5.** Determine the radiative heat flux between two isothermal gray concentric spheres with radii $R_1$ and $R_2$, temperatures $T_1$ and $T_2$, and emittances $\epsilon_1$ and $\epsilon_2$, respectively, as shown in Fig. 5-8*a*.

***Solution***
Again applying equation (5.33) for $i = 1$ (inner sphere) and $i = 2$ (outer sphere), we obtain:

$$i = 1: \quad \frac{q_1}{\epsilon_1} - \left(\frac{1}{\epsilon_1} - 1\right) F_{1-1}\, q_1 - \left(\frac{1}{\epsilon_2} - 1\right) F_{1-2}\, q_2 = F_{1-2}(E_{b1} - E_{b2}),$$
$$i = 2: \quad \frac{q_2}{\epsilon_2} - \left(\frac{1}{\epsilon_1} - 1\right) F_{2-1}\, q_1 - \left(\frac{1}{\epsilon_2} - 1\right) F_{2-2}\, q_2 = F_{2-1}(E_{b2} - E_{b1}).$$

With $F_{1-1} = 0$, $F_{1-2} = 1$, $F_{2-1} = A_1/A_2$, and $F_{2-2} = 1 - F_{2-1} = 1 - A_1/A_2$, these two equations reduce to

$$\frac{1}{\epsilon_1}\, q_1 - \left(\frac{1}{\epsilon_2} - 1\right) q_2 = \sigma(T_1^4 - T_2^4),$$

$$\left(\frac{1}{\epsilon_1} - 1\right) \frac{A_1}{A_2}\, q_1 + \left[\frac{1}{\epsilon_2} - \left(\frac{1}{\epsilon_2} - 1\right)\left(1 - \frac{A_1}{A_2}\right)\right] q_2 = -\frac{A_1}{A_2}\sigma(T_1^4 - T_2^4).$$

This may be solved for $q_1$ by eliminating $q_2$ (or using conservation of energy, i.e., $A_1q_1 + A_2q_2 = 0$), or

$$q_1 = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\epsilon_1} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\epsilon_2} - 1\right)}. \tag{5.35}$$

We note that equation (5.35) is not just limited to concentric spheres, but holds for any convex surface $A_1$ (i.e., with $F_{1-1} = 0$) that radiates only to $A_2$ (i.e., $F_{1-2} = 1$) as indicated in Fig. 5-8*b*. This is often convenient for a convex surface $A_i$ placed into a large, isothermal environment ($A_a \gg A_i$) at temperature $T_a$, leading to

$$q_i = \epsilon_i \sigma (T_i^4 - T_a^4). \tag{5.36}$$

Surface $A_i$ may also be a hypothetical one, closing an open configuration contained within a large environment.

**Example 5.6.** Repeat Example 5.3 for a groove whose surface is gray and diffuse, with emittance $\epsilon$, rather than black.

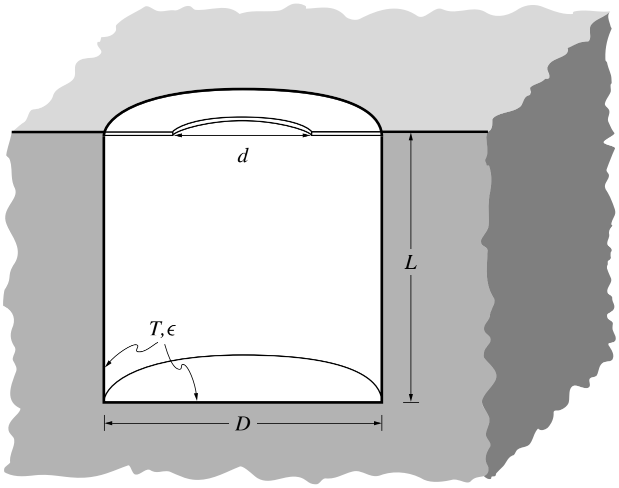



**FIGURE 5-9**
Cylindrical cavity with partial cover plate, Example 5.7.

***Solution***
Using equation (5.32) for the open configuration we obtain

$$i = 1: \quad \frac{q_1}{\epsilon} - \left(\frac{1}{\epsilon} - 1\right) F_{1-2}\, q_2 + H_{o1} = \sigma T^4 (1 - F_{1-2}),$$
$$i = 2: \quad \frac{q_2}{\epsilon} - \left(\frac{1}{\epsilon} - 1\right) F_{2-1}\, q_1 + H_{o2} = \sigma T^4 (1 - F_{2-1}),$$

where we have made use of the fact that $E_{b1} = E_{b2} = \sigma T^4$ and $\epsilon_1 = \epsilon_2 = \epsilon$. As in Example 5.3 we have $F_{1-2} = F_{2-1} = 1 - \sqrt{2}/2$ and $H_{o1} = q_{\rm sol} \cos\alpha$, $H_{o2} = q_{\rm sol} \sin\alpha$. Since we are only interested in the total heat loss we add the two equations, leading to

$$\left[\frac{1}{\epsilon} - \left(\frac{1}{\epsilon} - 1\right) F_{1-2}\right] (q_1 + q_2) = \sqrt{2}\sigma T^4 - q_{\rm sol}(\cos\alpha + \sin\alpha),$$

and

$$Q' = a(q_1 + q_2) = \frac{a\left[\sqrt{2}\sigma T^4 - q_{\rm sol}(\cos\alpha + \sin\alpha)\right]}{1 + \left(\dfrac{1}{\epsilon} - 1\right)\Big/ \sqrt{2}}.$$

Comparing this result with that of Example 5.3, we see that the heat loss due to emission is decreased (less emission, but more effective heat loss of emitted energy due to reflection from the opposing surface), as is the solar heat gain (since some of the irradiation is reflected back out of the cavity).

**Example 5.7.** Consider the cavity shown in Fig. 5-9, which consists of a cylindrical hole of diameter $D$ and length $L$. The top of the cavity is covered with a disk, which has a hole of diameter $d$. The entire inside of the cavity is isothermal at temperature $T$, and is covered with a gray, diffuse material of emittance $\epsilon$. Determine the amount of radiation escaping from the cavity.

***Solution***
For simplicity, since the entire surface is isothermal and has the same emittance, we use a single zone $A_1$, which comprises the entire groove surface (sides, bottom, and top). Therefore, equation (5.32) reduces to

$$\left[\frac{1}{\epsilon_1} - \left(\frac{1}{\epsilon_1} - 1\right) F_{1-1}\right] q_1 = (1 - F_{1-1}) E_{b1}.$$

Since the total radiative energy rate leaving the cavity is $Q_1 = A_1 q_1$, we get

$$Q_1 = \frac{1 - F_{1-1}}{\dfrac{1}{\epsilon_1} - \left(\dfrac{1}{\epsilon_1} - 1\right) F_{1-1}} A_1 E_{b1}.$$

The view factor $F_{1-1}$ is easily determined by recognizing that $F_{o-1} = 1$ (and $A_o$ is the opening at the top) and, by reciprocity,

$$F_{1-1} = 1 - F_{1-o} = 1 - \frac{A_o}{A_1} F_{o-1} = 1 - \frac{A_o}{A_1}.$$

Therefore, the radiative heat flux leaving the cavity, per unit area of opening, is

$$\frac{Q_1}{A_o} = \frac{\left(1 - 1 + \frac{A_o}{A_1}\right) \frac{A_1}{A_o} E_{b1}}{\frac{1}{\epsilon_1} - \left(\frac{1}{\epsilon_1} - 1\right)\left(1 - \frac{A_o}{A_1}\right)} = \frac{E_{b1}}{1 + \left(\frac{1}{\epsilon_1} - 1\right) \frac{A_o}{A_1}}.$$

Thus, if $A_o/A_1 \ll 1$, the opening of the cavity behaves like a blackbody with emissive power $E_{b1}$. Such cavities are commonly used in experimental methods in which blackbodies are needed for comparison. For example, a cavity with $d/D = 1/2$ and $L/D = 2$ has

$$\begin{aligned} \frac{A_o}{A_1} &= \frac{\pi d^2/4}{2\pi D^2/4 - \pi d^2/4 + \pi DL} = \frac{d^2}{2D^2 - d^2 + 4DL} \\ &= \frac{(d/D)^2}{2 - (d/D)^2 + 4(L/D)} = \frac{1/4}{2 - 1/4 + 4\times 2} = \frac{1}{39}. \end{aligned}$$

For $\epsilon_1 = 0.5$ this results in an *apparent emittance* of

$$\epsilon_a = \frac{Q_1}{A_o E_{b1}} = \frac{1}{1 + \left(\frac{1}{\epsilon_1} - 1\right) \frac{A_o}{A_1}} = \frac{1}{1 + \left(\frac{1}{0.5} - 1\right) \frac{1}{39}} = \frac{39}{40} = 0.975.$$

For computer calculations the Kronecker delta is introduced into equation (5.32), as was done for a black enclosure, leading to

$$\sum_{j=1}^{N} \left[\frac{\delta_{ij}}{\epsilon_j} - \left(\frac{1}{\epsilon_j} - 1\right) F_{i-j}\right] q_j = \sum_{j=1}^{N} \left[\delta_{ij} - F_{i-j}\right] E_{bj} - H_{oi}. \tag{5.37}$$

If all the temperatures are known and the radiative heat fluxes are to be determined, equation (5.37) may be cast in matrix form as

$$\mathbf{C} \cdot \mathbf{q} = \mathbf{A} \cdot \mathbf{e_b} - \mathbf{h_o}, \tag{5.38}$$

where $\mathbf{C}$ and $\mathbf{A}$ are matrices with elements

$$\begin{aligned} C_{ij} &= \frac{\delta_{ij}}{\epsilon_j} - \left(\frac{1}{\epsilon_j} - 1\right) F_{i-j}, \\ A_{ij} &= \delta_{ij} - F_{i-j}, \end{aligned}$$

and $\mathbf{q}$, $\mathbf{e_b}$, and $\mathbf{h_o}$ are vectors of the unknown heat fluxes $q_j$ and the known emissive powers $E_{bj}$ and external irradiations $H_{oj}$. Equation (5.38) is solved by matrix inversion as

$$\mathbf{q} = \mathbf{C}^{-1} \cdot [\mathbf{A} \cdot \mathbf{e_b} - \mathbf{h_o}]. \tag{5.39}$$

If the emissive power is known over only some of the surfaces, and the heat fluxes are specified elsewhere, equation (5.38) may be rearranged into a similar equation for the vector containing all the unknowns. Subroutine `graydiff` is provided in Appendix F for the solution of the simultaneous equations (5.38), requiring surface information and a partial view factor matrix as input. The solution to a three-dimensional version of Example 5.4 is also given in the form of a program `graydiffxch`, which may be used as a starting point for the solution to other problems. Fortran90, C++ as well as MATLAB® versions are provided. Several commercial solvers are also available, usually including software for view factor evaluation, such as TRASYS [1] and TSS [2].

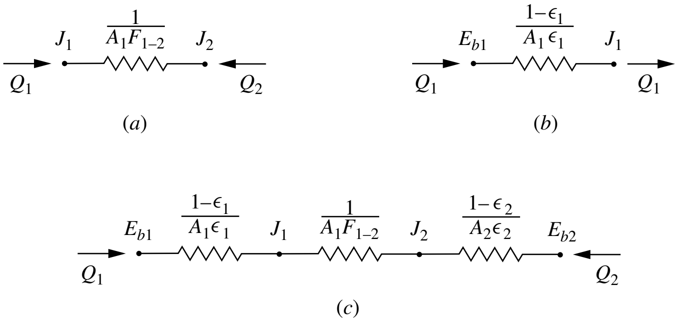



**FIGURE 5-10**
Electrical network analogy for infinite parallel plates: (*a*) space resistance, (*b*) surface resistance, (*c*) total resistance.

## 5.4 ELECTRICAL NETWORK ANALOGY

While equation (5.37) represents the most convenient set of governing equations for numerical calculations on today's digital computers, some people prefer to get a physical feeling for the radiative exchange problem by representing it through an analogous electrical network, a method more suitable for analog computers—now nearly extinct. For completeness, we shall briefly present this electrical network method, which was first introduced by Oppenheim [3].

From equation (5.20) we have

$$q_i = J_i - H_i, \quad i = 1, 2, \ldots, N, \tag{5.40}$$

or, with equations (5.30) and (5.31),

$$q_i = J_i - \sum_{j=1}^{N} J_j F_{i-j} - H_{oi}, \tag{5.41}$$

$$= \sum_{j=1}^{N} (J_i - J_j) F_{i-j} - H_{oi}, \quad i = 1, 2, \ldots, N. \tag{5.42}$$

We shall first consider the simple case of two infinite parallel plates without external irradiation. Thus, $N = 2$, $H_{oi} = 0$, and

$$Q_1 = A_1 q_1 = \frac{J_1 - J_2}{\dfrac{1}{A_1 F_{1-2}}} = -Q_2. \tag{5.43}$$

As written, equation (5.43) may be interpreted as follows: If the radiosities are considered *potentials*, $1/A_1F_{1-2}$ is a *radiative resistance between surfaces*, or a *space resistance*, and $Q$ is a radiative heat flow "current," then equation (5.43) is identical to the one governing an electrical current flowing across a resistor due to a voltage potential, as indicated in Fig. 5-10*a*. The space resistance is a measure of how easily a radiative heat flux flows from one surface to another: The larger $F_{1-2}$, the more easily heat can travel from $A_1$ to $A_2$, resulting in a smaller resistance. The same heat flux is also given by equation (5.31) as

$$Q_1 = \frac{E_{b1} - J_1}{\dfrac{1 - \epsilon_1}{A_1 \epsilon_1}} = \frac{J_2 - E_{b2}}{\dfrac{1 - \epsilon_2}{A_2 \epsilon_2}} = -Q_2, \tag{5.44}$$

where $(1 - \epsilon_i)/A_i\epsilon_i$ are *radiative surface resistances.* This situation is shown in Fig. 5-10*b*. The surface resistance describes a surface's ability to radiate. For the maximum radiator, a black surface, the resistance is zero. This fact implies that, for a finite heat flux, the potential drop

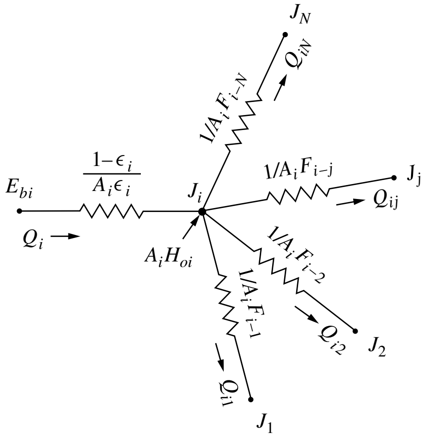



**FIGURE 5-11**
Network representation for radiative heat flux between surface $A_i$ and all other surfaces.

across a zero resistance must be zero, i.e., $J_i = E_{bi}$. Of course, the radiosities may be eliminated from equations (5.43) and (5.44), and

$$Q_1 = \frac{E_{b1} - E_{b2}}{\dfrac{1-\epsilon_1}{A_1\epsilon_1} + \dfrac{1}{A_1F_{1-2}} + \dfrac{1-\epsilon_2}{A_2\epsilon_2}} = -Q_2, \tag{5.45}$$

where the denominator is the *total radiative resistance* between surfaces $A_1$ and $A_2$. Since the three resistances are in series they simply add up as electrical resistances do; see Fig. 5-10*c*.

