CHAPTER 2

RADIATIVE PROPERTY PREDICTIONS FROM ELECTROMAGNETIC WAVE THEORY

2.1 INTRODUCTION

The basic radiative properties of surfaces forming an enclosure, i.e., emissivity, absorptivity, reflectivity, and transmissivity, must be known before any radiative heat transfer calculations can be carried out. Many of these properties vary with incoming direction, outgoing direction, and wavelength, and must usually be found through experiment. However, for pure, perfectly smooth surfaces these properties may be calculated from classical electromagnetic wave theory.[1] These predictions make experimental measurements unnecessary for some cases, and help interpolating as well as extrapolating experimental data in many other situations.

The first important discoveries with respect to light were made during the seventeenth century, such as the law of refraction (by Snell in 1621), the decomposition of white light into monochromatic components (by Newton in 1666), and the first determination of the speed of light (by Römer in 1675). However, the true nature of light was still unknown: The corpuscular theory (suggested by Newton) competed with a rudimentary wave theory. Not until the early nineteenth century was the wave theory finally accepted as the correct model for the description of light. Young proposed a model of purely transverse waves in 1817 (as opposed to the model prevalent until then of purely longitudinal waves), followed by Fresnel's comprehensive treatment of diffraction and other optical phenomena. In 1845 Faraday proved experimentally that there was a connection between magnetism and light. Based on these experiments, Maxwell presented in 1861 his famous set of equations for the complete description of electromagnetic waves, i.e., the interaction between electric and magnetic fields. Their success was truly remarkable, in particular because the theories of quantum mechanics and special relativity, with

[1]The National Institute of Standards and Technology (NIST, formerly NBS) has recommended to reserve the ending "-ivity" for radiative properties of pure, perfectly smooth materials (the ones discussed in this chapter), and "-ance" for rough and contaminated surfaces. Most real surfaces fall into the latter category, discussed in Chapter 3. While we will follow this convention throughout this book, the reader should be aware that many researchers in the field employ endings according to their own personal preference.

which electromagnetic waves are so strongly related, were not discovered until half a century later. To this day Maxwell's equations remain the basis for the study of light.*

2.2 THE MACROSCOPIC MAXWELL EQUATIONS

The original form of Maxwell's equations is based on electrical experiments available at the time, with their very coarse temporal and spatial resolution. Thus any of these measurements were spatial averages taken over many layers of atoms and temporal averages over many oscillations of an electromagnetic wave. For this reason the original set of equations is termed *macroscopic*. Today we know that electromagnetic waves interact with matter at the molecular level, with strong field fluctuations over each wave period. Therefore, more detailed treatises on optics and electromagnetic waves now generally start with a *microscopic* description of the wave equations, for example, the book by Stone [1]. While there is little disagreement in the literature on the microscopic equations, the macroscopic equations often differ somewhat from book to book, depending on assumptions made and constitutive relations used. Following the development of Stone [1], we may state the *macroscopic Maxwell equations* as

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho_f, \tag{2.1}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0, \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{2.3}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma_e \mathbf{E}, \tag{2.4}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the *electric field* and *magnetic field* vectors, respectively, ϵ is the electrical permittivity, μ is the magnetic permeability, σ_e is the electrical conductivity, and ρ_f is the charge density due to free electrons, which is generally assumed to be related to the electric field by the equation

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot (\sigma_e \mathbf{E}). \tag{2.5}$$

The *phenomenological coefficients* σ_e, μ, and ϵ depend on the medium under consideration, but may be assumed independent of the fields (for a *linear medium*) and independent of position and direction (for a *homogeneous and isotropic medium*); they may, however, depend on the wavelength of the electromagnetic waves [2].

2.3 ELECTROMAGNETIC WAVE PROPAGATION IN UNBOUNDED MEDIA

We seek a solution to the above set of equations in the form of a wave. The most general form of a *time-harmonic field* (i.e., a wave of constant frequency or wavelength) is

$$\mathbf{F} = \mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t = \mathbf{A} \cos 2\pi \nu t + \mathbf{B} \sin 2\pi \nu t, \tag{2.6}$$

where ω is the *angular frequency* (in radians/s), and $\nu = \omega/2\pi$ is the frequency in cycles per second. While a little less convenient, we will use the cyclical frequency ν in the following

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***James Clerk Maxwell (1831–1879)**
Scottish physicist. After attending the University of Edinburgh he obtained a mathematics degree from Trinity College in Cambridge. Following an appointment at Kings College in London he became the first Cavendish Professor of Physics at Cambridge. While best known for his electromagnetic theory, he made important contributions in many fields, such as thermodynamics, mechanics, and astronomy.

development in order to limit the number of different spectral variables employed in this book. When it comes to the time-harmonic solution of linear partial differential equations, it is usually advantageous to introduce a *complex representation* of the real field. Thus, setting

$$\mathbf{F}_c = \overline{\mathbf{F}}_c\, e^{2\pi i\nu t}, \quad \overline{\mathbf{F}}_c = \mathbf{A} - i\mathbf{B}, \tag{2.7}$$

where $\overline{\mathbf{F}}_c$ is the time-average of the complex field, results in

$$\mathbf{F} = \Re\{\mathbf{F}_c\}, \tag{2.8}$$

where the symbol $\Re$ denotes that the real part of the complex vector $\mathbf{F}_c$ is to be taken. Since the Maxwell equations are linear in the fields $\mathbf{E}$ and $\mathbf{H}$, one may solve them for their complex fields, and then extract their real parts after a solution has been found. Therefore, setting

$$\mathbf{E} = \Re\{\mathbf{E}_c\} = \Re\{\overline{\mathbf{E}}_c\, e^{2\pi i\nu t}\}, \tag{2.9}$$

$$\mathbf{H} = \Re\{\mathbf{H}_c\} = \Re\{\overline{\mathbf{H}}_c\, e^{2\pi i\nu t}\}, \tag{2.10}$$

results in

$$\nabla \cdot (\gamma \mathbf{E}_c) = 0, \tag{2.11}$$

$$\nabla \cdot \mathbf{H}_c = 0, \tag{2.12}$$

$$\nabla \times \mathbf{E}_c = -2\pi i\nu\mu \mathbf{H}_c, \tag{2.13}$$

$$\nabla \times \mathbf{H}_c = 2\pi i\nu\gamma \mathbf{E}_c, \tag{2.14}$$

where

$$\gamma = \epsilon - i\frac{\sigma_e}{2\pi\nu} \tag{2.15}$$

is the *complex permittivity*. If $\gamma \neq 0$, then it can be shown that the solution to the above set of equations must be *plane waves*, i.e., the electric and magnetic fields are *transverse* to the direction of propagation (have no component in the direction of propagation). Thus, the solution of equations (2.11) through (2.14) will be of the form

$$\mathbf{E} = \Re\{\overline{\mathbf{E}}_c\, e^{2\pi i\nu t}\} = \Re\{\mathbf{E}_0\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)}\}, \tag{2.16}$$

$$\mathbf{H} = \Re\{\overline{\mathbf{H}}_c\, e^{2\pi i\nu t}\} = \Re\{\mathbf{H}_0\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)}\}, \tag{2.17}$$

where $\mathbf{r}$ is a vector pointing to an arbitrary point in space, $\mathbf{w}$ is known as the *wave vector*[2] and $\mathbf{E}_0$ and $\mathbf{H}_0$ are constant vectors. In general $\mathbf{w}$ is a complex vector,

$$\mathbf{w} = \mathbf{w}' - i\mathbf{w}'', \tag{2.18}$$

where $\mathbf{w}'$ turns out to be a vector whose magnitude is the *wavenumber*, and $\mathbf{w}''$ is known as the *attenuation vector*. Employing equation (2.18), equations (2.16) and (2.17) may be rewritten as

$$\mathbf{E}_c = \mathbf{E}_0\, e^{-2\pi\mathbf{w}''\cdot\mathbf{r}}\, e^{-2\pi i(\mathbf{w}'\cdot\mathbf{r}-\nu t)}, \tag{2.19}$$

$$\mathbf{H}_c = \mathbf{H}_0\, e^{-2\pi\mathbf{w}''\cdot\mathbf{r}}\, e^{-2\pi i(\mathbf{w}'\cdot\mathbf{r}-\nu t)}. \tag{2.20}$$

Thus, the complex electric and magnetic fields have local amplitude vectors $\mathbf{E}_0\, e^{-2\pi\mathbf{w}''\cdot\mathbf{r}}$ and $\mathbf{H}_0\, e^{-2\pi\mathbf{w}''\cdot\mathbf{r}}$ and an oscillatory part $e^{-2\pi i(\mathbf{w}'\cdot\mathbf{r}-\nu t)}$ with *phase angle* $\phi = 2\pi(\mathbf{w}' \cdot \mathbf{r} - \nu t)$. The position vector $\mathbf{r}$ may be considered to have two components: one parallel to $\mathbf{w}'$, and the other perpendicular to it. The vector product $\mathbf{w}' \cdot \mathbf{r}$ is constant for all vectors $\mathbf{r}$ that have the same component parallel to $\mathbf{w}'$, i.e., on planes normal to the vector $\mathbf{w}'$; these planes are known as *planes of equal phase*. To see how the wave travels let us look at the phase angle at two different times and locations (Fig. 2-1). First, consider the point $\mathbf{r} = 0$ at time $t = 0$ with a zero phase

[2]The present definition of the *wave vector* differs by a factor of 2π and in name from the definition $\mathbf{k} = 2\pi\mathbf{w}$ in most optics texts in order to conform with our definition of *wavenumber*.