This network analogy is readily extended to more complicated situations by rewriting equation (5.42) as

$$Q_i = \frac{E_{bi} - J_i}{\dfrac{1-\epsilon_i}{A_i\epsilon_i}} = \sum_{j=1}^{N} \frac{J_i - J_j}{\dfrac{1}{A_iF_{i-j}}} - A_iH_{oi} = \sum_{j=1}^{N} Q_{i-j} - A_iH_{oi}. \tag{5.46}$$

Thus, the total heat flux at surface $i$ is the net radiative exchange between $A_i$ and all the other surfaces in the enclosure. The electrical analog is shown in Fig. 5-11, where the current flowing from $E_{bi}$ to $J_i$ is divided into $N$ parallel lines, each with a different potential difference and with different resistors.

**Example 5.8.** Consider a solar collector shown in Fig. 5-12*a*. The collector consists of a glass cover plate, a collector plate, and side walls. We shall assume that the glass is totally transparent to solar irradiation, which penetrates through the glass and hits the absorber plate with a strength of 1000 W/m$^2$. The absorber plate is black and is kept at a constant temperature $T_1 = 77°$C by heating water flowing underneath it. The side walls are insulated and made of a material with emittance $\epsilon_2 = 0.5$. The glass cover may be considered opaque to thermal (i.e., infrared) radiation with an emittance $\epsilon_3 = 0.9$. The collector is 1 m × 1 m × 10 cm in dimension and is reasonably evacuated to suppress free convection between absorber plate and glass cover. The convective heat transfer coefficient at the top of the glass cover is known to be $h = 5.0$ W/m$^2$ K, and the temperature of the ambient is $T_a = 17°$C. Estimate the collected energy for normal solar incidence.

***Solution***
We may construct an equivalent network (Fig. 5-12*b*), leading to

$$Q_1 = \frac{\sigma(T_1^4 - T_a^4)}{R_{13} + \dfrac{1-\epsilon_3}{A_3\epsilon_3} + R_{3a}} - A_1q_s,$$

where $R_{13}$ is the total resistance between surfaces $A_1$ and $A_3$, and $R_{3a}$ is the resistance, by radiation as well as free convection, between glass cover and environment. We note that, since $A_2$ is insulated, there

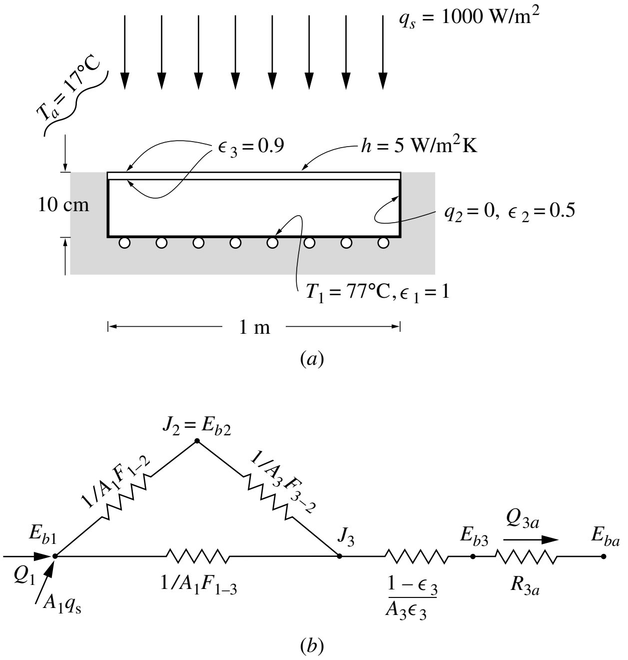



**FIGURE 5-12**
Schematics for Example 5.8: (*a*) geometry, (*b*) network.

is no heat flux entering/leaving at $E_{b2}$ and, from equation (5.44), $J_2 = E_{b2}$. Thus, the total resistance between $A_1$ and $A_3$ comes from two parallel circuits, one with resistance $1/(A_1F_{1-3})$ and the other with two resistances in series, $1/(A_1F_{1-2})$ and $1/(A_3F_{3-2})$, or

$$\frac{1}{R_{13}} = \frac{1}{1/(A_1F_{1-3})} + \frac{1}{1/(A_1F_{1-2}) + 1/(A_3F_{3-2})}$$
$$= A_1F_{1-3} + \tfrac{1}{2}A_1F_{1-2} = A_1\left(F_{1-3} + \tfrac{1}{2}F_{1-2}\right),$$

where we have used the fact that $A_1F_{1-2} = A_3F_{3-2}$ by symmetry. From Configuration 38 in Appendix D we obtain, with $X = Y = 10$, $F_{1-3} = 0.827$ and $F_{1-2} = 1 - F_{1-3} = 0.173$, and

$$R_{13} = 1\Big/\left[1\,\text{m}^2 \times (0.827 + 0.5 \times 0.173)\right] = 1.095\,\text{m}^{-2}.$$

The resistance between glass cover and ambient is a little more complicated. The total heat loss from the cover plate, by free convection and radiation, is

$$Q_{3a} = \epsilon_3 A_3 \sigma(T_3^4 - T_a^4) + hA_3(T_3 - T_a),$$

where we have assumed that the environment (sky) radiates to the collector with the ambient temperature $T_a$. To convert this to the correct form we rewrite it as

$$Q_{3a} = \sigma(T_3^4 - T_a^4)A_3\left[\epsilon_3 + \frac{h(T_3 - T_a)}{\sigma(T_3^4 - T_a^4)}\right],$$

or

$$\frac{1}{R_{3a}} = A_3\left[\epsilon_3 + \frac{h}{\sigma}\frac{T_3 - T_a}{T_3^4 - T_a^4}\right] = A_3\left[\epsilon_3 + \frac{h}{\sigma}\frac{1}{T_3^3 + T_3^2T_a + T_3T_a^2 + T_a^3}\right].$$

As a first approximation, if $T_3$ is not too different from $T_a$,

$$\frac{1}{R_{3a}} \simeq A_3\left(\epsilon_3 + \frac{h}{4\sigma T_a^3}\right) = 1\,\text{m}^2\left(0.9 + \frac{5\,\text{W/m}^2\,\text{K}}{4\times 5.670\times 10^{-8}\,\text{W/m}^2\,\text{K}^4\times(273+17)^3\,\text{K}^3}\right) = \frac{1}{0.554}\text{m}^2.$$

Finally, substituting the resistances into the expression for $Q_1$ we get

$$Q_1 = \frac{5.670\times 10^{-8}\,\text{W/m}^2\,\text{K}^4\left[(273+77)^4 - (273+17)^4\right]\text{K}^4}{1.095\,\text{m}^{-2} + \dfrac{1-0.9}{0.9\,\text{m}^2} + 0.554\,\text{m}^{-2}} - 1\,\text{m}^2 \times 1000\,\text{W/m}^2$$
$$= -744\,\text{W}.$$

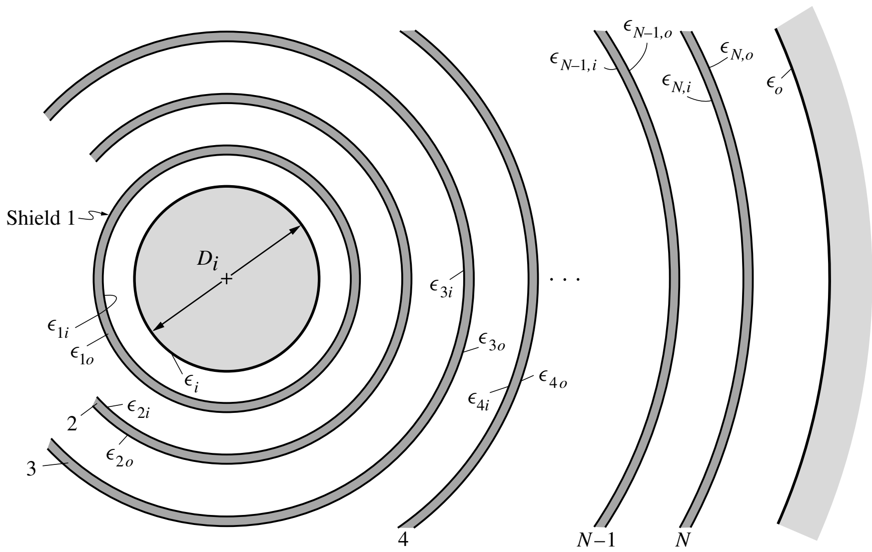



**FIGURE 5-13**
Concentric cylinders (or spheres) with $N$ radiation shields between them.

Since the system could collect a theoretical maximum of −1000 W, the collector efficiency is

$$\eta_{\text{collector}} = \frac{Q_1}{A_1 q_s} = \frac{744}{1000} = 0.744 = 74.4\%.$$

This efficiency should be compared with an uncovered black collector plate, whose net heat flux would be

$$\begin{aligned} Q_1 &= A_1\left[\sigma(T_1^4 - T_a^4) + h(T_1 - T_a) - q_s\right] \\ &= 1\,\text{m}^2\left[5.670\times10^{-8}\times(350^4-290^4)+5\times(350-290) - 1000\right]\,\text{W/m}^2 \\ &= -250\,\text{W}. \end{aligned}$$

Thus, an unprotected collector at that temperature would have an efficiency of only 25%.

The electrical network analogy is a very simple and physically appealing approach for simple two- and three-surface enclosures, such as the one of the previous example. However, in more complicated enclosures with multiple surfaces the method quickly becomes tedious and intractable.

## 5.5 RADIATION SHIELDS

In high-performance insulating materials it is common to suppress conductive and convective heat transfer by evacuating the space between two surfaces. This leaves thermal radiation as the dominant heat loss mode even for low-temperature applications such as insulation in cryogenic storage tanks. The radiation loss may be minimized by placing a multitude of closely spaced, parallel, highly reflective *radiation shields* between the surfaces. The radiation shields are generally made of thin metallic foils or, to reduce conductive losses further, of dielectric foils coated with metallic films. In either case radiation shields tend to be very specular reflectors. However, for closely spaced shields the directional behavior of the reflectance tends to be irrelevant and assuming diffuse reflectances gives excellent accuracy (see also Example 6.9 in the following chapter).

A typical arrangement for $N$ radiation shields between two concentric cylinders (or concentric spheres) is shown in Fig. 5-13. This geometry includes the case of parallel plates for large (and nearly equal) radii. Let the inner cylinder have temperature $T_i$, surface area $A_i$, and emittance $\epsilon_i$. Similarly, each shield has temperature $T_n$ (unknown), $A_n$, $\epsilon_{ni}$ (on its inner surface), and $\epsilon_{no}$ (on its outer surface). The last shield, $A_N$, faces the outer cylinder with $T_o$, $A_o$ and $\epsilon_o$. The net radiative heat rate leaving $A_i$ is, of course, equal to the heat rate going through each shield and to the one arriving at $A_o$. This net heat rate may be readily determined from the electrical network analogy, or by repeated application of the enclosure relations, equation (5.32). However, this is the type of problem for which the network analogy truly shines and we will use this method here. The case of concentric surfaces was already evaluated in Example 5.5, so that the net heat rate between any two of the concentric cylinders is then

$$Q = \frac{E_{bj} - E_{bk}}{R_{j-k}}, \quad R_{j-k} = \frac{1}{\epsilon_j A_j} + \frac{1}{A_k}\left(\frac{1}{\epsilon_k} - 1\right). \tag{5.47}$$

Therefore, we may write

$$QR_{i-1i} = E_{bi} - E_{b1},$$

$$QR_{1o-2i} = E_{b1} - E_{b2},$$

$$\vdots$$

$$QR_{No-o} = E_{bN} - E_{bo}.$$

Adding all these equations eliminates all the unknown shield temperatures, and, after solving for the heat flux, we obtain

$$Q = \frac{E_{bi} - E_{bo}}{R_{i-1i} + \sum_{n=1}^{N-1} R_{no-n+1,i} + R_{No-o}}. \tag{5.48}$$

**Example 5.9.** A Dewar holding 4 liters of liquid helium at 4.2 K consists essentially of two concentric stainless steel ($\epsilon = 0.3$) cylinders of 50 cm length, and inner and outer diameters of $D_i = 10$ cm and $D_o = 20$ cm, respectively. The space between the cylinders is evacuated to a high vacuum to eliminate conductive/convective heat losses. Radiation shields are to be placed between the Dewar walls to reduce radiative losses to the point that it takes 24 hours for the 4-liter filling to evaporate if the Dewar is placed into an environment at 298 K. For the purpose of this example the following may be assumed: (*i*) End losses as well as conduction/convection losses are negligible, (*ii*) the wall temperatures are at $T_i = 4.2$ K and $T_o = 298$ K, respectively, and (*iii*) radiation is one-dimensional. Thin plastic sheets coated on both sides with aluminum ($\epsilon = 0.05$) are available as shield material. Estimate the number of shields required. The heat of evaporation for helium at atmospheric pressure is $h_{fg,\text{He}} = 20.94$ J/g (which is a very low value compared with other liquids), and the liquid density is $\rho_{\text{He}} = 0.125\,\text{g/cm}^3$ [4].

***Solution***
The total heat required to evaporate 4 liters of liquid helium is

$$Q = \rho_{\text{He}} V_{\text{He}} h_{fg,\text{He}} = 0.125\,\frac{\text{g}}{\text{cm}^3} \times 4\,\text{liters} \times \frac{10^3\,\text{cm}^3}{\text{liter}} \times 20.94\,\frac{\text{J}}{\text{g}} = 10.47\,\text{kJ}.$$

If all of this energy is supplied through radial radiation over a time period of 24 hours, one infers that the heat flux in equation (5.48) must be held at or below $\dot{Q} = Q/24\,\text{h} = 10{,}470\,\text{J}/24\,\text{h} \times (1\,\text{h}/3600\,\text{s}) = 0.1212\,\text{W}$, or $q_i = \dot{Q}/A_i = 0.1212\,\text{W}/(\pi \times 10\,\text{cm} \times 50\,\text{cm}) = 7.71 \times 10^{-5}\,\text{W/cm}^2$. Therefore, the total resistance must, from equation (5.48), be a minimum of

$$\begin{aligned} A_i R_{\text{tot}} &= |E_{bi} - E_{bo}|/q_i = 5.670 \times 10^{-12} \times |4.2^4 - 298^4|/7.71 \times 10^{-5} \\ &= 580.0. \end{aligned}$$

We note from equation (5.47) that the resistances are inversely proportional to shield area. Therefore, it is best to place the shields as close to the inner cylinder as possible. We will assume that the shields can be so closely spaced that $A_i \simeq A_2 \simeq \ldots \simeq A_N = A_s = \pi D_s L$, with $D_s = 11\,\text{cm}$. Evaluating the total resistance from equations (5.47) and (5.48), we find

$$A_i R_{\text{tot}} = \frac{1}{\epsilon_w} + \left(\frac{1}{\epsilon_s} - 1\right)\frac{A_i}{A_s} + \sum_{n=1}^{N-1}\left(\frac{2}{\epsilon_s} - 1\right)\frac{A_i}{A_s} + \frac{1}{\epsilon_s}\frac{A_i}{A_s} + \left(\frac{1}{\epsilon_w} - 1\right)\frac{A_i}{A_o},$$

where $\epsilon_w = 0.3$ is the emittance of the (stainless steel) walls, and $\epsilon_s = 0.05$ is the emittance of the (aluminized) shields. Since the elements of the series in the last equation do not depend on $n$, we may solve for $N$ as

$$\begin{aligned} N &= \frac{A_i R_{\text{tot}} - \dfrac{1}{\epsilon_w} - \left(\dfrac{1}{\epsilon_w} - 1\right)\dfrac{A_i}{A_o}}{\left(\dfrac{2}{\epsilon_s} - 1\right)\dfrac{A_i}{A_s}} \\ &= \frac{580.0 - \frac{1}{0.3} - \left(\frac{1}{0.3} - 1\right)\frac{10}{20}}{\left(\frac{2}{0.05} - 1\right)\frac{10}{11}} \\ &= 16.23. \end{aligned}$$

Therefore, a minimum of 17 radiation shields would be required. Note from equation (5.35) that, without radiation shields,

$$\begin{aligned} q_i &= \frac{|E_{bi} - E_{bo}|}{\dfrac{1}{\epsilon_w} + \left(\dfrac{1}{\epsilon_w} - 1\right)\dfrac{A_i}{A_o}} = \frac{5.670 \times 10^{-12}|4.2^4 - 298^4|}{\dfrac{1}{0.3} + \left(\dfrac{1}{0.3} - 1\right) \times \dfrac{1}{2}} \\ &= 9.94 \times 10^{-3}\,\text{W/cm}^2, \end{aligned}$$

that is, the heat loss is approximately 100 times larger!

## 5.6 SOLUTION METHODS FOR THE GOVERNING INTEGRAL EQUATIONS

The usefulness of the method described in the previous sections is limited by the fact that it requires the radiosity to be constant over each subsurface. This is rarely the case if the subsurfaces of the enclosure are relatively large (as compared with typical distances between surfaces). Today, with the advent of powerful digital computers, more accurate solutions are usually obtained by increasing the number of subsurfaces, $N$, in equation (5.37), which then become simply a finite-difference solution to the integral equation (5.28). Still, there are times when more accurate methods for the solution of equation (5.28) are desired (for computational efficiency), or when exact or approximate solutions are sought in explicit form. Therefore, we shall give here a very brief outline of such solution methods.

If radiosity $J$ is to be determined, the governing equation that needs to be solved is either equation (5.24), if the surface temperature is given, or equation (5.25), if surface heat flux is specified. If unknown temperatures or heat fluxes are to be determined directly, equation (5.28) must be solved. In all cases the governing equation may be written as a *Fredholm integral equation of the second kind,*

$$\phi(\mathbf{r}) = f(\mathbf{r}) + \int_A K(\mathbf{r}, \mathbf{r}')\,\phi(\mathbf{r}')\,dA', \tag{5.49}$$

where $K(\mathbf{r}, \mathbf{r}')$ is called the *kernel* of the integral equation, $f(\mathbf{r})$ is a known function, and $\phi(\mathbf{r})$ is the function to be determined (e.g., radiosity or heat flux). Comprehensive discussions for the treatment of such integral equations are given in mathematical texts such as Courant and Hilbert [5] or Hildebrand [6]. A number of radiative heat transfer examples have been discussed by Özişik [7].

Numerical solutions to equation (5.49) may be found in a number of ways. In the *method of successive approximation* a first guess of $\phi(\mathbf{r}) = f(\mathbf{r})$ is made with which the integral in equation (5.49) is evaluated (analytically in some simple situations, but more often through numerical quadrature). This leads to an improved value for $\phi(\mathbf{r})$, which is substituted back into the integral, and so on. This scheme is known to converge for all surface radiation problems. Another possible solution method is *reduction to algebraic equations* by using numerical quadrature for the integral, i.e., replacing it by a series of quadrature coefficients and nodal values. This leads to a set of equations similar to equation (5.37), but of higher accuracy. This type of solution method is most easily extended to arbitrary, three-dimensional geometries, for example, as recently demonstrated by Daun and Hollands [8], who employed nonuniform rational B-splines (NURBS) to express the surfaces. A third method of solution has been given by Sparrow and Haji-Sheikh [9], who demonstrated that the method of *variational calculus* may be applied to general problems governed by a Fredholm integral equation.

Most early numerical solutions in the literature dealt with two very basic systems. The problem of two-dimensional parallel plates of finite width was studied in some detail by Sparrow and coworkers [9–11], using the variational method. The majority of studies have concentrated on radiation from cylindrical holes because of the importance of this geometry for cylindrical tube flow, as well as for the preparation of a blackbody for calibrating radiative property measurements. The problem of an infinitely long isothermal hole radiating from its opening was first studied by Buckley [12] and by Eckert [13]. Buckley's work appears to be the first employing the kernel approximation method. Much later, the same problem was solved exactly through the method of successive approximation (with numerical quadrature) by Sparrow and Albers [14]. A finite hole, but with both ends open, was studied by a number of investigators. Usiskin and Siegel [15] considered the constant wall heat flux case, using the kernel approximation as well as a variational approach. The constant wall temperature case was studied by Lin and Sparrow [16], and combined convection/surface radiation was investigated by Perlmutter and Siegel [17, 18]. Of greater importance for the manufacture of a blackbody is the isothermal cylindrical cavity of finite depth, which was studied by Sparrow and coworkers [19, 20] using successive approximations. If part of the opening is covered by a flat ring with a smaller hole, such a cavity behaves like a blackbody for very small $L/R$ ratios. This problem was studied by Alfano [21] and Alfano and Sarno [22]. Because of their importance for the manufacture of blackbody cavities these results are summarized in Table 5.1. A detector removed from the cavity will sense a signal proportional to the intensity leaving the bottom center of the cavity in the normal direction. Thus the effectiveness of the blackbody is measured by how close to unity the ratio $I_n/I_b(T)$ is. For perfectly diffuse reflectors, $I_n = J/\pi$, and with $I_b = \sigma T^4/\pi$ an apparent emittance is defined as

$$\epsilon_a = I_n/I_b(T) = J/\sigma T^4. \tag{5.50}$$

To give an outline of how the different methods may be applied we shall, over the following few pages, solve the same simple example by three different methods, the first two being "exact," and the third being the kernel approximation.

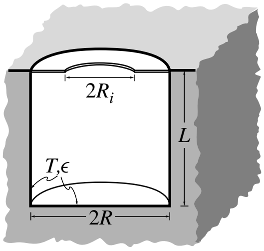
**Example 5.10.** Consider two long parallel plates of width $w$ as shown in Fig. 5-14. Both plates are isothermal at the (same) temperature $T$, and both have a gray, diffuse emittance of $\epsilon$. The plates are separated by a distance $h$ and are placed in a large, cold environment. Determine the local radiative heat fluxes along the plate using the method of successive approximation.

***Solution***

From equation (5.24) we find, with $dF_{di-di} = 0$,

$$\begin{aligned} J_1(x_1) &= \epsilon\sigma T^4 + (1-\epsilon)\int_0^w J_2(x_2)\,dF_{d1-d2}, \\ J_2(x_2) &= \epsilon\sigma T^4 + (1-\epsilon)\int_0^w J_1(x_1)\,dF_{d2-d1}, \end{aligned}$$

**TABLE 5.1**
**Apparent emittance, $\epsilon_a = J/\sigma T^4$, at the bottom center of an isothermal partially covered cylindrical cavity [21, 22].**




| $\epsilon$ | $R_i/R$ | $\epsilon_a$ $(L/R = 2)$ | $(L/R = 4)$ | $(L/R = 8)$ |
|---|---|---|---|---|
| 0.25 | 0.4 | 0.916 | 0.968 | 0.990 |
| | 0.6 | 0.829 | 0.931 | 0.981 |
| | 0.8 | 0.732 | 0.888 | 0.969 |
| | 1.0 | 0.640 | 0.844 | 0.965 |
| 0.50 | 0.4 | 0.968 | 0.990 | 0.998 |
| | 0.6 | 0.932 | 0.979 | 0.995 |
| | 0.8 | 0.887 | 0.964 | 0.992 |
| | 1.0 | 0.839 | 0.946 | 0.989 |
| 0.75 | 0.4 | 0.988 | 0.997 | 0.999 |
| | 0.6 | 0.975 | 0.997 | 0.998 |
| | 0.8 | 0.958 | 0.988 | 0.997 |
| | 1.0 | 0.939 | 0.982 | 0.996 |

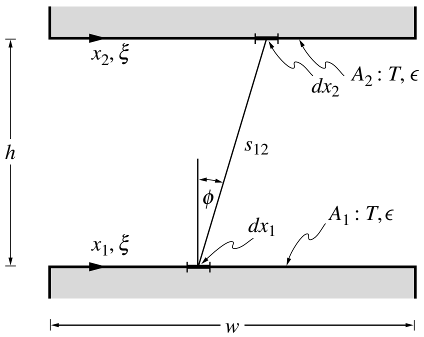



**FIGURE 5-14**
Radiative exchange between two long isothermal parallel plates.

and, from Configuration 1 in Appendix D, with $s_{12} = h/\cos\phi$, $s_{12}\,d\phi = dx_2\cos\phi$, and $\cos\phi = h/\sqrt{h^2 + (x_2 - x_1)^2}$,

$$dx_1\,dF_{d1-d2} = dx_2\,dF_{d2-d1} = \frac{1}{2}\cos\phi\,d\phi\,dx_1 = \frac{\cos^3\phi}{2h}\,dx_1\,dx_2 = \frac{1}{2}\frac{h^2\,dx_1\,dx_2}{[h^2 + (x_1 - x_2)^2]^{3/2}}.$$

Introducing nondimensional variables $W = w/h$, $\xi = x/h$, and $\mathscr{J}(x) = J(x)/\sigma T^4$, and realizing that, as a result of symmetry, $J_1 = J_2$ (and $q_1 = q_2$), we may simplify the governing integral equation to

$$\mathscr{J}(\xi) = \epsilon + \frac{1}{2}(1-\epsilon)\int_0^W \mathscr{J}(\xi')\frac{d\xi'}{[1 + (\xi' - \xi)^2]^{3/2}}. \tag{5.51}$$

Making a first guess of $\mathscr{J}^{(1)} = \epsilon$ we obtain a second guess by substitution,

$$\begin{aligned}\mathscr{J}^{(2)}(\xi) &= \epsilon\left\{1 + \frac{1}{2}(1-\epsilon)\int_0^W \frac{d\xi'}{[1 + (\xi' - \xi)^2]^{3/2}}\right\}\\ &= \epsilon\left\{1 + \frac{1}{2}(1-\epsilon)\left[\frac{W-\xi}{\sqrt{1 + (W-\xi)^2}} + \frac{\xi}{\sqrt{1+\xi^2}}\right]\right\}.\end{aligned}$$