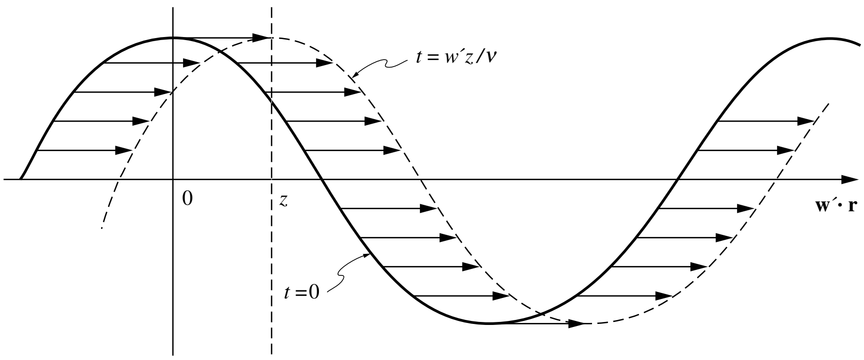

FIGURE 2-1
Phase propagation of an electromagnetic wave.

angle. Second, consider another point a distance z away into the direction of $\mathbf{w}'$; we see that the phase angle is zero at that point when $t = |\mathbf{w}'|z/\nu$. Thus, the phase velocity with which the wave travels from one point to the other is $c = z/t = \nu/w'$. We conclude that the wave propagates into the direction of $\mathbf{w}'$, and that the vector's magnitude, w', is equal to the wavenumber η. Examining the amplitude vectors we see that $\mathbf{w}'' \cdot \mathbf{r} = \text{const}$ are *planes of equal amplitude*, and that the amplitude of the fields diminishes into the direction of $\mathbf{w}''$. If planes of equal phase and equal amplitude coincide (i.e., if $\mathbf{w}'$ and $\mathbf{w}''$ are parallel) we say the wave is *homogeneous*, otherwise the wave is said to be *inhomogeneous*. Since $\mathbf{E}_0$ and $\mathbf{w}$ are independent of position, we can substitute equation (2.19) into equation (2.11) and, assuming γ to be also invariant with space, find that

$$\begin{aligned}\nabla \cdot (\gamma \mathbf{E}_c) &= \gamma \nabla \cdot \left(\mathbf{E}_0\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)}\right) = \gamma \mathbf{E}_0 \cdot \nabla \left(e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)}\right) \\ &= \gamma \mathbf{E}_0\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)} \cdot \nabla(-2\pi i \mathbf{w}\cdot\mathbf{r}) = -2\pi i \gamma \mathbf{w}\cdot\mathbf{E}_0\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)} = 0. \end{aligned} \tag{2.21}$$

Similarly, substituting equation (2.19) into equation (2.13) results in

$$\begin{aligned}\nabla \times \mathbf{E}_c &= \nabla \times \left(\mathbf{E}_0\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)}\right) = \nabla\left(e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)}\right) \times \mathbf{E}_0 \\ &= -2\pi i \mathbf{w}\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)} \times \mathbf{E}_0 = -2\pi i \nu \mu \mathbf{H}_0\, e^{-2\pi i(\mathbf{w}\cdot\mathbf{r}-\nu t)}. \end{aligned} \tag{2.22}$$

Thus, the partial differential equations (2.11) through (2.14) may be replaced by a set of algebraic equations,

$$\mathbf{w} \cdot \mathbf{E}_0 = 0, \tag{2.23}$$

$$\mathbf{w} \cdot \mathbf{H}_0 = 0, \tag{2.24}$$

$$\mathbf{w} \times \mathbf{E}_0 = \nu \mu \mathbf{H}_0, \tag{2.25}$$

$$\mathbf{w} \times \mathbf{H}_0 = -\nu \gamma \mathbf{E}_0. \tag{2.26}$$

It is clear from equations (2.23) and (2.24) that both $\mathbf{E}_0$ and $\mathbf{H}_0$ are perpendicular to $\mathbf{w}$, and it follows then from equations (2.25) and (2.26) that they are also perpendicular to each other.[3] If the wave is homogeneous, then $\mathbf{w}$ points into the direction of wave propagation, and the electric and magnetic fields lie in planes perpendicular to this direction, as indicated in Fig. 2-2.

It remains to relate the complex wave vector $\mathbf{w}$ to the properties of the medium. Taking the vector product of equation (2.25) with $\mathbf{w}$ and recalling the vector identity derived, for example, in Wylie [3],

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}), \tag{2.27}$$

[3] Remember that all three vectors are complex and, therefore, the interpretation of "perpendicular" is not straightforward.

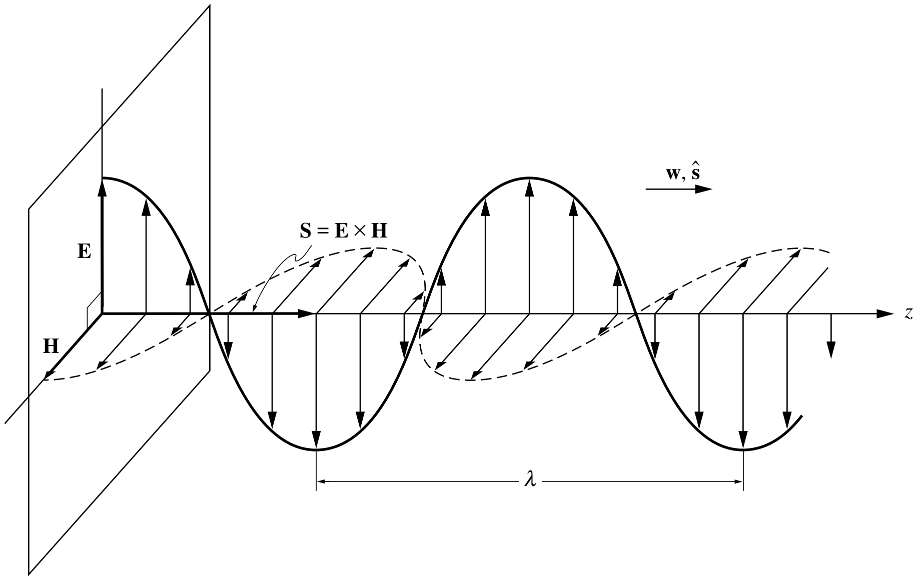

FIGURE 2-2
Electric and magnetic fields of a homogeneous wave.

which leads to

$$\mathbf{w} \times (\mathbf{w} \times \mathbf{E}_0) = \mathbf{w}(\mathbf{w} \cdot \mathbf{E}_0) - \mathbf{E}_0 \mathbf{w} \cdot \mathbf{w} = \nu\mu \mathbf{w} \times \mathbf{H}_0 = -\nu^2 \mu\gamma \mathbf{E}_0,$$

or

$$\mathbf{w} \cdot \mathbf{w} = \nu^2 \mu\gamma. \tag{2.28}$$

If the wave travels through vacuum there can be no attenuation ($\mathbf{w}'' = 0$) and $\mu = \mu_0$, $\gamma = \epsilon_0$. We thus obtain the *speed of light in vacuum* as

$$c_0 = \nu/w' = \nu/\sqrt{\mathbf{w} \cdot \mathbf{w}} = \frac{1}{\sqrt{\epsilon_0 \mu_0}}. \tag{2.29}$$

It is customary to introduce the *complex index of refraction*

$$m = n - ik \tag{2.30}$$

into equation (2.28) such that

$$\mathbf{w} \cdot \mathbf{w} = \nu^2 \mu\gamma = \nu^2 \epsilon_0 \mu_0 \left(\frac{\epsilon\mu}{\epsilon_0 \mu_0} - i \frac{\sigma_e \mu}{2\pi\nu\epsilon_0 \mu_0} \right) = \eta_0^2 m^2, \tag{2.31}$$

where $\eta_0 = \nu/c_0$ is the wavenumber of a wave with frequency ν and phase velocity c_0, i.e., of a wave traveling through vacuum. This definition of m demands that

$$n^2 - k^2 = \frac{\epsilon\mu}{\epsilon_0 \mu_0} = \epsilon\mu c_0^2, \tag{2.32}$$

$$nk = \frac{\sigma_e \mu}{4\pi\nu\epsilon_0 \mu_0} = \frac{\sigma_e \mu \lambda_0 c_0}{4\pi}, \tag{2.33}$$

where $\lambda_0 = 1/\eta_0 = c_0/\nu$ is the wavelength for the wave in vacuum. Equations (2.32) and (2.33) may be solved for the *refractive index n* and the *absorptive index*[4] *k* as

[4]The *absorptive index* is often referred to as *extinction coefficient* in the literature. Since the term *extinction coefficient* is also employed for another, related property we will always use the term *absorptive index* in this book to describe the imaginary part of the index of refraction.