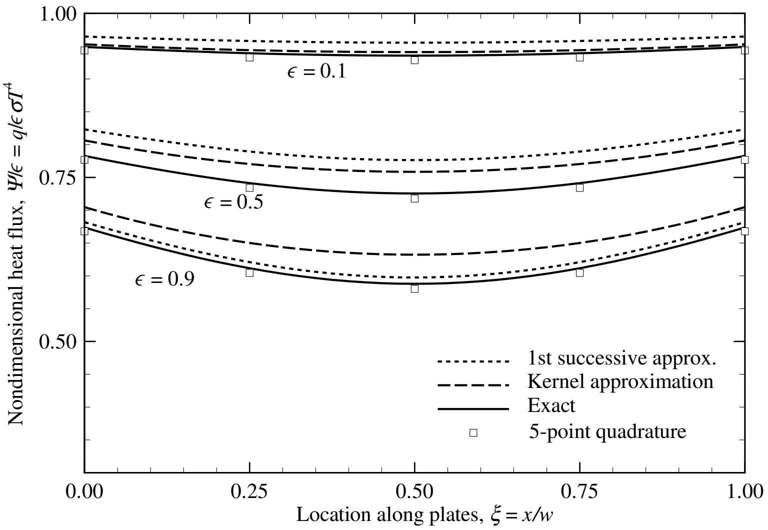



**FIGURE 5-15**
Local radiative heat flux on long, isothermal parallel plates, determined by various methods.

Repeating the procedure we get

$$\mathscr{J}^{(3)}(\xi) = \epsilon\left\{1 + \frac{1}{2}(1-\epsilon)\left[\frac{W-\xi}{\sqrt{1+(W-\epsilon)^2}} + \frac{\xi}{\sqrt{1+\xi^2}}\right]\right.$$
$$\left. + \frac{1}{4}(1-\epsilon)^2\int_0^W\left[\frac{W-\xi'}{\sqrt{1+(W-\xi')^2}} + \frac{\xi'}{\sqrt{1+\xi'^2}}\right]\frac{d\xi'}{[1+(\xi'-\xi)^2]^{3/2}}\right\},$$

where the last integral becomes quite involved. We shall stop at this point since further successive integrations would have to be carried out numerically. It is clear from the above expression that the terms in the series diminish as $\epsilon\,[(1-\epsilon)W]^n$, i.e., few successive iterations are necessary for surfaces with low reflectances and/or $w/h$ ratios. Once the radiosity has been determined the local heat flux follows from equation (5.26). Limiting ourselves to $\mathscr{J}^{(2)}$ (single successive approximation), this yields

$$\begin{aligned}\Psi(\xi) &= \frac{q(\xi)}{\sigma T^4} = \frac{\epsilon}{1-\epsilon}[1-\mathscr{J}(\xi)]\\ &= \epsilon - \frac{\epsilon^2}{2}\left[\frac{W-\xi}{\sqrt{1+(W-\xi)^2}} + \frac{\xi}{\sqrt{1+\xi^2}}\right] - \mathscr{O}\Big(\epsilon^2(1-\epsilon)W^2\Big),\end{aligned}$$

where $\mathscr{O}(z)$ is shorthand for "order of magnitude $z$." Some results are shown in Fig. 5-15 and compared with other solution methods for the case of $W = w/h = 1$ and three values of the emittance. Observe that the heat loss is a minimum at the center of the plate, since this location receives maximum irradiation from the other plate (i.e., the view factor from this location to the opposing plate is maximum). For decreasing $\epsilon$ the heat loss increases, of course, since more is emitted; however, this increase is less than linear since also more energy is coming in, of which a larger fraction is absorbed. The first successive approximation does very well for small and large $\epsilon$ as expected from the order of magnitude of the neglected terms.

**Example 5.11.** Repeat Example 5.10 using numerical quadrature.

***Solution***
The governing equation is, of course, again equation (5.51). We shall approximate the integral on the right-hand side by a series obtained through numerical integration, or *quadrature.* In this method an integral is approximated by a weighted series of the integrand evaluated at a number of nodal points; or

$$\int_a^b f(\xi,\xi')\,d\xi' \simeq (b-a)\sum_{j=1}^{J} c_j f(\xi,\xi_j), \qquad \sum_{j=1}^{J} c_j = 1. \tag{5.52}$$

Here the $\xi_j$ represent $J$ locations between $a$ and $b$, and the $c_j$ are weight coefficients. The nodal points $\xi_j$ may be equally spaced for easy presentation of results *(Newton–Cotes quadrature)*, or their location may be optimized for increased accuracy *(Gaussian quadrature)*; for a detailed treatment of quadrature see, for example, the book by Fröberg [23].

Using equation (5.52) in equation (5.51) we obtain

$$\mathscr{J}_i = \epsilon + (1-\epsilon)W\sum_{j=1}^{J} c_j\ \mathscr{J}_j f_{ij}, \quad i = 1, 2, \ldots, J,$$

where

$$f_{ij} = \frac{1}{2}\Big/\left[1 + (\xi_j - \xi_i)^2\right]^{3/2}.$$

This system of equations may be further simplified by utilizing the symmetry of the problem, i.e., $\mathscr{J}(\xi) = \mathscr{J}(W-\xi)$. Assuming that nodes are placed symmetrically about the centerline, $\xi_{J+1-j} = \xi_j$, leads to $c_{J+1-j} = c_j$ and $\mathscr{J}_{J+1-j} = \mathscr{J}_j$, or

$$J\ \text{odd:}\quad \mathscr{J}_i = \epsilon + (1-\epsilon)W\left\{\sum_{j=1}^{(J-1)/2} c_j\ \mathscr{J}_j\left[f_{ij} + f_{i,J+1-j}\right] + c_{(J+1)/2}\ \mathscr{J}_{(J+1)/2}\, f_{i,(J+1)/2}\right\}, \quad i = 1, 2, \ldots, \frac{J+1}{2},$$

$$J\ \text{even:}\quad \mathscr{J}_i = \epsilon + (1-\epsilon)W\sum_{j=1}^{J/2} c_j\ \mathscr{J}_j\,(f_{ij} + f_{i,J+1-j}), \quad i = 1, 2, \ldots, \frac{J}{2}.$$

The values of the radiosities may be determined by successive approximation, or by direct matrix inversion. In Fig. 5-15 the simple case of $J = 5$ (resulting in three simultaneous equations) is included, using Newton–Cotes quadrature with $\xi_j = W(j-1)/4$ and $c_1 = c_5 = 7/90$, $c_2 = c_4 = 32/90$, and $c_3 = 12/90$ [23].

Exact analytical solutions that yield explicit relations for the unknown radiosity are rare and limited to a few special geometries. However, approximate analytical solutions may be found for many geometries through the *kernel approximation method.* In this method the kernel $K(x, x')$ is approximated by a linear series of special functions such as $e^{-ax'}$, $\cos ax'$, $\cosh ax'$, and so on (i.e., functions that, after one or two differentiations with respect to $x'$, turn back into the original function except for a constant factor). It is then often possible to convert integral equation (5.49) into a differential equation that may be solved explicitly. The method is best illustrated through an example.

**Example 5.12.** Repeat Example 5.11 using the kernel approximation method.

***Solution***

We again need to solve equation (5.51), this time by approximating the kernel. For convenience we shall choose a simple exponential form,

$$K(\xi,\xi') = \frac{1}{[1 + (\xi' - \xi)^2]^{3/2}} \simeq a\, e^{-b|\xi'-\xi|}.$$

We shall determine "optimum" parameters $a$ and $b$ by letting the approximation satisfy the *0th* and *1st moments*. This implies multiplying the expression by $|\xi' - \xi|$ raised to the *0th* and *1st* powers, followed by integration over the entire domain for $|\xi' - \xi|$, i.e., from 0 to $\infty$ (since $W$ could be arbitrarily large).[3]

[3]Using the actual $W$ at hand will result in a better approximation, but new values for $a$ and $b$ must be determined if $W$ is changed; in addition, the mathematics become considerably more involved.

Thus,

$$\textit{0th}\text{ moment:}\qquad \int_0^\infty \frac{dx}{(1+x^2)^{3/2}} = 1 = \int_0^\infty a\,e^{-bx}\,dx = \frac{a}{b},$$

$$\textit{1st}\text{ moment:}\qquad \int_0^\infty \frac{x\,dx}{(1+x^2)^{3/2}} = 1 = \int_0^x a\,e^{-bx}x\,dx = \frac{a}{b^2},$$

leading to $a = b = 1$ and

$$K(\xi,\xi') \simeq e^{-|\xi'-\xi|}.$$

Substituting this expression into equation (5.51) leads to

$$\mathscr{J}(\xi) \simeq \epsilon + \frac{1}{2}(1-\epsilon)\left[\int_0^\xi \mathscr{J}(\xi')\,e^{-(\xi-\xi')}\,d\xi' + \int_\xi^W \mathscr{J}(\xi')\,e^{-(\xi'-\xi)}\,d\xi'\right].$$

We shall now differentiate this expression twice with respect to $\xi$, for which we need to employ *Leibniz's rule*, equation (3.106). Therefore,

$$\frac{d\mathscr{J}}{d\xi} = \frac{1}{2}(1-\epsilon)\left[\mathscr{J}(\xi) - \int_0^\xi \mathscr{J}(\xi')\,e^{-(\xi-\xi')}\,d\xi' - \mathscr{J}(\xi) + \int_\xi^W \mathscr{J}(\xi')\,e^{-(\xi'-\xi)}\,d\xi'\right],$$

$$\frac{d^2\mathscr{J}}{d\xi^2} = \frac{1}{2}(1-\epsilon)\left[-\mathscr{J}(\xi) + \int_0^\xi \mathscr{J}(\xi')\,e^{-(\xi-\xi')}\,d\xi' - \mathscr{J}(\xi) + \int_\xi^W \mathscr{J}(\xi')\,e^{-(\xi'-\xi)}\,d\xi'\right],$$

or, by comparison with the expression for $\mathscr{J}(\xi)$,

$$\frac{d^2\mathscr{J}}{d\xi^2} = \mathscr{J} - \epsilon - (1-\epsilon)\,\mathscr{J} = \epsilon(\mathscr{J} - 1).$$

Thus, the governing integral equation has been converted into a second-order ordinary differential equation, which is readily solved as

$$\mathscr{J}(\xi) = 1 + C_1\,e^{-\sqrt{\epsilon}\xi} + C_2\,e^{+\sqrt{\epsilon}\xi}.$$

While an integral equation does not require any boundary conditions, we have converted the governing equation into a differential equation that requires two boundary conditions in order to determine $C_1$ and $C_2$. The dilemma is overcome by substituting the general solution back into the governing integral equation (with approximated kernel). This calculation can be done for variable values of $\xi$ by comparing coefficients of independent functions of $\xi$, or simply for two arbitrarily selected values for $\xi$. The first method gives the engineer proof that his analysis is without mistake, but is usually considerably more tedious. Often it is also possible to employ symmetry, as is the case here, since $\mathscr{J}(\xi) = \mathscr{J}(W-\xi)$ or

$$C_1\left[e^{-\sqrt{\epsilon}\xi} - e^{-\sqrt{\epsilon}(W-\xi)}\right] = -C_2\left[e^{\sqrt{\epsilon}\xi} - e^{\sqrt{\epsilon}(W-\xi)}\right] = C_2\,e^{\sqrt{\epsilon}W}\left[e^{-\sqrt{\epsilon}\xi} - e^{-\sqrt{\epsilon}(W-\xi)}\right],$$

or

$$C_1 = C_2\,e^{\sqrt{\epsilon}W}.$$

Consequently,

$$\mathscr{J}(\xi) = 1 + C_1\left[e^{-\sqrt{\epsilon}\xi} + e^{-\sqrt{\epsilon}(W-\xi)}\right],$$

and substituting this expression into the governing equation at $\xi = 0$ gives

$$\begin{aligned}
\mathscr{J}(0) &= 1 + C_1\left(1 + e^{-\sqrt{\epsilon}W}\right)\\
&= \epsilon + \frac{1}{2}(1-\epsilon)\int_0^W \left\{1 + C_1\left[e^{-\sqrt{\epsilon}\xi'} + e^{-\sqrt{\epsilon}(W-\xi')}\right]\right\}e^{-\xi'}\,d\xi'\\
&= \epsilon + \frac{1}{2}(1-\epsilon)\int_0^W \left\{e^{-\xi'} + C_1\left[e^{-(1+\sqrt{\epsilon})\xi'} + e^{-\xi'-\sqrt{\epsilon}(W-\xi')}\right]\right\}d\xi'\\
&= \epsilon - \frac{1}{2}(1-\epsilon)\left\{e^{-\xi'} + C_1\left[\frac{e^{-(1+\sqrt{\epsilon})\xi'}}{1+\sqrt{\epsilon}} + \frac{e^{-\xi'-\sqrt{\epsilon}(W-\xi')}}{1-\sqrt{\epsilon}}\right]\right\}\Bigg|_0^W\\
&= \epsilon + \frac{1}{2}(1-\epsilon)\left\{1 - e^{-W} + C_1\left[\frac{1 - e^{-(1+\sqrt{\epsilon})W}}{1+\sqrt{\epsilon}} + \frac{e^{-\sqrt{\epsilon}W} - e^{-W}}{1-\sqrt{\epsilon}}\right]\right\}.
\end{aligned}$$

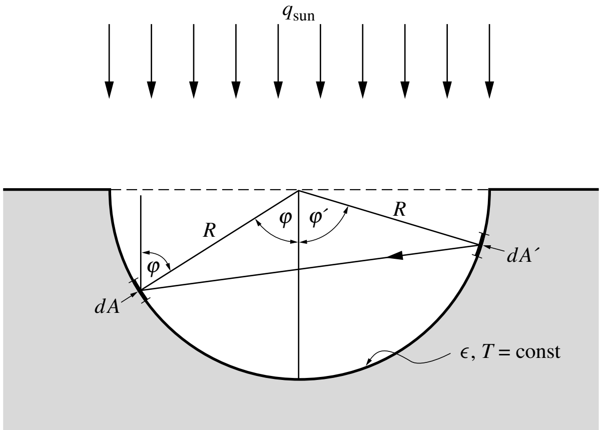



**FIGURE 5-16**
Isothermal hemispherical cavity irradiated normally by the sun, Example 5.13.

Solving this for $C_1$ gives

$$1-\epsilon-\tfrac{1}{2}(1-\epsilon)(1-e^{-W}) = C_1\left[\tfrac{1}{2}\left(1-\sqrt{\epsilon}\right)\left(1-e^{-(1+\sqrt{\epsilon})W}\right)+\tfrac{1}{2}\left(1+\sqrt{\epsilon}\right)\left(e^{-\sqrt{\epsilon}W}-e^{-W}\right)-\left(1-e^{-\sqrt{\epsilon}W}\right)\right]$$

$$\tfrac{1}{2}(1-\epsilon)\left(1+e^{-W}\right) = C_1\left\{\tfrac{1}{2}\left(1-\sqrt{\epsilon}\right)+\tfrac{1}{2}\left(1+\sqrt{\epsilon}\right)e^{-\sqrt{\epsilon}W}-1-e^{-\sqrt{\epsilon}W}-\left[\tfrac{1}{2}\left(1-\sqrt{\epsilon}\right)e^{-\sqrt{\epsilon}W}+\tfrac{1}{2}\left(1+\sqrt{\epsilon}\right)\right]e^{-W}\right\},$$

or

$$C_1 = -\frac{1-\epsilon}{(1+\sqrt{\epsilon})+(1-\sqrt{\epsilon})\,e^{-\sqrt{\epsilon}W}}$$

and

$$\mathscr{J}(\xi) = 1-(1-\epsilon)\frac{e^{-\sqrt{\epsilon}\xi}+e^{-\sqrt{\epsilon}(W-\xi)}}{(1+\sqrt{\epsilon})+(1-\sqrt{\epsilon})\,e^{-\sqrt{\epsilon}W}}.$$

Finally, the nondimensional heat flux follows as

$$\Psi(\xi) = \frac{\epsilon}{1-\epsilon}[1-\mathscr{J}(\xi)] = \frac{\epsilon\left[e^{-\sqrt{\epsilon}\xi}+e^{-\sqrt{\epsilon}(W-\xi)}\right]}{(1+\sqrt{\epsilon})+(1-\sqrt{\epsilon})\,e^{-\sqrt{\epsilon}W}},$$

which is also included in Fig. 5-15.

Note that $e^{-|\xi'-\xi|}$ is not a particularly good approximation for the kernel, since the actual kernel has a zero first derivative at $\xi' = \xi$. A better approximation can be obtained by using

$$K(\xi,\xi') \simeq a_1\,e^{-b_1|\xi'-\xi|} + a_2\,e^{-b_2|\xi'-\xi|}$$

(with $a_1 > 1$ and $a_2 < 0$). If $W$ is relatively small, say $< \frac{1}{2}$, a good approximation may be obtained using

$$K(\xi,\xi') \simeq \cos a(\xi'-\xi)$$

(since the kernel has an inflection point at $|\xi'-\xi| = \frac{1}{2}$).

We shall conclude this chapter with two examples that demonstrate that exact analytical solutions are possible for a few simple geometries for which the view factors between area elements attain certain special forms.

**Example 5.13.** Consider a hemispherical cavity irradiated by the sun as shown in Fig. 5-16. The surface of the cavity is kept isothermal at temperature $T$ and is coated with a gray, diffuse material with emittance $\epsilon$. Assuming that the cavity is, aside from the solar irradiation, exposed to cold surroundings, determine the local heat flux rates that are necessary to maintain the cavity surface at constant temperature.

***Solution***
From equation (5.24) the local radiosity at position $(\varphi, \psi)$ is determined as

$$\begin{aligned} J(\varphi) &= \epsilon\sigma T^4 + (1-\epsilon)H(\varphi) \\ &= \epsilon\sigma T^4 + (1-\epsilon)\left[\int_A J(\varphi')\, dF_{dA-dA'} + H_o(\varphi)\right], \end{aligned}$$

where we have already stated that radiosity is a function of $\varphi$ only, i.e., there is no dependence on azimuthal angle $\psi$. The view factor between infinitesimal areas on a sphere is known from the *inside sphere method,* equation (4.33), as

$$dF_{dA-dA'} = \frac{dA'}{4\pi R^2} = \frac{R^2 \sin\varphi'\, d\varphi'\, d\psi'}{4\pi R^2}.$$

The external irradiation at $dA$ is readily determined as $H_o(\varphi) = q_{\text{sun}} \cos\varphi$, and the expression for radiosity becomes

$$\begin{aligned} J(\varphi) &= \epsilon\sigma T^4 + (1-\epsilon)\left[\int_0^{2\pi}\int_0^{\pi/2} J(\varphi')\,\frac{\sin\varphi'\, d\varphi'\, d\psi'}{4\pi} + q_{\text{sun}}\cos\varphi\right] \\ &= \epsilon\sigma T^4 + \frac{1-\epsilon}{2}\int_0^{\pi/2} J(\varphi')\sin\varphi'\, d\varphi' + (1-\epsilon)q_{\text{sun}}\cos\varphi. \end{aligned}$$

Because of the unique behavior of view factors between sphere surface elements we note that the irradiation at location $\varphi$ that arrives from other parts of the sphere, $H_s$, *does not depend on* $\varphi$. Thus,

$$H_s = \frac{1}{2}\int_0^{\pi/2} J(\varphi')\sin\varphi'\, d\varphi' = \text{const},$$

and

$$J(\varphi) = \epsilon\sigma T^4 + (1-\epsilon)H_s + (1-\epsilon)q_{\text{sun}}\cos\varphi.$$

Substituting this equation into the expression for $H_s$ leads to

$$\begin{aligned} H_s &= \frac{1}{2}\int_0^{\pi/2}\left[\epsilon\sigma T^4 + (1-\epsilon)H_s + (1-\epsilon)q_{\text{sun}}\cos\varphi'\right]\sin\varphi'\, d\varphi' \\ &= \tfrac{1}{2}\epsilon\sigma T^4 + \tfrac{1}{2}(1-\epsilon)H_s + \tfrac{1}{4}(1-\epsilon)q_{\text{sun}}, \end{aligned}$$

or

$$H_s = \frac{\epsilon}{1+\epsilon}\sigma T^4 + \frac{1-\epsilon}{2(1+\epsilon)}q_{\text{sun}}.$$

An energy balance at $dA$ gives

$$q(\varphi) = \epsilon\sigma T^4 - \epsilon H(\varphi) = \epsilon(\sigma T^4 - H_s - q_{\text{sun}}\cos\varphi)$$

or

$$q(\varphi) = \epsilon\left[\frac{\sigma T^4}{1+\epsilon} - \left(\frac{1-\epsilon}{2(1+\epsilon)} + \cos\varphi\right)q_{\text{sun}}\right].$$

We observe from this example that in problems where all radiating surfaces are part of a sphere, none of the view factors involved depend on the location of the originating surface, and an exact analytical solution can always be found in a similar fashion. Apparently, this was first recognized by Jensen [24] and reported in the book by Jakob [25].

Exact analytical solutions are also possible for such configurations where all relevant view factors have repeating derivatives (as in the kernel approximation).

**Example 5.14.** A long thin radiating wire is to be employed as an infrared light source. To maximize the output of infrared energy into the desired direction, the wire is fitted with an insulated, highly reflective sheath as shown in Fig. 5-17. The sheath is cylindrical with radius $R$ (which is much larger than the diameter of the wire), and has a cutout of half-angle $\varphi$ to let the concentrated infrared light escape. Assuming that the wire is heated with a power of $Q'$ W/m length of wire, and that the sheath

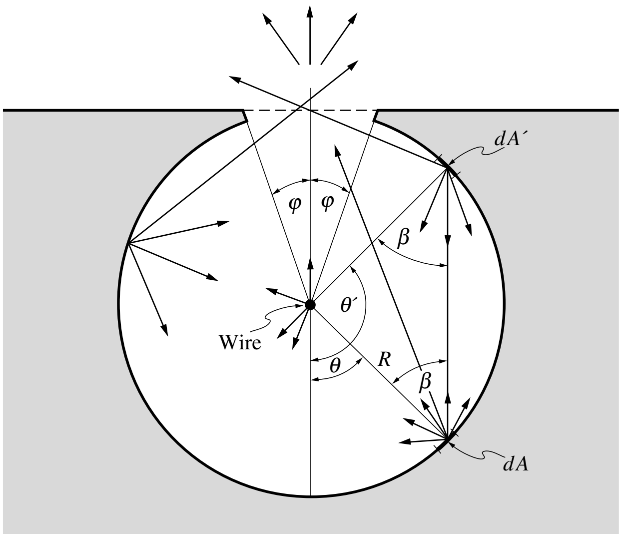



**FIGURE 5-17**
Thin radiating wire with radiating sheath, Example 5.14.

can lose heat only by radiation and only from its inside surface, determine the temperature distribution across the sheath.

***Solution***
From an energy balance on a surface element $dA$ it follows from equation (5.20) that, with $q(\theta) = 0$,

$$\sigma T^4(\theta) = J(\theta) = H(\theta),$$

and

$$H(\theta) = \int_A J(\theta')\, dF_{dA-dA'} + H_o(\theta).$$

We may treat the energy emitted from the wire as external radiation (neglecting absorption by the wire since it is so small). Since the total released energy will spread equally into all directions, we find

$$H_o(\theta) = Q'/2\pi R = \text{const.}$$

The view factor $dF_{dA-dA'}$ between two infinitely long strips on the cylinder surface is given by Configuration 1 in Appendix D as

$$F_{dA-dA'} = \tfrac{1}{2}\cos\beta\, d\beta,$$

where the angle $\beta$ is indicated in Fig. 5-17 and may be related to $\theta$ through

$$2\beta + |\theta' - \theta| = \pi.$$

Differentiating $\beta$ with respect to $\theta'$ we obtain $d\beta = \pm d\theta'/2$, depending on whether $\theta'$ is larger or less than $\theta$. Substituting for $\beta$ in the view factor, this becomes

$$F_{dA-dA'} = \frac{1}{2}\cos\left(\frac{\pi}{2} - \left|\frac{\theta' - \theta}{2}\right|\right)\frac{1}{2}\, d\theta' = \frac{1}{4}\sin\left|\frac{\theta' - \theta}{2}\right| d\theta',$$

where the ± has been omitted since the view factor is always positive (i.e., $|d\beta|$ is to be used). Substituting this into the above relationship for radiosity we obtain

$$\begin{aligned} J(\theta) &= \frac{1}{4}\int_{-\pi+\varphi}^{\pi-\varphi} J(\theta')\sin\left|\frac{\theta' - \theta}{2}\right| d\theta' + H_o \\ &= \frac{1}{4}\int_{-\pi+\varphi}^{\theta} J(\theta')\sin\frac{\theta - \theta'}{2}\, d\theta' + \frac{1}{4}\int_{\theta}^{\pi-\varphi} J(\theta')\sin\frac{\theta' - \theta}{2}\, d\theta' + H_o. \end{aligned}$$

Since the view factor in the integrand has repetitive derivatives we may convert this integral equation into a second-order differential equation, as was done in the kernel approximation method. Differentiating twice, we have

$$\frac{dJ}{d\theta} = \frac{1}{8}\int_{-\pi+\varphi}^{\theta} J(\theta')\cos\frac{\theta-\theta'}{2}\,d\theta' - \frac{1}{8}\int_{\theta}^{\pi-\varphi} J(\theta')\cos\frac{\theta'-\theta}{2}\,d\theta',$$

$$\frac{d^2J}{d\theta^2} = \frac{1}{8}J(\theta) - \frac{1}{16}\int_{-\pi+\varphi}^{\theta} J(\theta')\sin\frac{\theta-\theta'}{2}\,d\theta' + \frac{1}{8}J(\theta) - \frac{1}{16}\int_{\theta}^{\pi-\varphi} J(\theta')\sin\frac{\theta'-\theta}{2}\,d\theta'.$$

Comparing this result with the above integral equation for $J(\theta)$ we find

$$\frac{d^2J}{d\theta^2} = \tfrac{1}{4}J(\theta) - \tfrac{1}{4}[J(\theta) - H_o] = \tfrac{1}{4}H_o.$$

This equation is readily solved as

$$J(\theta) = \tfrac{1}{8}H_o\,\theta^2 + C_1\theta + C_2.$$

The two integration constants must now be determined by substituting the solution back into the governing integral equation. However, $C_1$ may be determined from symmetry since, for this problem, $J(\theta) = J(-\theta)$ and $C_1 = 0$. To determine $C_2$ we evaluate $J$ at $\theta = 0$:

$$\begin{aligned}
J(0) = C_2 &= \frac{1}{4}\int_{-\pi+\varphi}^{0} J(\theta')\sin\left(-\frac{\theta'}{2}\right)d\theta' + \frac{1}{4}\int_{0}^{\pi-\varphi} J(\theta')\sin\frac{\theta'}{2}\,d\theta' + H_o\\
&= \frac{1}{2}\int_{0}^{\pi-\varphi} J(\theta')\sin\frac{\theta'}{2}\,d\theta' + H_o\\
&= \frac{1}{2}\int_{0}^{\pi-\varphi}\left(C_2 + \frac{H_o}{8}\theta'^2\right)\sin\frac{\theta'}{2}\,d\theta' + H_o.
\end{aligned}$$

Integrating twice by parts we obtain

$$\begin{aligned}
C_2 &= H_o - \left(C_2 + \frac{H_o}{8}\theta'^2\right)\cos\frac{\theta'}{2}\bigg|_0^{\pi-\varphi} + \frac{H_o}{4}\int_0^{\pi-\varphi}\theta'\cos\frac{\theta'}{2}\,d\theta'\\
&= H_o - \left[C_2 + \frac{H_o}{8}(\pi-\varphi)^2\right]\cos\left(\frac{\pi}{2}-\frac{\varphi}{2}\right) + C_2 + \frac{H_o}{2}\left(\theta'\sin\frac{\theta'}{2}\bigg|_0^{\pi-\varphi} - \int_0^{\pi-\varphi}\sin\frac{\theta'}{2}\,d\theta'\right)\\
&= H_o + C_2 - \left[C_2 + \frac{H_o}{8}(\pi-\varphi)^2\right]\sin\frac{\varphi}{2} + \frac{H_o}{2}\left((\pi-\varphi)\sin\left(\frac{\pi}{2}-\frac{\varphi}{2}\right) + 2\cos\frac{\theta'}{2}\bigg|_0^{\pi-\varphi}\right)\\
&= H_o + C_2 - \left[C_2 + \frac{H_o}{8}(\pi-\varphi)^2\right]\sin\frac{\varphi}{2} + \frac{H_o}{2}(\pi-\varphi)\cos\frac{\varphi}{2} + H_o\sin\frac{\varphi}{2} - H_o.
\end{aligned}$$

Solving this equation for $C_2$ we get

$$C_2 = H_o\left[1 + \frac{\pi-\varphi}{2}\cos\frac{\varphi}{2} - \frac{1}{8}(\pi-\varphi)^2\right].$$

Therefore,

$$T^4(\theta) = \frac{J}{\sigma} = \frac{Q'}{2\pi R\sigma}\left\{1 + \frac{\pi-\varphi}{2}\cos\frac{\varphi}{2} - \frac{1}{8}\left[(\pi-\varphi)^2 - \theta^2\right]\right\}.$$

We find that the temperature has a minimum at $\theta = 0$, since around that location the view factor to the opening is maximum, resulting in a maximum of escaping energy. The temperature level increases as $\varphi$ decreases (since less energy can escape) and reaches $T \to \infty$ as $\varphi = 0$ (since this produces an insulated closed enclosure with internal heat production).