$$n^2 = \frac{1}{2}\left[\frac{\epsilon}{\epsilon_0} + \sqrt{\left(\frac{\epsilon}{\epsilon_0}\right)^2 + \left(\frac{\lambda_0\sigma_e}{2\pi c_0\epsilon_0}\right)^2}\right], \tag{2.34}$$

$$k^2 = \frac{1}{2}\left[-\frac{\epsilon}{\epsilon_0} + \sqrt{\left(\frac{\epsilon}{\epsilon_0}\right)^2 + \left(\frac{\lambda_0\sigma_e}{2\pi c_0\epsilon_0}\right)^2}\right], \tag{2.35}$$

where we have assumed the material to be nonmagnetic, or $\mu = \mu_0$. These relations do not reveal the frequency (wavelength) dependence of the complex index of refraction, since the phenomenological coefficientss ϵ and σ_e may depend on frequency. If the wave is homogeneous the wave vector may be written as $\mathbf{w} = (w' - iw'')\hat{\mathbf{s}}$, where $\hat{\mathbf{s}}$ is a unit vector in the direction of wave propagation, and it follows from equation (2.31) that $w' - iw'' = \eta_0(n - ik)$, so that the electric and magnetic fields reduce to

$$\mathbf{E}_c = \mathbf{E}_0\, e^{-2\pi\eta_0 kz}\, e^{-2\pi i\eta_0 n(z - c_0 t/n)}, \tag{2.36}$$

$$\mathbf{H}_c = \mathbf{H}_0\, e^{-2\pi\eta_0 kz}\, e^{-2\pi i\eta_0 n(z - c_0 t/n)}, \tag{2.37}$$

where $z = \hat{\mathbf{s}}\cdot\mathbf{r}$ is distance along the direction of propagation. For a nonvacuum, the *phase velocity* c of an electromagnetic wave is[5]

$$c = \frac{c_0}{n}. \tag{2.38}$$

Further, the field strengths decay exponentially for nonzero values of k; thus, the absorptive index gives an indication of how quickly a wave is absorbed within the medium. Inspection of equation (2.35) shows that a large absorptive index k corresponds to a large electrical conductivity σ_e: Electromagnetic waves tend to be attenuated rapidly in good electrical conductors, such as metals, but are often transmitted with weak attenuation in media with poor electrical conductivity, or *dielectrics*, such as glass.

The magnitude and direction of the transfer of electromagnetic energy is given by the Poynting vector, i.e., a vector of magnitude EH pointing into the direction of propagation (cf. Fig. 2-2),[6]

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \Re\{\mathbf{E}_c\}\times\Re\{\mathbf{H}_c\}. \tag{2.39}$$

The instantaneous value for the Poynting* vector is a rapidly varying function of time. Of greater value to the engineer is a time-averaged value of the Poynting vector, say

$$\overline{\mathbf{S}} = \frac{1}{\delta t}\int_t^{t+\delta t} \mathbf{S}(t)\, dt, \tag{2.40}$$

where δt is a very small amount of time, but significantly larger than the duration of a period, $1/\nu$; since $\mathbf{S}$ repeats itself after each period (if no attenuation occurs) a δt equal to any multiple of $1/\nu$ will give the same result for $\overline{\mathbf{S}}$, namely

$$\overline{\mathbf{S}} = \tfrac{1}{2}\Re\{\mathbf{E}_c\times\mathbf{H}_c^*\}, \tag{2.41}$$

[5]Since there are materials that have $n < 1$ it is possible to have *phase velocities* (i.e., the velocity with which the amplitude of continuous waves penetrates through a medium) larger than c_0; these should be distinguished from the *signal velocities* (i.e., the velocity with which the energy contained in the waves travels), which can never exceed the speed of light in vacuum. The difference between the two may be grasped more easily by visualizing the movement of ocean waves: The wave crests move at a certain speed across the ocean surface (phase velocity), while the actual velocity of the water (signal velocity) is relatively slow.

[6]Note that, since the vector cross-product is a nonlinear operation, the Poynting vector may **not** be calculated from $\mathbf{S} = \Re\{\mathbf{E}_c\times\mathbf{H}_c\}$.

***John Henry Poynting (1852–1914)**

British physicist. He served as professor of physics at the University of Birmingham from 1880 until his death. His discovery that electromagnetic energy is proportional to the product of electric and magnetic field strength is known as Poynting's theorem.

where $\mathbf{H}^*$ denotes the complex conjugate of $\mathbf{H}$, and the factor of $1/2$ results from integrating over $\cos^2(2\pi\eta_0 c_0 t)$ and $\sin^2(2\pi\eta_0 c_0 t)$ terms. Thus using equation (2.25) and the vector identity (2.27), the Poynting vector may be expressed as

$$\overline{\mathbf{S}} = \frac{1}{2\nu\mu}\Re\{\mathbf{E}_c \times (\mathbf{w}^* \times \mathbf{E}_c^*)\} = \frac{1}{2\nu\mu}\Re\{\mathbf{w}^*(\mathbf{E}_c \cdot \mathbf{E}_c^*)\}$$
$$= \frac{n}{2c_0\mu}|\mathbf{E}_0|^2\, e^{-4\pi\eta_0 kz}\hat{\mathbf{s}}. \tag{2.42}$$

The vector $\mathbf{S}$ points into the direction of propagation, and—as the wave traverses the medium—its energy content is attenuated exponentially, where the attenuation factor

$$\kappa = 4\pi\eta_0 k \tag{2.43}$$

is known as the *absorption coefficient* of the medium.

Example 2.1. A plane homogeneous wave propagates through a perfect dielectric medium ($n = 2$) in the direction of $\hat{\mathbf{s}} = 0.8\hat{\mathbf{i}} + 0.6\hat{\mathbf{k}}$ with a wavenumber of $\eta_0 = 2500\,\mathrm{cm}^{-1}$ and an electric field amplitude vector of $\mathbf{E}_0 = E_0[(6+3i)\hat{\mathbf{i}} + (2-5i)\hat{\mathbf{j}} - (8+4i)\hat{\mathbf{k}}]/\sqrt{154}$, where $E_0 = 600\,\mathrm{N/C}$, and the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors in the x-, y- and z-directions. Determine the magnetic field amplitude vector and the energy contained in the wave, assuming that the medium is nonmagnetic.

Solution

Since $\mathbf{w} = \mathbf{w}'$ is colinear with $\hat{\mathbf{s}}$, we find from equation (2.31) that $\mathbf{w} = w\hat{\mathbf{s}} = \eta_0 n\hat{\mathbf{s}}$ and, from equation (2.25),

$$\mathbf{H}_0 = \frac{1}{\nu\mu}\mathbf{w}\times\mathbf{E}_0 = \frac{1}{\nu\mu_0}\mathbf{w}\times\mathbf{E}_0 = \frac{n}{c_0\mu_0}\hat{\mathbf{s}}\times\mathbf{E}_0$$
$$= \frac{nE_0}{c_0\mu_0\sqrt{154}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0.8 & 0.0 & 0.6 \\ 6+3i & 2-5i & -8-4i \end{vmatrix}$$
$$= \frac{nE_0}{c_0\mu_0 5\sqrt{154}}[(-6+15i)\hat{\mathbf{i}} + (50+25i)\hat{\mathbf{j}} + (8-20i)\hat{\mathbf{k}}]$$
$$= \frac{H_0}{\sqrt{3850}}[(-6+15i)\hat{\mathbf{i}} + (50+25i)\hat{\mathbf{j}} + (8-20i)\hat{\mathbf{k}}],$$

where

$$H_0 = \frac{nE_0}{c_0\mu_0} = \frac{2\times 600\,\mathrm{N/C}}{2.998\times10^8\,\mathrm{m/s}\times4\pi\times10^{-7}\,\mathrm{N\,s^2/C^2}} = 3.185\,\mathrm{C/m\,s},$$

and it is assumed that, for a nonmagnetic medium, the magnetic permeability is equal to the one in vacuum, $\mu = \mu_0$ (from Table A.1). The energy content of the wave is given by the Poynting vector, either equation (2.41) or equation (2.42). Choosing the latter, we get

$$\overline{\mathbf{S}} = \frac{n}{2c_0\mu_0}E_0^2\hat{\mathbf{s}} = \overline{S}\hat{\mathbf{s}}, \qquad \overline{S} = \frac{2\times600^2\,\mathrm{N^2/C^2}}{2\times2.2998\times10^{-8}\,\mathrm{m/s}\times4\pi\times10^{-7}\,\mathrm{N\,s^2/C^2}} = 955.6\,\mathrm{W/m^2}.$$

2.4 POLARIZATION

Knowledge of the frequency, direction of propagation, and the energy content [i.e., the magnitude of the Poynting vector, equation (2.42)] does not completely describe a monochromatic (or time-harmonic) electromagnetic wave. Every train of electromagnetic waves has a property known as the *state of polarization*. Polarization effects are generally not very important to the heat transfer engineer since emitted light generally is randomly polarized. In some applications partially or fully polarized light is employed, for example, from laser sources; and the engineer needs to know *(i)* how the reflective behavior of a surface depends on the polarization of incoming light, and *(ii)* how reflection from a surface tends to alter the state of polarization. We shall give here only a very brief introduction to polarization, based heavily on the excellent short description in Bohren and Huffman [2]. More detailed accounts on the subject may be found in the books by van de Hulst [4], Chandrasekhar [5], and others.

Consider a plane monochromatic wave with wavenumber η propagating through a nonabsorbing medium ($k \equiv 0$) in the z-direction. When describing polarization, it is customary to relate parameters to the electric field (keeping in mind that the magnetic field is simply perpendicular to it), which follows from equation (2.36) as

$$\mathbf{E} = \Re\{\mathbf{E}_c\} = \Re\{(\mathbf{A} - i\mathbf{B})\, e^{-2\pi i \eta n(z-ct)}\} = \mathbf{A}\cos 2\pi\eta n(z-ct) - \mathbf{B}\sin 2\pi\eta n(z-ct), \tag{2.44}$$

where the vector $\mathbf{E}_0$ and its real components $\mathbf{A}$ and $\mathbf{B}$ are independent of position and lie, at any position z, in the plane normal to the direction of propagation. At any given location, say $z = 0$, the tip of the electric field vector traces out the curve

$$\mathbf{E}(z=0,t) = \mathbf{A}\cos 2\pi\nu t + \mathbf{B}\sin 2\pi\nu t. \tag{2.45}$$

This curve, shown in Fig. 2-3, describes an ellipse that is known as the *vibration ellipse*. The ellipse collapses into a straight line if either $\mathbf{A}$ or $\mathbf{B}$ vanishes, in which case the wave is said to be *linearly polarized* (sometimes also called *plane polarized*). If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular to one another and are of equal magnitude, the vibration ellipse becomes a circle and the wave is known as *circularly polarized*. In general, the wave in equation (2.44) is *elliptically polarized*.

At any given time, say $t = 0$, the curve described by the tip of the electric field vector is a helix (Fig. 2-4), or

$$\mathbf{E}(z,t=0) = \mathbf{A}\cos 2\pi n\eta z - \mathbf{B}\sin 2\pi n\eta z. \tag{2.46}$$

Equation (2.46) describes the electric field at any one particular time. As time increases the helix moves into the direction of propagation, and its intersection with any plane z = const describes the local vibration ellipse.

The state of polarization, which is characterized by its vibration ellipse, is defined by its *ellipticity*, b/a (the ratio of the length of its semiminor axis to that of its semimajor axis, as shown in Fig. 2-3), its *azimuth* γ (the angle between an arbitrary reference direction and its semimajor axis), and its *handedness* (i.e., the direction with which the tip of the electric field vector traverses through the vibration ellipse, clockwise or counterclockwise). These three parameters together with the magnitude of the Poynting vector are the *ellipsometric parameters* of a plane wave.

Example 2.2. Calculate the ellipsometric parameters a, b, and γ for the wave considered in Example 2.1.

Solution
From equation (2.44) we find

$$\mathbf{A} = E_0(6\hat{\imath} + 2\hat{\jmath} - 8\hat{\mathbf{k}})/\sqrt{154}, \quad \mathbf{B} = -E_0(3\hat{\imath} - 5\hat{\jmath} - 4\hat{\mathbf{k}})/\sqrt{154},$$

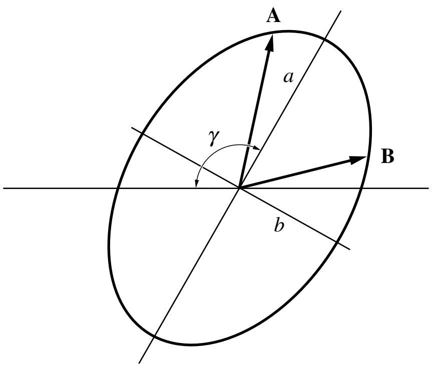

FIGURE 2-3
Vibration ellipse for a monochromatic wave.

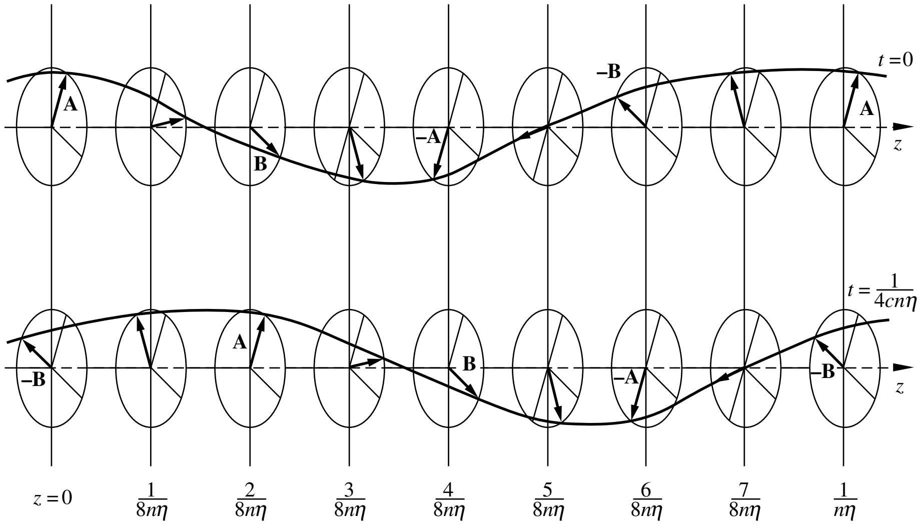

FIGURE 2-4
Space variation of electric field at fixed times.

and at any given location, say $z = 0$, the electric field vector may be written as

$$\mathbf{E} = E_0\left[(6\cos 2\pi\nu t - 3\sin 2\pi\nu t)\hat{\mathbf{i}} + (2\cos 2\pi\nu t + 5\sin 2\pi\nu t)\hat{\mathbf{j}} - (8\cos 2\pi\nu t - 4\sin 2\pi\nu t)\hat{\mathbf{k}}\right] / \sqrt{154}.$$

The time-varying magnitude $|\mathbf{E}|$ at this location then is

$$\begin{aligned}|\mathbf{E}|^2 = \mathbf{E}\cdot\mathbf{E} = \frac{E_0^2}{154}&(36\cos^2 2\pi\nu t - 36\cos 2\pi\nu t\sin 2\pi\nu t + 9\sin^2 2\pi\nu t\\ &+ 4\cos 2\pi\nu t + 20\cos^2 2\pi\nu t\sin 2\pi\nu t + 25\sin^2 2\pi\nu t\\ &+ 64\cos 2\pi\nu t - 64\cos^2 2\pi\nu t\sin 2\pi\nu t + 16\sin^2 2\pi\nu t)\\ =\ & E_0^2(50 - 80\cos 2\pi\nu t\sin 2\pi\nu t + 54\cos^2 2\pi\nu t)/154.\end{aligned}$$

The maximum (a) and minimum (b) of $|\mathbf{E}|$ may be found by differentiating the last expression with respect to t and setting the result equal to zero. This operation leads to

$$-80(\cos^2 2\pi\nu t - \sin^2 2\pi\nu t) = 108\sin 2\pi\nu t\cos 2\pi\nu t$$
$$-80\cos 4\pi\nu t = 54\sin 4\pi\nu t$$

or

$$2\pi\nu t = 0.5\tan^{-1}\left(-\frac{80}{54}\right).$$

This function is double-valued, leading to $(2\pi\nu t)_1 = -27.99°$ and $(2\pi\nu t)_2 = 62.01°$. Substituting these values into the expression for $\mathbf{E}$ gives

$$\mathbf{E}_1 = E_0(0.5404\hat{\mathbf{i}} - 0.0468\hat{\mathbf{j}} - 0.7205\hat{\mathbf{k}}), \quad |\mathbf{E}| = a = 0.9009E_0$$

and

$$\mathbf{E}_2 = E_0(0.0134\hat{\mathbf{i}} + 0.4314\hat{\mathbf{j}} - 0.0179\hat{\mathbf{k}}), \quad |\mathbf{E}| = b = 0.4339E_0.$$

The evaluation of the azimuth depends on the choice of a reference axis in the plane of the vibration ellipse. In the present problem the y-axis lies in this plane and is, therefore, the natural choice. Thus,

$$\cos\gamma = \frac{\mathbf{E}\cdot\hat{\mathbf{j}}}{|\mathbf{E}|} = -\frac{0.0468}{0.9009} = -0.0519, \quad \gamma = 92.97°.$$

While the ellipsometric parameters completely describe any monochromatic wave, they are difficult to measure directly (with the exception of the Poynting vector). In addition, when two or more waves of the same frequency but different polarization are superposed, only their strengths are additive: The other three ellipsometric parameters must be calculated anew. For these reasons a different but equivalent description of polarized light, known as *Stokes' parameters*, is usually preferred. The Stokes' parameters are defined by separating the wave train into two perpendicular components:

$$\mathbf{E}_c = \mathbf{E}_0\, e^{-2\pi i\eta n(z-ct)}; \quad \mathbf{E}_0 = E_\parallel \hat{\mathbf{e}}_\parallel + E_\perp \hat{\mathbf{e}}_\perp, \tag{2.47}$$

where $\hat{\mathbf{e}}_\parallel$ and $\hat{\mathbf{e}}_\perp$ are *real* orthogonal unit vectors in the plane normal to wave propagation, such that $\hat{\mathbf{e}}_\parallel$ lies in an arbitrary reference plane that includes the wave propagation vector, and $\hat{\mathbf{e}}_\perp$ is perpendicular to it.[7] The *parallel* ($E_\parallel$) and *perpendicular* ($E_\perp$) *polarization* components are generally complex and may be written as

$$E_\parallel = a_\parallel\, e^{-i\delta_\parallel}, \quad E_\perp = a_\perp\, e^{-i\delta_\perp}, \tag{2.48}$$

where a is the magnitude of the electric field and δ is the *phase angle of polarization*. Waves with parallel polarization (i.e., with electric field in the plane of incidence, and magnetic field normal to it) are also called *transverse magnetic* (TM) waves; and perpendicular polarization is *transverse electric* (TE). Substitution into equation (2.44) leads to

$$\begin{aligned}\mathbf{E} &= \Re\{a_\parallel\, e^{-i\delta_\parallel - 2\pi i\eta n(z-ct)}\hat{\mathbf{e}}_\parallel + a_\perp\, e^{-i\delta_\perp - 2\pi i\eta n(z-ct)}\hat{\mathbf{e}}_\perp\}\\ &= a_\parallel \cos[\delta_\parallel + 2\pi\eta n(z-ct)]\hat{\mathbf{e}}_\parallel + a_\perp \cos[\delta_\perp + 2\pi\eta n(z-ct)]\hat{\mathbf{e}}_\perp.\end{aligned} \tag{2.49}$$