The fact that long cylindrical surfaces lend themselves to exact analysis was apparently first recognized by Sparrow [26]. The preceding two examples have shown that exact solutions may be found for a number of special geometries, namely, (*i*) enclosures whose surfaces all lie on a single sphere, and (*ii*) enclosures for which view factors between surface elements have repetitive derivatives. For other still fairly simple geometries an approximate analytical solution may be determined from the *kernel approximation method.* However, the vast majority of radiative heat transfer problems in enclosures without a participating medium must be solved by numerical methods. A large majority of these are solved using the *net radiation method* described in the first few sections of this chapter. If greater accuracy or better numerical efficiency is desired, one of the numerical methods briefly described in this section needs to be used, such as *numerical quadrature* leading to a set of linear algebraic equations (as in the net radiation method).

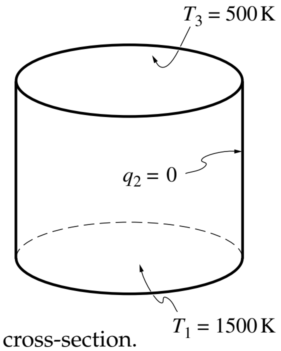
## References

1. Jensen, C. L.: "TRASYS-II user's manual–thermal radiation analysis system," Technical report, Martin Marietta Aerospace Corp., Denver, 1987.
2. Chin, J. H., T. D. Panczak, and L. Fried: "Spacecraft thermal modeling," *Int. J. Numer. Methods Eng.*, vol. 35, pp. 641–653, 1992.
3. Oppenheim, A. K.: "Radiation analysis by the network method," *Transactions of ASME, Journal of Heat Transfer*, vol. 78, pp. 725–735, 1956.
4. Kropschot, R. H., B. W. Birmingham, and D. B. Mann (eds.): *Technology of Liquid Helium*, National Bureau of Standards, Monograph 111, Washington, D.C., 1968.
5. Courant, R., and D. Hilbert: *Methods of Mathematical Physics*, Interscience Publishers, New York, 1953.
6. Hildebrand, F. B.: *Methods of Applied Mathematics*, Prentice Hall, Englewood Cliffs, NJ, 1952.
7. Özişik, M. N.: *Radiative Transfer and Interactions With Conduction and Convection*, John Wiley & Sons, New York, 1973.
8. Daun, K. J., and K. G. T. Hollands: "Infinitesimal-area radiative analysis using parametric surface representation, through NURBS," *ASME Journal of Heat Transfer*, vol. 123, no. 2, pp. 249–256, 2001.
9. Sparrow, E. M., and A. Haji-Sheikh: "A generalized variational method for calculating radiant interchange between surfaces," *ASME Journal of Heat Transfer*, vol. 87, pp. 103–109, 1965.
10. Sparrow, E. M.: "Application of variational methods to radiation heat transfer calculations," *ASME Journal of Heat Transfer*, vol. 82, pp. 375–380, 1960.
11. Sparrow, E. M., J. L. Gregg, J. V. Szel, and P. Manos: "Analysis, results, and interpretation for radiation between simply arranged gray surfaces," *ASME Journal of Heat Transfer*, vol. 83, pp. 207–214, 1961.
12. Buckley, H.: "On the radiation from the inside of a circular cylinder," *Phil. Mag.*, vol. 4, no. 23, pp. 753–762, 1927.
13. Eckert, E. R. G.: "Das Strahlungsverhältnis von Flächen mit Einbuchtungen und von zylindrischen Bohrungen," *Arch. Wärmewirtschaft*, vol. 16, pp. 135–138, 1935.
14. Sparrow, E. M., and L. U. Albers: "Apparent emissivity and heat transfer in a long cylindrical hole," *ASME Journal of Heat Transfer*, vol. 82, pp. 253–255, 1960.
15. Usiskin, C. M., and R. Siegel: "Thermal radiation from a cylindrical enclosure with specified wall heat flux," *ASME Journal of Heat Transfer*, vol. 82, pp. 369–374, 1960.
16. Lin, S. H., and E. M. Sparrow: "Radiant interchange among curved specularly reflecting surfaces, application to cylindrical and conical cavities," *ASME Journal of Heat Transfer*, vol. 87, pp. 299–307, 1965.
17. Perlmutter, M., and R. Siegel: "Effect of specularly reflecting gray surface on thermal radiation through a tube and from its heated wall," *ASME Journal of Heat Transfer*, vol. 85, pp. 55–62, 1963.
18. Siegel, R., and M. Perlmutter: "Convective and radiant heat transfer for flow of a transparent gas in a tube with a gray wall," *International Journal of Heat and Mass Transfer*, vol. 5, pp. 639–660, 1962.
19. Sparrow, E. M., L. U. Albers, and E. R. G. Eckert: "Thermal radiation characteristics of cylindrical enclosures," *ASME Journal of Heat Transfer*, vol. 84, pp. 73–81, 1962.
20. Sparrow, E. M., and R. P. Heinisch: "The normal emittance of circular cylindrical cavities," *Applied Optics*, vol. 9, pp. 2569–2572, 1970.
21. Alfano, G.: "Apparent thermal emittance of cylindrical enclosures with and without diaphragms," *International Journal of Heat and Mass Transfer*, vol. 15, no. 12, pp. 2671–2674, 1972.
22. Alfano, G., and A. Sarno: "Normal and hemispherical thermal emittances of cylindrical cavities," *ASME Journal of Heat Transfer*, vol. 97, no. 3, pp. 387–390, 1975.
23. Fröberg, C. E.: *Introduction to Numerical Analysis*, Addison-Wesley, Reading, MA, 1969.
24. Jensen, H. H.: "Some notes on heat transfer by radiation," *Kgl. Danske Videnskab. Selskab. Mat.-Fys. Medd.*, vol. 24, no. 8, pp. 1–26, 1948.
25. Jakob, M.: *Heat Transfer*, vol. 2, John Wiley & Sons, New York, 1957.
26. Sparrow, E. M.: "Radiant absorption characteristics of concave cylindrical surfaces," *ASME Journal of Heat Transfer*, vol. 84, pp. 283–293, 1962.

## Problems

**5.1** A firefighter (approximated by a two-sided black surface at 310 K 180 cm long and 40 cm wide) is facing a large fire at a distance of 10 m (approximated by a semi-infinite black surface at 1500 K). Ground and sky are at 0°C (and may also be approximated as black). What are the net radiative heat fluxes on the front and back of the firefighter? Compare these with heat rates by free convection ($h$ = 10 W/m$^2$ K, $T_{\text{amb}}$ = 0°C).

**5.2** A small furnace consists of a cylindrical, black-walled enclosure, 20 cm long and with a diameter of 10 cm. The bottom surface is electrically heated to 1500 K, while the cylindrical sidewall is insulated. The top plate is exposed to the environment, such that its temperature is 500 K. Estimate the heating requirements for the bottom wall, and the temperature of the cylindrical sidewall, by treating the sidewall as ($a$) a single zone, ($b$) two equal rings of 10 cm height each.

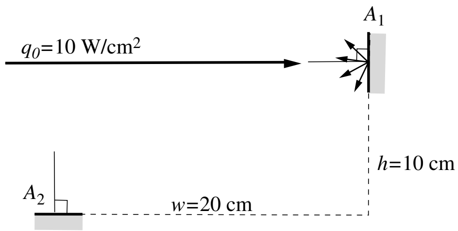



**5.3** Repeat Problem 5.2 for a 20 cm high furnace of quadratic (10 cm × 10 cm) cross-section.

**5.4** A small star has a radius of 100,000 km. Suppose that the star is originally at a uniform temperature of 1,000,000 K before it "dies," i.e., before nuclear fusion stops supplying heat. If it is assumed that the star has a constant heat capacity of $\rho c_p$ = 1 kJ/m$^3$ K, and that it remains isothermal during cool-down, estimate the time required until the star has cooled to 10,000 K. Note: A body of such proportions radiates like a blackbody (Why?).

**5.5** A collimated light beam of $q_0$ = 10 W/cm$^2$ originating from a blackbody source at 1250 K is aimed at a small target $A_1$ = 1 cm$^2$ as shown. The target is coated with a diffusely reflecting material, whose emittance is

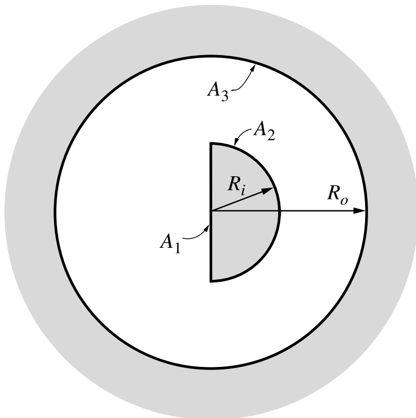
$$\epsilon'_\lambda = \begin{cases} 0.9\cos\theta, & \lambda < 4\,\mu\text{m}, \\ 0.2, & \lambda > 4\,\mu\text{m}. \end{cases}$$



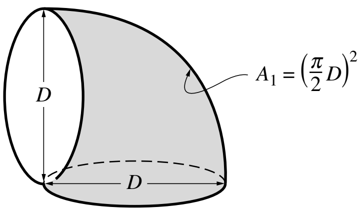

Light reflected from $A_1$ travels on to a detector $A_2$ = 1 cm$^2$, coated with the same material as $A_1$. How much of the collimated energy $q_0$ is absorbed by detector $A_2$?

**5.6** Repeat Problem 5.2 for the case that the top surface of the furnace is coated with a gray, diffuse material with emittance $\epsilon_3$= 0.5 (other surfaces remain black).

**5.7** A long half-cylindrical rod is enclosed by a long diffuse, gray isothermal cylinder as shown. Both rod and cylinder may be considered isothermal ($T_1$ = $T_2$, $\epsilon_1$ = $\epsilon_2$, $T_3$, $\epsilon_3$) and gray, diffuse reflectors. Give an expression for the heat lost from the rod (per unit length).



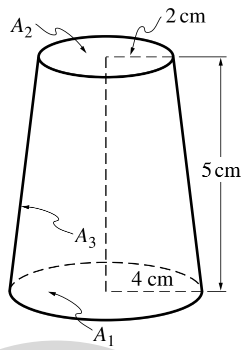

**5.8** Consider a 90° pipe elbow as shown in the figure (pipe diameter = $D$ = 1 m; inner elbow radius = 0, outer elbow radius = $D$). The elbow is isothermal at temperature $T$ = 1000 K, has a gray diffuse emittance $\epsilon$ = 0.4, and is placed in a cool environment. What is the total heat loss from the isothermal elbow (inside and outside)?



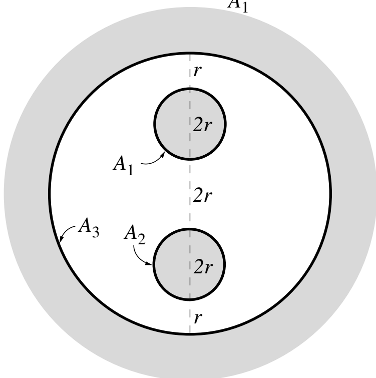
**5.9** For the configuration shown in the figure, determine the temperature of Surface 2 with the following data:

$$\begin{aligned}
\text{Surface 1:}\quad T_1 &= 1000\,\text{K},\\
q_1 &= -1\,\text{W/cm}^2,\\
\epsilon_1 &= 0.6;\\
\text{Surface 2:}\quad \epsilon_2 &= 0.2;\\
\text{Surface 3:}\quad \epsilon_3 &= 0.3,\ \text{perfectly insulated.}
\end{aligned}$$

All configurations are gray and diffuse.

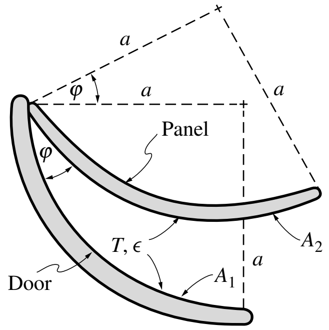



**5.10** Two pipes carrying hot combustion gases are enclosed in a cylindrical duct as shown. Assuming both pipes to be isothermal at 2000 K and diffusely emitting and reflecting ($\epsilon = 0.5$), and the duct wall to be isothermal at 500 K and diffusely emitting and reflecting ($\epsilon = 0.2$), determine the radiative heat loss from the pipes.




**5.11** A cubical enclosure has gray, diffuse walls which interchange energy. Four of the walls are isothermal at $T_s$ with emittance $\epsilon_s$, the other two are isothermal at $T_t$ with emittance $\epsilon_t$. Calculate the heat flux rates per unit time and area.

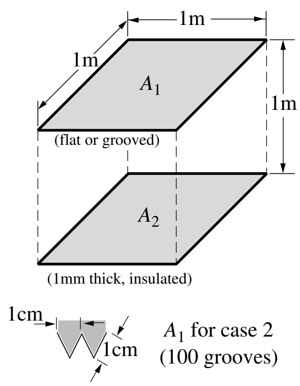
**5.12** During launch the heat rejector radiative panels of the Space Shuttle are folded against the inside of the Shuttle doors. During orbit the doors are opened and the panels are rotated out by an angle $\varphi$ as shown in the figure. Assuming door and panel can be approximated by infinitely long, isothermal quarter-cylinders of radius $a$ and emittance $\epsilon = 0.8$, calculate the necessary rotation angle $\varphi$ so that half the total energy emitted by panel (2) and door (1) escapes through the opening. At what opening angle will a maximum amount of energy be rejected? How much and why?




**5.13** Consider two $1 \times 1\,\text{m}^2$, thin, gray, diffuse plates located a distance $h = 1$ m apart. The temperature of the top plate is maintained at $T_1 = 1200$ K, whereas the bottom plate is initially at $T_2 = 300$ K and insulated on the outside. In case 1, the surface of the top plate is flat, whereas in case 2 grooves, whose dimensions are indicated below, have been machined in the plate's surface. In either case the surfaces are gray and diffuse, and the surroundings may be considered as black and having a temperature $T_\infty = 500$ K; convective heat transfer effects may be neglected.