Thus, the arbitrary wave given by equation (2.44) has been decomposed into two linearly polarized waves that are perpendicular to one another. The four *Stokes' parameters I, Q, U*, and *V* are defined by

$$I = E_\parallel E_\parallel^* + E_\perp E_\perp^* = a_\parallel^2 + a_\perp^2, \tag{2.50}$$

$$Q = E_\parallel E_\parallel^* - E_\perp E_\perp^* = a_\parallel^2 - a_\perp^2, \tag{2.51}$$

$$U = E_\parallel E_\perp^* + E_\perp E_\parallel^* = 2a_\parallel a_\perp \cos(\delta_\parallel - \delta_\perp), \tag{2.52}$$

$$V = i(E_\parallel E_\perp^* - E_\perp E_\parallel^*) = 2a_\parallel a_\perp \sin(\delta_\parallel - \delta_\perp), \tag{2.53}$$

where the asterisks again denote complex conjugates. It can be shown that these four parameters may be determined through power measurements either directly (I), using a linear polarizer (arranged in the parallel and perpendicular directions for Q, rotated 45° for U), or a circular polarizer (V) (see, for example, Bohren and Huffman [2]). It is clear that only three of the Stokes' parameters are independent, since

$$I^2 = Q^2 + U^2 + V^2. \tag{2.54}$$

Since the Stokes' parameters of a wave train are expressed in terms of the energy contents of its component waves [which can be seen by comparison with equation (2.42)], it follows that the Stokes' parameters for a collection of waves are additive.

The Stokes' parameters may also be related to the ellipsometric parameters by

$$I = a^2 + b^2, \tag{2.55}$$

$$Q = (a^2 - b^2)\cos 2\gamma, \tag{2.56}$$

$$U = (a^2 - b^2)\sin 2\gamma, \tag{2.57}$$

$$V = \pm 2ab, \tag{2.58}$$

[7]In the literature subscripts p and s are also commonly used, from the German words "parallel" and "senkrecht" (perpendicular).

TABLE 2.1
Stokes' parameters for several cases of polarized light.

Linearly Polarized				
0°	90°	+45°	−45°	γ
↔	↕	↘	↙	
$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}1\\\cos 2\gamma\\\sin 2\gamma\\0\end{pmatrix}$
Circularly Polarized				
	Right		Left	
	↻		↺	
	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$		$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$	

where the azimuth γ is measured from $\hat{\mathbf{e}}_{\|}$, and the sign of V specifies the handedness of the vibration ellipse. The sets of Stokes' parameters for a few special cases of polarization are shown—normalized, and written as column vectors—in Table 2.1 (from [2]). The parameters Q and U show the degree of *linear polarization* (plus its orientation), while V is related to the degree of *circular polarization*.

The above definition of the Stokes' parameters is correct for strictly monochromatic waves as given by equation (2.47). Most natural light sources, such as the sun, lightbulbs, fires, and so on, produce light whose amplitude, $\mathbf{E}_0$, is a slowly varying function of time (i.e., in comparison with a full wave period, $1/\nu$), or

$$\mathbf{E}_0(t) = E_{\|}(t)\hat{\mathbf{e}}_{\|} + E_{\perp}(t)\hat{\mathbf{e}}_{\perp}. \tag{2.59}$$

Such waves are called *quasi-monochromatic*. If, through their slow respective variations with time, $E_{\|}$ and $E_{\perp}$ are *uncorrelated*, then the wave is said to be *unpolarized*. In such a case the vibration ellipse changes slowly with time, eventually tracing out ellipses of all shapes, orientations, and handedness. All waves discussed so far had a fixed relationship between $E_{\|}$ and $E_{\perp}$, and are known as *(completely) polarized*. If some correlation between $E_{\|}$ and $E_{\perp}$ exists (for example, a wave of constant handedness, ellipticity, or azimuth), then the wave is called *partially polarized*. For quasi-monochromatic waves the Stokes' parameters are defined in terms of time-averaged values, and equation (2.54) must be replaced by

$$I^2 \geq Q^2 + U^2 + V^2, \tag{2.60}$$

where the equality sign holds only for polarized light. For unpolarized light one gets $Q = U = V = 0$, while for partially polarized light the magnitudes of Q, U, and V give the following:

$$\begin{aligned}
&\textit{degree of polarization} && = \sqrt{Q^2 + U^2 + V^2}/I,\\
&\textit{degree of linear polarization} && = \sqrt{Q^2 + U^2}/I,\\
&\textit{degree of circular polarization} && = V/I.
\end{aligned}$$

Example 2.3. Reconsider the plane wave of the last two examples. Decompose the wave into two linearly polarized waves, one in the x-z-plane, and the other perpendicular to it. What are the Stokes' coefficients, the phase differences between the two polarizations, and the different degrees of polarization?

Solution
With $\hat{\mathbf{s}} = 0.8\hat{\mathbf{\imath}} + 0.6\hat{\mathbf{k}}$ and the knowledge that $\hat{\mathbf{e}}_\|$ must lie in the x-z-plane, i.e., $\hat{\mathbf{e}}_\| \cdot \hat{\mathbf{\jmath}} = 0$, and that $\hat{\mathbf{e}}_\|$ must be normal to $\hat{\mathbf{s}}$, or $\hat{\mathbf{e}}_\| \cdot \hat{\mathbf{s}} = 0$, and finally that $\hat{\mathbf{e}}_\perp$ must be perpendicular to both of them, we get

$$\hat{\mathbf{e}}_\| = 0.6\hat{\mathbf{\imath}} - 0.8\hat{\mathbf{k}}, \quad \hat{\mathbf{e}}_\perp = \hat{\mathbf{\jmath}},$$

where the choice of sign for both vectors is arbitrary (and we have chosen to let $\hat{\mathbf{e}}_\|$, $\hat{\mathbf{e}}_\perp$, and $\hat{\mathbf{s}}$ form a right-handed coordinate system). Thus, from equation (2.47) and

$$\mathbf{E}_0 = E_0[(6+3i)\hat{\mathbf{\imath}} + (2-5i)\hat{\mathbf{\jmath}} - (8+4i)\hat{\mathbf{k}}]/\sqrt{154}$$

it follows immediately that

$$\begin{aligned}\mathbf{E}_\| &= E_0(2+i)(3\hat{\mathbf{\imath}} - 4\hat{\mathbf{k}})/\sqrt{154} = \left(5/\sqrt{154}\right)(2+i)E_0\hat{\mathbf{e}}_\|,\\ \mathbf{E}_\perp &= E_0(2-5i)\hat{\mathbf{\jmath}}/\sqrt{154} = \left[(2-5i)/\sqrt{154}\right]E_0\hat{\mathbf{e}}_\perp,\end{aligned}$$

or

$$\begin{aligned}E_\| &= \left(5/\sqrt{154}\right)(2+i)E_0 = \sqrt{\frac{125}{154}}E_0\,e^{-i\delta_\|},\\ E_\perp &= \left[(2-5i)/\sqrt{154}\right]E_0 = \sqrt{\frac{29}{154}}E_0\,e^{-i\delta_\perp},\end{aligned}$$

with

$$\begin{aligned}\delta_\| &= -\tan^{-1}\left(\frac{1}{2}\right) = -26.565°,\\ \delta_\perp &= -\tan^{-1}\left(-\frac{5}{2}\right) = 68.199°,\end{aligned}$$

and a phase difference between the two polarizations of

$$\delta_\| - \delta_\perp = -94.76°$$

(since $\tan^{-1}$ is a double-valued function, the correct value is determined by checking the signs of the real and imaginary parts of E). The Stokes' parameters can be calculated either directly from equations (2.50) through (2.53), or from equations (2.55) through (2.58) (using the ellipsometric parameters calculated in the last example). We use here the first approach so that we get

$$\begin{aligned}I &= (125+29)E_0^2/154 = E_0^2,\\ Q &= (125-29)E_0^2/154 = 48E_0^2/77,\\ U &= 5(4+2i+10i-5+4-2i-10i-5)E_0^2/154 = -5E_0^2/77,\\ V &= 5i(4+2i+10i-5-4+2i+10i+5)E_0^2/154 = -60E_0^2/77.\end{aligned}$$

Finally, the degrees of polarization follow as $\sqrt{Q^2+U^2+V^2}/I = 100\%$ total polarization, $\sqrt{Q^2+U^2}/I = 62.7\%$ linear polarization, and $|V|/I = 77.9\%$ circular polarization.

In general, the state of polarization of an electromagnetic wave train is changed when it interacts with an optical element (which may be a polarizer or reflector, but can also be a reflecting surface in an enclosure, or a scattering element, such as suspended particles). While a polarized beam is characterized by its four-element Stokes vector, it is possible to represent the effects of an optical element by a 4×4 matrix, known as the *Mueller matrix*, which describes the relations between incident and transmitted Stokes vectors. Details can be found, e.g., in Bohren and Huffman [2].

2.5 REFLECTION AND TRANSMISSION

When an electromagnetic wave is incident on the interface between two homogeneous media, the wave will be partially reflected and partially transmitted into the second medium. We will

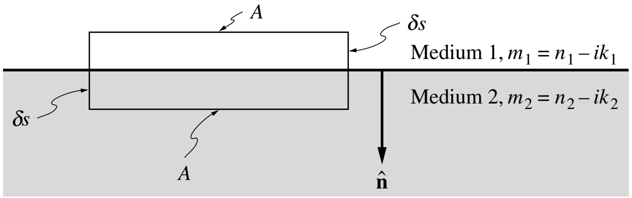

FIGURE 2-5
Geometry for derivation of interface conditions.

limit our discussion here to plane interfaces, i.e., to cases where the local radius of curvature is much greater than the wavelength of the incoming light, λ, for which the problem may be reduced to algebraic equations. Some discussion on strongly curved surfaces in the form of small particles will be given in Chapter 12, which deals with radiative properties of particulate clouds.

In the following, after first establishing the general conditions for Maxwell's equations at the interface, we shall consider a wave traveling from one nonabsorbing medium into another nonabsorbing medium, followed by a short discussion of a wave incident from a nonabsorbing onto an absorbing medium.

Interface Conditions for Maxwell's Equations

To establish boundary conditions for **E** and **H** at an interface between two media, we shall apply the theorems of Gauss and Stokes to Maxwell's equations. Both theorems convert volume integrals to surface integrals and are discussed in detail in standard mathematical texts such as Wylie [3]. Given a vector function **F**, defined within a volume V and on its boundary Γ, the theorems may be stated as

Gauss' theorem:

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_\Gamma \mathbf{F} \cdot d\mathbf{\Gamma}, \tag{2.61}$$

Stokes' theorem:

$$\int_V \nabla \times \mathbf{F}\, dV = -\int_\Gamma \mathbf{F} \times d\mathbf{\Gamma}, \tag{2.62}$$

where $d\mathbf{\Gamma} = \hat{\mathbf{n}}\, d\Gamma$ and $\hat{\mathbf{n}}$ is a unit surface normal pointing out of the volume.

Now consider a thin volume element $\delta V = A\, \delta s$ containing part of the interface as shown in Fig. 2-5. Applying Gauss' theorem to the first of Maxwell's equations, equation (2.11) yields

$$\int_{\delta V} \nabla \cdot (\gamma \mathbf{E}_c)\, dV = \int_\Gamma \gamma \mathbf{E}_c \cdot d\mathbf{\Gamma} \approx \int_A [(\gamma \mathbf{E}_c)_1 \cdot (-\hat{\mathbf{n}}) + (\gamma \mathbf{E}_c)_2 \cdot \hat{\mathbf{n}}]\, dA = 0, \tag{2.63}$$

where Γ is the total surface area of δV, and contributions to the surface integral come mainly from the two sides parallel to the interface since δs is small. Also, shrinking A to an arbitrarily small area, we conclude that, everywhere along the interface,

$$m_1^2\, \mathbf{E}_{c1} \cdot \hat{\mathbf{n}} = m_2^2\, \mathbf{E}_{c2} \cdot \hat{\mathbf{n}}, \tag{2.64}$$

where equation (2.31) has been used, together with assuming nonmagnetic media, to eliminate the complex permittivity γ. Similarly, from equation (2.12)

$$\mathbf{H}_{c1} \cdot \hat{\mathbf{n}} = \mathbf{H}_{c2} \cdot \hat{\mathbf{n}}. \tag{2.65}$$

Thus, the normal components of $m^2\mathbf{E}_c$ and $\mathbf{H}_c$ are conserved across a plane boundary. Stokes' theorem may be applied to equations (2.13) and (2.14), again for the volume element shown in Fig. 2-5. For example,

$$\int_{\delta V} \nabla \times \mathbf{H}_c\, dV = -\int_\Gamma \mathbf{H}_c \times d\mathbf{\Gamma} \approx \int_A (\mathbf{H}_{c1} - \mathbf{H}_{c2}) \times \hat{\mathbf{n}}\, dA = \int_V 2\pi i \nu \gamma \mathbf{E}_c\, dV, \tag{2.66}$$

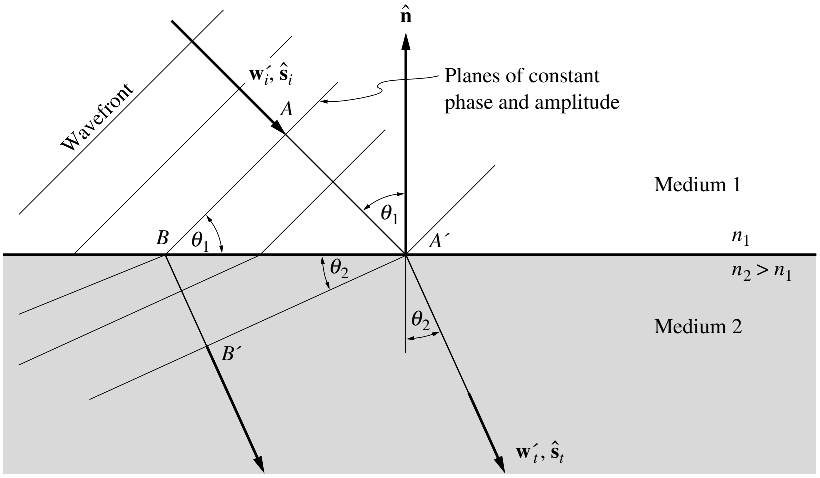

FIGURE 2-6
Transmission and reflection of a plane wave at the interface between two nonabsorbing media.

or, after shrinking $\delta s \to 0$ and A to a small value,

$$\mathbf{E}_{c1} \times \hat{\mathbf{n}} = \mathbf{E}_{c2} \times \hat{\mathbf{n}} \tag{2.67}$$

and

$$\mathbf{H}_{c1} \times \hat{\mathbf{n}} = \mathbf{H}_{c2} \times \hat{\mathbf{n}}. \tag{2.68}$$

Therefore, the tangential components of both $\mathbf{E}_c$ and $\mathbf{H}_c$ are conserved across a plane boundary.

Given the incident wave, it is possible to find the complete fields from Maxwell's equations and the above interface conditions. However, it is obvious that there will be a reflected wave in the medium of incidence, and a transmitted wave in the other medium. We may also assume that all waves remain plane waves. A consequence of having guessed the solution to this point is that conditions (2.67) and (2.68) are sufficient to specify the reflected and transmitted waves, and it turns out that conditions (2.64) and (2.65) are automatically satisfied (Stone [1]).

The Interface between Two Nonabsorbing Media

The reflection and transmission relationships become particularly simple if homogeneous plane waves reach the plane interface between two nonabsorbing media. For such a wave train the planes of equal phase and equal amplitude coincide and are normal to the direction of propagation, as shown in Fig. 2-6. This plane, also called the *wavefront*, moves at constant speed $c_1 = c_0/n_1$ through Medium 1, and at a constant but speed $c_2 = c_0/n_2$ through Medium 2. If $n_2 > n_1$ then, as shown in Fig. 2-6, the wavefront will move more slowly through Medium 2, lagging behind the wavefront traveling through Medium 1. This is readily put in mathematical terms by looking at points A and B on the wavefront at a certain time t. At time $t + \Delta t$ the part of the wavefront initially at A will have reached point A' on the interface while the wavefront at point B, traveling a shorter distance through Medium 2, will have reached point B', where

$$\Delta t = \frac{\overline{AA'}}{c_1} = \frac{\overline{BB'}}{c_2}. \tag{2.69}$$

Using geometric relations for $\overline{AA'}$ and $\overline{BB'}$ and substituting for the phase velocities, we obtain

$$\Delta t = \frac{\overline{BA'}\sin\theta_i}{c_0/n_1} = \frac{\overline{BA'}\sin\theta_2}{c_0/n_2} = \frac{\overline{BA'}\sin\theta_r}{c_0/n_1}, \tag{2.70}$$

where the last term pertains to reflection, for which a similar relationship must exist (but which is not shown to avoid overcrowding of the figure). Thus we conclude that

$$\theta_r = \theta_i = \theta_1, \tag{2.71}$$

that is, according to electromagnetic wave theory, reflection of light is always purely specular. This is a direct consequence of a "plane" interface, i.e., a surface that is not only flat (with infinite radius of curvature) but also perfectly smooth. Equation (2.70) also gives a relationship between the directions of the incoming and transmitted waves as

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}, \tag{2.72}$$

which is known as *Snell's law*.* The angles $\theta_1 = \theta_i$ and $\theta_2 = \theta_r$ are called the *angles of incidence and refraction*. The present derivation of Snell's law was based on geometric principles and is valid only for plane homogeneous waves, which limits its applicability to the interface between two nonabsorbing media, i.e., two perfect dielectrics. A more rigorous derivation of a generalized version of Snell's law is given when incidence on an absorbing medium is considered.