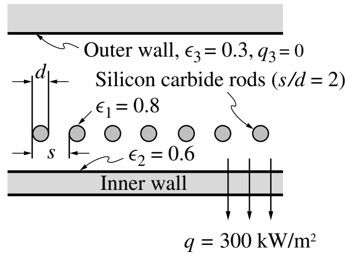
(*a*) Estimate the effect of the surface preparation of the top surface on the initial temperature change of the bottom plate ($dT_2/dt$ at $t = 0$).

(*b*) Justify, then use, a lumped-capacity analysis for the bottom plate to predict the history of temperature and heating rates of the bottom plate until steady state is reached.

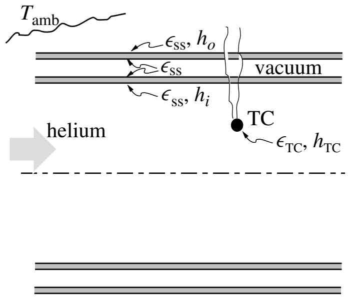



The following properties are known: top plate: $\epsilon_1 = 0.6$, $T_1 = 1200$ K; bottom plate: $T_2(t = 0) = 300$ K, $\epsilon_2 = 0.5$, $\rho_2 = 800\,\text{kg/m}^3$(*density*), $c_{p2} = 440$ J/kg K, $k_2 = 200$ W/m K.

**5.14** A row of equally spaced, cylindrical heating elements ($s = 2d$) is used to heat the inside of a furnace as shown. Assuming that the outer wall is made of firebrick with $\epsilon_3 = 0.3$ and is perfectly insulated, that the heating rods are made of silicon carbide ($\epsilon_1 = 0.8$), and that the inner wall has an emittance of $\epsilon_2 = 0.6$, what must the operating temperature of the rods be to supply a net heat flux of 300 kW/m$^2$ to the furnace, if the inner wall is at a temperature of 1300 K?

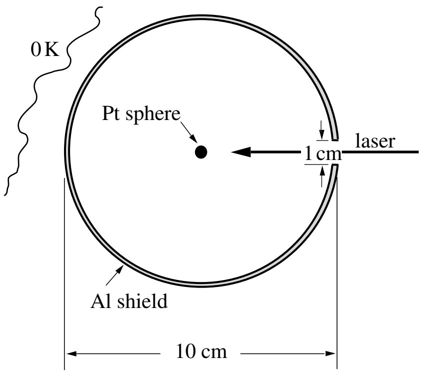



**5.15** A thermocouple used to measure the temperature of cold, low-pressure helium flowing through a long duct shows a temperature reading of 10 K. To minimize heat losses from the duct to the surroundings the duct is made of two concentric thin layers of stainless steel with an evacuated space in between (inner diameter $d_i$ = 2 cm, outer diameter $d_o$ = 2.5 cm; stainless layers very thin and of high conductivity). The emittance of the thermocouple is $\epsilon_{TC} = 0.6$, the convection heat transfer coefficient between helium and tube wall is $h_i$ = 5 W/m$^2$ K, and between thermocouple and helium is $h_{TC}$ = 2 W/m$^2$ K, and the emittance of the stainless steel is $\epsilon_{ss} = 0.2$ (gray and diffuse, all four surfaces). The free convection heat transfer coefficient between the outer tube and the surroundings at $T_{\rm amb}$ = 300 K is $h_o$ = 5 W/m$^2$ K. To determine the actual temperature of the helium,

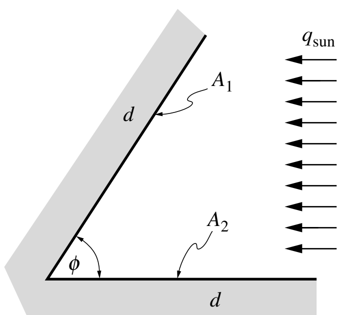



*(a)* Prepare an energy balance for the thermocouple.
*(b)* Prepare an energy balance for the heat loss through the duct wall (the only unknowns here should be $T_{\rm He}, T_i$, and $T_o$).
*(c)* Outline how to solve for the temperature of the helium (no need to carry out solution).
*(d)* Do you expect the thermocouple to be accurate? (Hint: Check the magnitudes of the terms in *(a)*.)

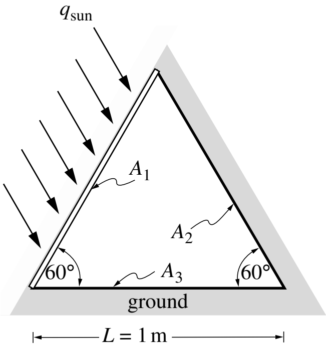
**5.16** During a materials processing experiment on the Space Shuttle (under microgravity conditions), a platinum sphere of 3 mm diameter is levitated in a large, cold black vacuum chamber. A spherical aluminum shield (with a circular cutout) is placed around the sphere as shown, to reduce heat loss from the sphere. Initially, the sphere is at 200 K and is suddenly irradiated with a laser providing an irradiation of 100 W (normal to beam) to raise its temperature rapidly to its melting point (2741 K). Determine the time required to reach the melting point. You may assume the platinum and aluminum to be gray and diffuse ($\epsilon_{Pt} = 0.25$, $\epsilon_{Al} = 0.1$), the sphere to be essentially isothermal at all times, and the shield to have zero heat capacity.




**5.17** Two identical circular disks are connected at one point of their periphery by a hinge. The configuration is then opened by an angle $\phi$ as shown in the figure. Assuming the opening angle to be $\phi = 60°$, $d$ = 1 m, calculate the average equilibrium temperature for each of the two disks, with solar radiation entering the configuration parallel to Disk 2 with a strength of $q_{\rm sun}$ = 1000 W/m$^2$. Disk 1 is gray and diffuse with $\alpha = \epsilon = 0.5$, Disk 2 is black. Both disks are insulated.



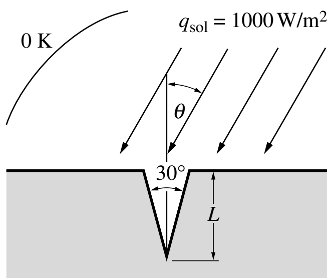
**5.18** A long greenhouse has the cross-section of an equilateral triangle as shown. The side exposed to the sun consists of a thin sheet of glass ($A_1$) with reflectivity $\rho_1 = 0.1$. The glass may be assumed perfectly transparent to solar radiation, and totally opaque to radiation emitted inside the greenhouse. The other side wall ($A_2$) is opaque with emittance $\epsilon_2 = 0.2$, while the floor ($A_3$) has $\epsilon_3 = 0.8$. All surfaces reflect diffusely. For simplicity, you may assume surfaces $A_1$ and $A_2$ to be perfectly insulated, while the floor loses heat to the ground according to



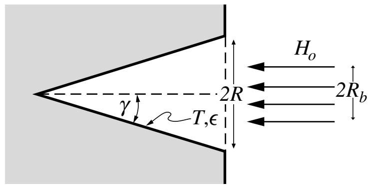

$$q_{3,\text{conduction}} = U(T_3 - T_\infty)$$

where $T_\infty = 280\,\text{K}$ is the temperature of the ground, and $U = 19.5\,\text{W/m}^2\,\text{K}$ is an overall heat transfer coefficient. Determine the temperatures of all three surfaces for the case that the sun shines onto the greenhouse with strength $q_{\text{sun}} = 1000\,\text{W/m}^2$ in a direction parallel to surface $A_2$.

**5.19** A long, black V-groove is irradiated by the sun as shown. Assuming the groove to be perfectly insulated, and radiation to be the only mode of heat transfer, determine the average groove temperature as a function of solar incidence angle $\theta$ (give values for $\theta = 0°,\ 15°,\ 30°,\ 60°,\ 90°$). For simplicity the V-groove wall may be taken as a single zone.




**5.20** Consider the conical cavity shown (radius of opening $R$, opening angle $\gamma = 30°$), which has a gray, diffusely reflective coating ($\epsilon = 0.6$) and is perfectly insulated. The cavity is irradiated by a collimated beam of strength $H_0$ and radius $R_b = 0.5R$).

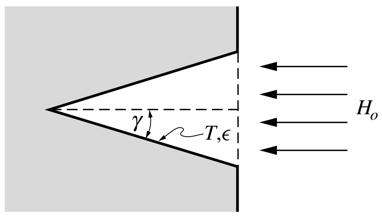



(*a*) Using a single node analysis, develop an expression relating $H_0$ to the average cavity temperature $T$.

(*b*) For a more accurate analysis a two-node analysis is to be performed. What nodes would you choose? Develop expressions for the necessary view factors in terms of known ones (including those given in App. D) and surface areas, then relate the two temperatures to $H_0$.

(*c*) Qualitatively, what happens to the cavity's overall average temperature, if the beam is turned away by an angle $\alpha$?

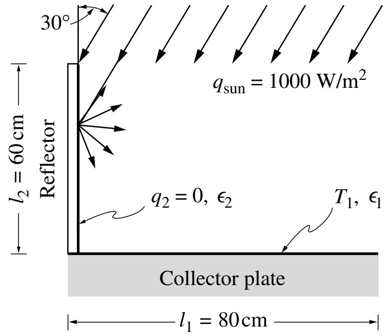
**5.21** A (simplified) radiation heat flux meter consists of a conical cavity coated with a gray, diffuse material, as shown in the figure. To measure the radiative heat flux, the cavity is perfectly insulated.



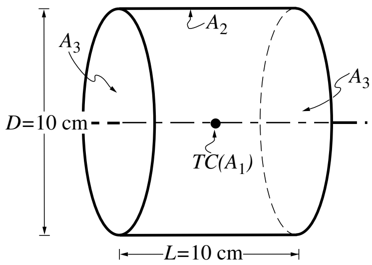

(*a*) Develop an expression that relates the flux, $H_o$, to the cavity temperature, $T$.

(*b*) If the cavity is turned away from the incoming flux by an angle $\alpha$, what happens to the cavity temperature?

**5.22** A very long solar collector plate is to collect energy at a temperature of $T_1 = 350\,\text{K}$. To improve its performance for off-normal solar incidence, a highly reflective surface is placed next to the collector as shown in the adjacent figure. How much energy (per unit length) does the collector plate collect for a solar incidence angle of 30°? For simplicity you may make the following assumptions: The collector is isothermal and gray-diffuse with emittance $\epsilon_1 = 0.8$; the reflector is gray-diffuse with $\epsilon_2 = 0.1$, and heat losses from the reflector by convection as well as all losses from the collector ends may be neglected.



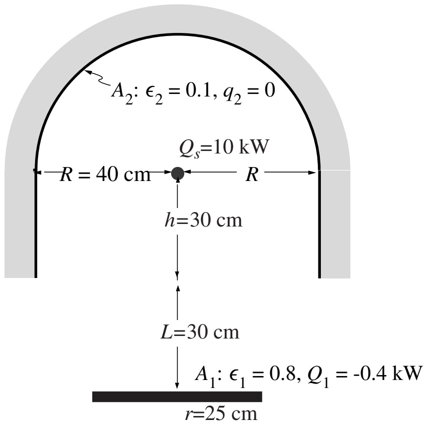
**5.23** A thermocouple (approximated by a 1 mm diameter sphere with gray-diffuse emittance $\epsilon_1 = 0.5$) is suspended inside a tube through which a hot, nonparticipating gas at $T_g = 2000\,\text{K}$ is flowing. In the vicinity of the thermocouple the tube temperature is known to be $T_2 = 1000\,\text{K}$ (wall emittance $\epsilon_2 = 0.5$). For the purpose of this problem you may assume both ends of the tube to be closed with a black surface at the temperature of the gas, $T_3 = 2000\,\text{K}$. Again, for the purpose of this problem, you may assume that the thermocouple gains a heat flux of $10^4\,\text{W/m}^2$ of thermocouple surface area, which it must reject again in the form of radiation. Estimate the temperature of the thermocouple. Hints:




(*a*) Treat the tube ends together as a single surface $A_3$.

(*b*) Note that the thermocouple is small, i.e., $F_{x-1} \ll 1$.

**5.24** A small spherical heat source outputting $Q_s = 10$ kW power, spreading equally into all directions, is encased in a reflector as shown, consisting of a hemisphere of radius $R = 40\,\text{cm}$, plus a ring of radius $R$ and height $h = 30\,\text{cm}$. The arrangement is used to heat a disk of radius = 25 cm a distance of $L = 30\,\text{cm}$ below the reflector. All surfaces are gray and diffuse, with emittances of $\epsilon_1 = 0.8$ and $\epsilon_2 = 0.1$. Reflector $A_2$ is insulated.

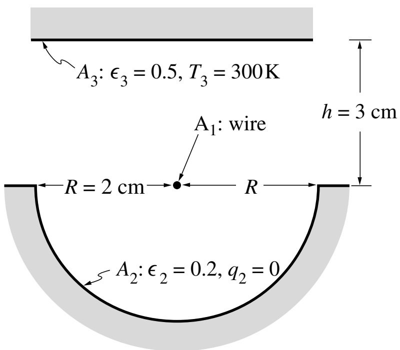



(*a*) Determine (per unit area of receiving surface) the irradiation from heat source to reflector and to disk;

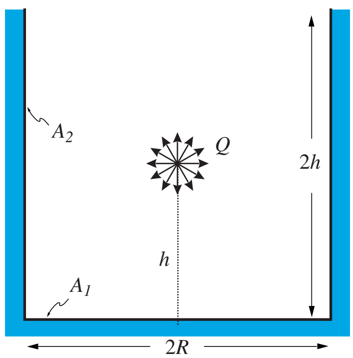
(*b*) all relevant view factors; and

(*c*) the temperature of the disk, if 0.4 kW of power is extracted from the disk.