Besides the directions of reflection and transmission we should like to be able to determine the amounts of reflected and transmitted light. From equations (2.19) and (2.20) we can write expressions for the electric and magnetic fields in Medium 1 (consisting of incident and reflected waves) by setting $\mathbf{w}'' = 0$ for a nonabsorbing medium as

$$\mathbf{E}_{c1} = \mathbf{E}_{0i}\, e^{-2\pi i(\mathbf{w}_\mathrm{i}'\cdot\mathbf{r}-\nu t)} + \mathbf{E}_{0r}\, e^{-2\pi i(\mathbf{w}_\mathrm{r}'\cdot\mathbf{r}-\nu t)}, \tag{2.73}$$

$$\mathbf{H}_{c1} = \mathbf{H}_{0i}\, e^{-2\pi i(\mathbf{w}_\mathrm{i}'\cdot\mathbf{r}-\nu t)} + \mathbf{H}_{0r}\, e^{-2\pi i(\mathbf{w}_\mathrm{r}'\cdot\mathbf{r}-\nu t)}. \tag{2.74}$$

Similarly for Medium 2,

$$\mathbf{E}_{c2} = \mathbf{E}_{0t}\, e^{-2\pi i(\mathbf{w}_\mathrm{t}'\cdot\mathbf{r}-\nu t)}, \tag{2.75}$$

$$\mathbf{H}_{c2} = \mathbf{H}_{0t}\, e^{-2\pi i(\mathbf{w}_\mathrm{t}'\cdot\mathbf{r}-\nu t)}. \tag{2.76}$$

For convenience we place the coordinate origin at that point of the boundary where reflection and transmission are to be considered. Thus, at that point of the interface, with $\mathbf{r} = 0$, using boundary conditions (2.67) and (2.68),

$$(\mathbf{E}_{0i} + \mathbf{E}_{0r}) \times \hat{\mathbf{n}} = \mathbf{E}_{0t} \times \hat{\mathbf{n}}, \tag{2.77}$$

$$(\mathbf{H}_{0i} + \mathbf{H}_{0r}) \times \hat{\mathbf{n}} = \mathbf{H}_{0t} \times \hat{\mathbf{n}}. \tag{2.78}$$

To evaluate the tangential components of the electric and magnetic fields at the interface, it is advantageous to break up the fields (which, in general, may be unpolarized or elliptically polarized) into two linearly polarized waves, one parallel to the *plane of incidence* (formed by the incident wave vector $\mathbf{w}_i$ and the surface normal $\hat{\mathbf{n}}$), and the other perpendicular to it, or

$$\mathbf{E}_0 = E_\parallel \hat{\mathbf{e}}_\parallel + E_\perp \hat{\mathbf{e}}_\perp, \quad \mathbf{H}_0 = H_\parallel \hat{\mathbf{e}}_\parallel + H_\perp \hat{\mathbf{e}}_\perp. \tag{2.79}$$

This is shown schematically in Fig. 2-7. It is readily apparent from the figure that, in the plane of incidence, the unit vectors normal to the interface ($\hat{\mathbf{n}}$) and tangential to the interface ($\hat{\mathbf{t}}$) may be expressed as

$$\hat{\mathbf{n}} = \hat{\mathbf{s}}_i \cos\theta_1 - \hat{\mathbf{e}}_{i\parallel} \sin\theta_1 = -\hat{\mathbf{s}}_r \cos\theta_1 + \hat{\mathbf{e}}_{r\parallel} \sin\theta_1 = \hat{\mathbf{s}}_t \cos\theta_2 - \hat{\mathbf{e}}_{t\parallel} \sin\theta_2, \tag{2.80a}$$

$$\hat{\mathbf{t}} = \hat{\mathbf{s}}_i \sin\theta_1 + \hat{\mathbf{e}}_{i\parallel} \cos\theta_1 = \hat{\mathbf{s}}_r \sin\theta_1 + \hat{\mathbf{e}}_{r\parallel} \cos\theta_1 = \hat{\mathbf{s}}_t \sin\theta_2 + \hat{\mathbf{e}}_{t\parallel} \cos\theta_2. \tag{2.80b}$$

* **Willebrord van Snel van Royen (1580–1626)**
Dutch astronomer and mathematician, who discovered Snell's law in 1621.

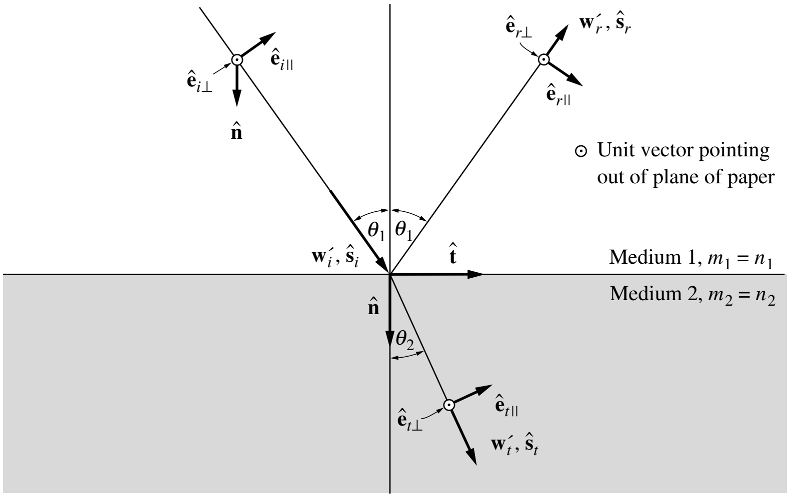

FIGURE 2-7
Orientation of wave vectors at an interface.

As defined in Fig. 2-7 the unit vectors $\hat{\mathbf{e}}_\parallel$, $\hat{\mathbf{e}}_\perp$ and $\hat{\mathbf{s}}$ form right-handed coordinate systems for the incident and transmitted waves, i.e.,

$$\hat{\mathbf{e}}_\parallel = \hat{\mathbf{e}}_\perp \times \hat{\mathbf{s}}, \quad \hat{\mathbf{e}}_\perp = \hat{\mathbf{s}} \times \hat{\mathbf{e}}_\parallel, \quad \hat{\mathbf{s}} = \hat{\mathbf{e}}_\parallel \times \hat{\mathbf{e}}_\perp, \tag{2.81}$$

and a left-handed coordinate system for the reflected wave (leading to opposite signs for the above cross-products of unit vectors).[8]

Therefore, from equation (2.80)

$$\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{n}} = \pm\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{s}} \cos\theta = -\hat{\mathbf{e}}_\perp \cos\theta,$$
$$\hat{\mathbf{e}}_\perp \times \hat{\mathbf{n}} = \pm\hat{\mathbf{e}}_\perp \times \hat{\mathbf{s}} \cos\theta \mp \hat{\mathbf{e}}_\perp \times \hat{\mathbf{e}}_\parallel \sin\theta = \hat{\mathbf{e}}_\parallel \cos\theta + \hat{\mathbf{s}} \sin\theta = \hat{\mathbf{t}},$$

where the top sign applies to the incident and transmitted waves, while the lower sign applies to the reflected component. The second of these relations can also be obtained directly from Fig. 2-7. Using these relations, equations (2.77) and (2.78) may be rewritten in terms of polarized components as

$$(E_{i\parallel} + E_{r\parallel}) \cos\theta_1 = E_{t\parallel} \cos\theta_2, \tag{2.82}$$
$$E_{i\perp} + E_{r\perp} = E_{t\perp}, \tag{2.83}$$
$$(H_{i\parallel} + H_{r\parallel}) \cos\theta_1 = H_{t\parallel} \cos\theta_2, \tag{2.84}$$
$$H_{i\perp} + H_{r\perp} = H_{t\perp}. \tag{2.85}$$

The magnetic field may be eliminated through the use of equation (2.25): With $\mathbf{w} = \eta_0 m\hat{\mathbf{s}} = (\nu/c_0)m\hat{\mathbf{s}}$ from equation (2.31) we have

$$\begin{aligned}\mathbf{H}_0 &= \frac{m}{c_0\mu}\hat{\mathbf{s}} \times \mathbf{E}_0 = \pm\frac{m}{c_0\mu\cos\theta}(\hat{\mathbf{n}} \pm \hat{\mathbf{e}}_\parallel \sin\theta) \times (E_\parallel\hat{\mathbf{e}}_\parallel + E_\perp\hat{\mathbf{e}}_\perp)\\ &= \pm\frac{m}{c_0\mu\cos\theta}\left[E_\parallel \cos\theta\hat{\mathbf{e}}_\perp - E_\perp(\hat{\mathbf{t}} - \hat{\mathbf{s}}\sin\theta)\right]\\ &= \pm\frac{m}{c_0\mu}(E_\parallel\hat{\mathbf{e}}_\perp - E_\perp\hat{\mathbf{e}}_\parallel).\end{aligned} \tag{2.86}$$

[8]This is necessary for consistency, i.e., for normal incidence there should not be any difference between parallel and perpendicular polarized waves.