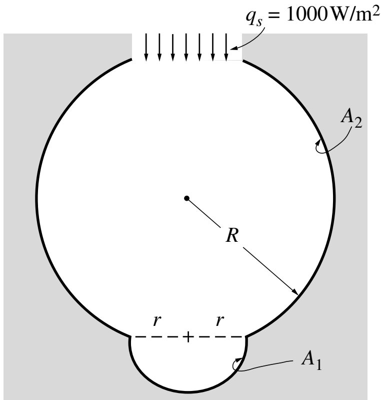
**5.25** A long thin black heating wire radiates 300 W per cm length of wire and is used to heat a flat surface by thermal radiation. To increase its efficiency the wire is surrounded by an insulated half-cylinder as shown in the figure. Both surfaces are gray and diffuse with emittances $\epsilon_2$ and $\epsilon_3$, respectively. What is the net heat flux at Surface 3? How does this compare with the case without cylinder?
Hint: You may either treat the heating wire as a thin cylinder whose radius you eventually shrink to zero, or treat radiation from the wire as external radiation (the second approach being somewhat simpler).



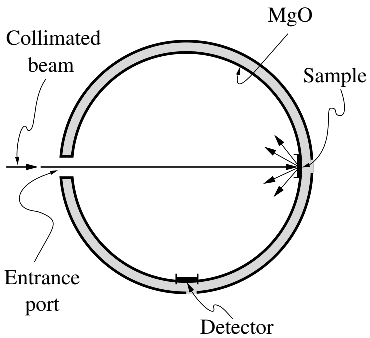

**5.26** Consider the configuration shown, consisting of a cylindrical cavity $A_2$, a circular disk $A_1$ at the bottom, and a small spherical radiation source (blackbody at 4000K) of strength $Q = 10{,}000$ W as shown ($R = 10$ cm, $h = 10$ cm). The cylinder wall $A_2$ is covered with a gray, diffuse material with $\epsilon_2 = 0.1$, and is perfectly insulated. Surface $A_1$ is kept at a constant temperature of 400 K. No other external surfaces or sources affect the heat transfer. Assuming surface $A_1$ to be gray and diffuse with $\epsilon_1 = 0.3$ determine the amount of heat that needs to be removed from $A_1$ ($Q_1$).



**5.27** Determine $F_{1-2}$ for the rotationally symmetric configuration shown in the figure (i.e., a big sphere, $R = 13$ cm, with a circular hole, $r = 5$ cm, and a hemispherical cavity, $r = 5$ cm). Assuming Surface 2 to be gray and diffuse ($\epsilon = 0.5$) and insulated and Surface 1 to be black and also insulated, what is the average temperature of the black cavity if collimated irradiation of 1000 W/m$^2$ is penetrating through the hole as shown?



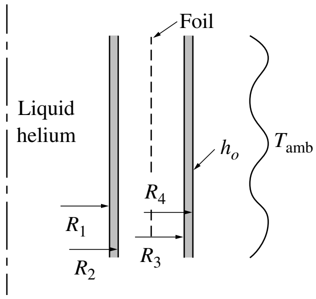

**5.28** An integrating sphere (a device to measure surface properties) is 10 cm in radius. It contains on its inside wall a 1 cm$^2$ black detector, a $1 \times 2$ cm entrance port, and a $1 \times 1$ cm sample as shown. The remaining portion of the sphere is smoked with magnesium oxide having a short-wavelength reflectance of 0.98, which is almost perfectly diffuse. A collimated beam of radiant energy (i.e., all energy is contained within a very small cone of solid angles) enters the sphere through the entrance port, falls onto the sample, and then is reflected and interreflected, giving rise to a sphere wall radiosity and irradiation. Radiation emitted from the walls is not detected because the source radiation is chopped, and the detector–amplifier system responds only to the chopped radiation. Find the fractions of the chopped incoming radiation that are




(*a*) lost out the entrance port,
(*b*) absorbed by the MgO-smoked wall, and
(*c*) absorbed by the detector.

[Item (*c*) is called the "sphere efficiency."]

**5.29** The side wall of a flask holding liquid helium may be approximated as a long double-walled cylinder as shown in the adjacent sketch. The container walls are made of 1 mm thick stainless steel ($k = 15$ W/m K, $\epsilon = 0.2$), and have outer radii of $R_2 = 10$ cm and $R_4 = 11$ cm. The space between walls is evacuated, and the outside is exposed to free convection with the ambient at $T_{\text{amb}} = 20°$C and a heat transfer coefficient of $h_o = 10$ W/m$^2$ K (for the combined effects of free convection and radiation). It is reasonable to assume that the temperature of the inner wall is at liquid helium temperature, or $T(R_2) = 4$ K.




(*a*) Determine the heat gain by the helium, per unit length of flask.
(*b*) To reduce the heat gain a thin silver foil ($\epsilon = 0.02$) is placed midway between the two walls. How does this affect the heat flux?

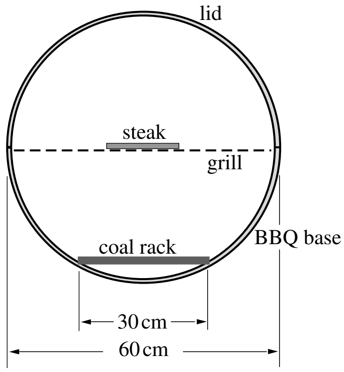
For the sake of the problem, you may assume both steel and silver to be diffuse reflectors.

**5.30** Repeat Problem 5.6, breaking up the sidewall into four equal ring elements. Use the view factors calculated in Problem 4.25 together with program `graydiffxch` of Appendix F.

**5.31** The inside surfaces of a furnace in the shape of a parallelepiped with dimensions 1 m × 2 m × 4 m are to be broken up into 28 1 m × 1 m subareas. The gray-diffuse side walls (of dimension 1 m × 2 m and 1 m × 4 m) have emittances of $\epsilon_s = 0.7$ and are perfectly insulated, the bottom surface has an emittance of $\epsilon_b = 0.9$ and a temperature $T_b = 1600$ K, while the top's emittance is $\epsilon_t = 0.2$ and its temperature is $T_t = 500$ K. Using the view factors calculated in Problem 4.26 and program `graydiffxch` of Appendix

F, calculate the heating/cooling requirements for bottom and top surfaces, as well as the temperature distribution along the side walls.

**5.32** For your Memorial Day barbecue you would like to broil a steak on your backyard BBQ, which consists of a base unit in the shape of a hemisphere ($D$= 60 cm), fitted with a disk-shaped coal rack, and a disk-shaped grill, as shown in the sketch. Hot coal may be assumed to cover the entire floor of the unit, with uniform temperature $T_c = 1200\,\text{K}$, and an emittance of $\epsilon_c = 1$. The side wall is soot-covered and black on the inside, but has an outside emittance of $\epsilon_o = 0.5$. The steak (modeled as a $d_s = 15\,\text{cm}$ disk, 1 cm thick, emittance $\epsilon_s = 0.8$, initially at $T_s = 280\,\text{K}$) is now placed on the grill (assumed to be so lightweight as to be totally transparent and not participating in the heat transfer). The environment is at 300 K, and free convection may be neglected.




*(a)* Assuming that the lid is not placed on top of the unit, estimate the initial heating rates on the two surfaces of the steak.

*(b)* How would the heating rates change, if the lid (also a hemisphere) is put on ($\epsilon_i = \epsilon_o = 0.5$)? Could one achieve a more even heating rate (top and bottom) if the emittance of the inside surface is increased or decreased?

Note: Part *(b)* will be quite tedious, unless program `graydiffxch` of Appendix F is used (which, in turn, will require iteration or a little trickery).

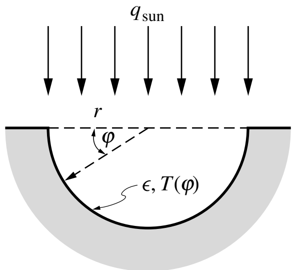
**5.33** Consider Configuration 33 in Appendix D with $h = w$. The bottom wall is at constant temperature $T_1$ and has emittance $\epsilon_1$; the side wall is at $T_2 = \text{const}$ and $\epsilon_2$. Find the exact expression for $q_1(x)$ if $\epsilon_2 = 1$.

**5.34** An infinitely long half-cylinder is irradiated by the sun as shown in the figure, with $q_{\text{sun}} = 1000\,\text{W/m}^2$. The inside of the cylinder is gray and diffuse, the outside is insulated. There is no radiation from the background. Determine the equilibrium temperature distribution along the cylinder periphery,

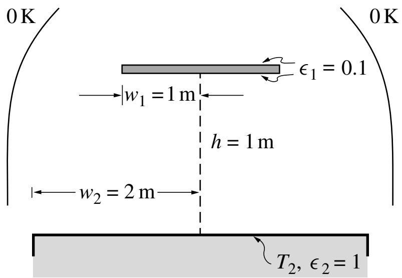
(*a*) using four isothermal zones of 45° each,

(*b*) using the exact relations.

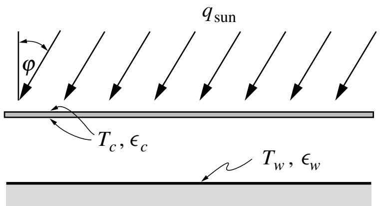
Hint: Use differentiation as in the kernel approximation method.




**5.35** To calculate the net heat loss from a part of a spacecraft, this part may be approximated by an infinitely long black plate at temperature $T_2 = 600\,\text{K}$, as shown. Parallel to this plate is another (infinitely long) thin plate that is gray and emits/reflects diffusely with the same emittance $\epsilon_1$ on both sides. You may assume the surroundings to be black at 0 K. Calculate the net heat loss from the black plate.




**5.36** A large isothermal surface (exposed to vacuum, temperature $T_w$, diffuse-gray emittance $\epsilon_w$) is irradiated by the sun. To reduce the heat gain/loss from the surface, a thin copper shield (emittance $\epsilon_c$ and initially at temperature $T_{c0}$) is placed between surface and sun as shown in the figure.



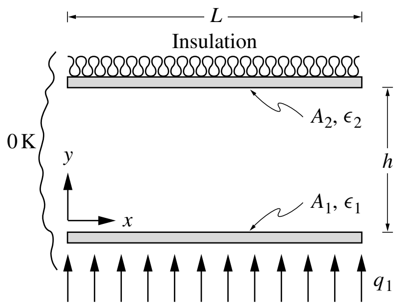

(*a*) Determine the relationship between $T_c$ and time $t$ (it is sufficient to leave the answer in implicit form with an unsolved integral).

(*b*) Give the steady state temperature for $T_c$ (i.e., for $t \to \infty$).

(*c*) Briefly discuss qualitatively the following effects:

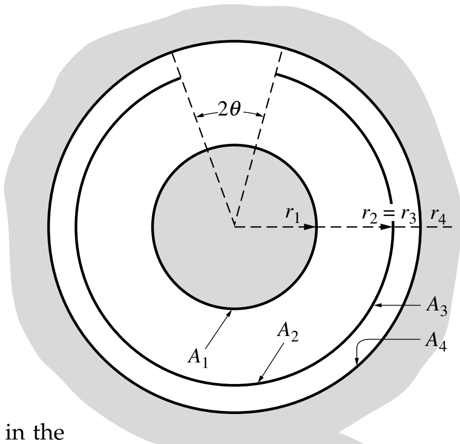
(*i*) The shield is replaced by a moderately thick slab of styrofoam coated on both sides with a very thin layer of copper.
(*ii*) The surfaces are finite in size.

**5.37** Consider two infinitely long, parallel, black plates of width $L$ as shown. The bottom plate is uniformly heated electrically with a heat flux of $q_1$ = const, while the top plate is insulated. The entire configuration is placed into a large cold environment.



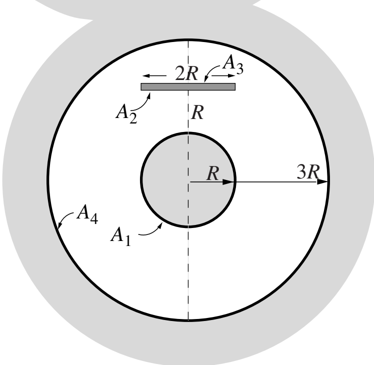

(*a*) Determine the governing equations for the temperature variation across the plates.
(*b*) Find the solution by the kernel substitution method. To avoid tedious algebra, you may leave the final result in terms of *two* constants to be determined, as long as you outline carefully how these constants may be found.
(*c*) If the plates are gray and diffuse with emittances $\epsilon_1$ and $\epsilon_2$, how can the temperature distribution be determined, using the solution from part *(b)*?

**5.38** To reduce heat transfer between two infinite concentric cylinders a third cylinder is placed between them as shown in the figure. The center cylinder has an opening of half-angle $\theta$. The inner cylinder is black and at temperature $T_1$ = 1000 K, while the outer cylinder is at $T_4$ = 300 K. The outer cylinder and both sides of the shield are coated with a reflective material, such that $\epsilon_c = \epsilon_2 = \epsilon_3 = \epsilon_4$. Determine the heat loss from the inner cylinder as function of coating emittance $\epsilon_c$, using



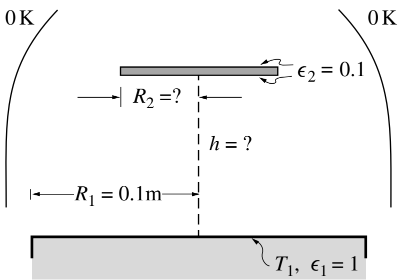

(*a*) the net radiation method,
(*b*) the network analogy.

**5.39** Consider the two long concentric cylinders as shown in the figure. Between the two cylinders is a long, thin flat plate as also indicated. The inner cylinder is black and generating heat on its inside in the amount of $Q_1'$ = 1 kW/m length of the cylinder, which must be removed by radiation. The plate is gray and diffuse with emittance $\epsilon_2 = \epsilon_3 = 0.5$, while the outer cylinder is black and cold ($T_4$ = 0 K). Determine the temperature of the inner cylinder, using




(*a*) the net radiation method,
(*b*) the network analogy.

**5.40** An isothermal black disk at $T_1$ = 500*K* is flush with the outer surface of a spacecraft and is thus exposed to outer space. To minimize heat loss from the disk a disk-shaped radiation shield is placed coaxially and parallel to the disk as shown; the shield radius is $R_2$ (which may be smaller or larger than $R_1$), and its distance from the black disk is a variable $h$. Determine an expression for the heat loss from the black disk as a function of shield radius and distance, using




(*a*) the net radiation method,
(*b*) the network analogy.