Again, the upper sign applies to incident and transmitted waves, and the lower sign to reflected waves. The last two conditions may now be rewritten in terms of the electric field. Assuming the magnetic permeability to be the same in both media, and setting $m = n$ (nonabsorbing media), this leads to

$$(E_{i\perp} - E_{r\perp})\, n_1 \cos\theta_1 = E_{t\perp} n_2 \cos\theta_2, \tag{2.87}$$

$$(E_{i\|} - E_{r\|})\, n_1 = E_{t\|} n_2. \tag{2.88}$$

From this one may calculate the *reflection coefficient* r and the *transmission coefficient* t as

$$r_\| = \frac{E_{r\|}}{E_{i\|}} = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}, \tag{2.89}$$

$$r_\perp = \frac{E_{r\perp}}{E_{i\perp}} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}, \tag{2.90}$$

$$t_\| = \frac{E_{t\|}}{E_{i\|}} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}, \tag{2.91}$$

$$t_\perp = \frac{E_{t\perp}}{E_{i\perp}} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}. \tag{2.92}$$

For an interface between two nonabsorbing media these coefficients turn out to be real, even though the electric field amplitudes are complex. The *reflectivity* ρ is defined as the fraction of *energy* in a wave that is reflected and must, therefore, be calculated from the Poynting vector, equation (2.42), so that

$$\rho_\| = \frac{\overline{S}_{r\|}}{\overline{S}_{i\|}} = \left(\frac{E_{r\|}}{E_{i\|}}\right)^2 = r_\|^2 \tag{2.93}$$

gives the reflectivity of that part of the wave whose electric field vector lies in the plane of incidence (with its magnetic field normal to it), and

$$\rho_\perp = \frac{\overline{S}_{r\perp}}{\overline{S}_{i\perp}} = \left(\frac{E_{r\perp}}{E_{i\perp}}\right)^2 = r_\perp^2 \tag{2.94}$$

is the reflectivity for the part whose electric field vector is normal to the plane of incidence. In terms of these polarized components the overall reflectivity may be stated as "reflected energy for both polarizations, divided by the total incoming energy," or

$$\rho = \frac{E_{i\|}E_{i\|}^*\rho_\| + E_{i\perp}E_{i\perp}^*\rho_\perp}{E_{i\|}E_{i\|}^* + E_{i\perp}E_{i\perp}^*}. \tag{2.95}$$

For *unpolarized* and *circularly polarized* light $E_{i\|} = E_{i\perp}$, and the reflectivity for the entire wave train is

$$\rho = \frac{1}{2}(\rho_\| + \rho_\perp) = \frac{1}{2}\left[\left(\frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}\right)^2 + \left(\frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos_1 + n_2\cos\theta_2}\right)^2\right]. \tag{2.96}$$

From this relationship the refractive indices may be eliminated through Snell's law, giving

$$\rho = \frac{1}{2}\left[\frac{\tan^2(\theta_1-\theta_2)}{\tan^2(\theta_1+\theta_2)} + \frac{\sin^2(\theta_1-\theta_2)}{\sin^2(\theta_1+\theta_2)}\right], \tag{2.97}$$

which is known as *Fresnel's relation*.* Subroutine `fresnel` in Appendix F is a generalized version of Fresnel's relation for an interface between a perfect dielectric and an absorbing medium (see following section), where `n` $= n_2/n_1$, `k` $= k_2/n_1$, and `th` $= \theta_1$.

***Augustin-Jean Fresnel (1788–1827)**
French physicist, and one of the early pioneers for the wave theory of light. Serving as an engineer for the French government he studied aberration of light and interference in polarized light. His optical theories earned him very little recognition during his lifetime.

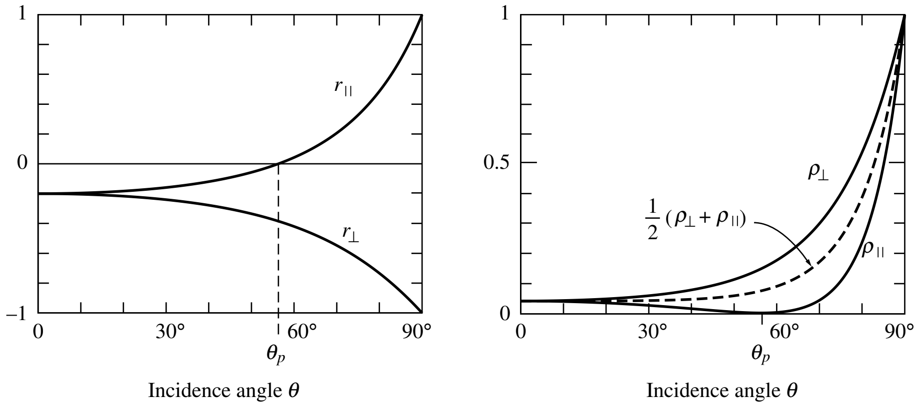

FIGURE 2-8
Reflection coefficients and reflectivities for the interface between two dielectrics ($n_2/n_1 = 1.5$).

The overall *transmissivity* τ may similarly be evaluated from the Poynting vector, equation (2.42), but the different refractive indices and wave propagation directions in the transmitting and incident media must be considered, so that

$$\tau = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} t^2 = 1 - \rho. \tag{2.98}$$

An example for the angular reflectivity at the interface between two dielectrics (with $n_2/n_1 = 1.5$) is given in Fig. 2-8. It is seen that, at an angle of incidence of $\theta_1 = \theta_p$, $r_{\|}$ passes through zero resulting in a zero reflectivity for the parallel component of the wave. This angle is known as the *polarizing angle* or *Brewster's angle*,* since light reflected from the surface—regardless of the incident polarization—will be completely polarized. Brewster's angle follows from equations (2.72) and (2.89) as

$$\tan\theta_p = \frac{n_2}{n_1}. \tag{2.99}$$

Different behavior is observed if light travels from one dielectric into another, optically less dense medium ($n_1 > n_2$),[9] shown in Fig. 2-9. Examination of equation (2.72) shows that θ_2 reaches the value of 90° for an angle of incidence θ_c, called the *critical angle*,

$$\sin\theta_c = \frac{n_2}{n_1}. \tag{2.100}$$

It is left as an exercise for the reader to show that, for $\theta_1 > \theta_c$, light of any polarization is reflected, and nothing is transmitted into the second medium.

It is important to realize that upon reflection a wave changes its state of polarization, since $E_{\|}$ and $E_{\perp}$ are attenuated by different amounts. If the incident wave is unpolarized (e.g., emission from a hot surface), $E_{\|}$ and $E_{\perp}$ are unrelated and will remain so after reflection. If the incident wave is polarized (e.g., laser radiation), the relationship between $E_{\|}$ and $E_{\perp}$ will change, causing a change in polarization.

Example 2.4. The plane homogeneous wave of the previous examples encounters the flat interface with another dielectric ($n_2 = 8/3$) that is described by the equation $z = 0$ (i.e., the x-y-plane at $z = 0$). Calculate

---

***Sir David Brewster (1781–1868)**
Scottish scientist, entered Edinburgh University at age 12 to study for the ministry. After completing his studies he turned his attention to science, particularly optics. In 1815, the year he discovered the law named after him, he was elected Fellow of the Royal Society.

[9]The optical density of a medium is related to the number of atoms contained over a distance equal to the wavelength of the light and is proportional to the refractive index.

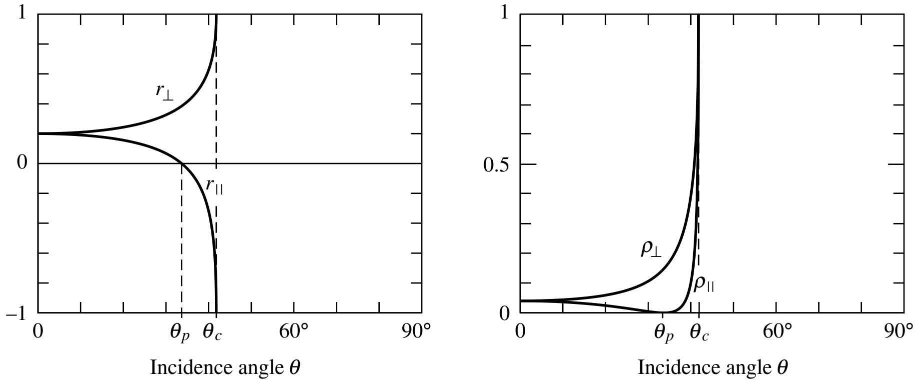

FIGURE 2-9
Reflection coefficients and reflectivities for the interface between two dielectrics ($n_1/n_2 = 1.5$).

the angles of incidence, reflection, and refraction. What fraction of energy of the wave is reflected, and how much is transmitted? In addition, determine the state of polarization of the reflected wave.

Solution
Since the interface is described by $z = 0$, the surface normal (pointing into Medium 2) is simply $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. From $\hat{\mathbf{s}} = 0.8\hat{\mathbf{i}} + 0.6\hat{\mathbf{k}}$ and $\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} = \cos\theta_1 = 0.6$, it follows that the angle of incidence is $\theta_1 = 53.13°$ off normal, which is equal to the angle of reflection, while the angle of refraction follows from Snell's law, equation (2.72), as

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = \frac{2}{8/3} \times 0.8 = 0.6, \quad \theta_2 = 36.87°.$$

It follows that $\cos\theta_2 = 0.8$ and the reflection coefficients are calculated from equations (2.89) and (2.90) as

$$r_\parallel = \frac{2 \times 0.8 - (8/3) \times 0.6}{2 \times 0.8 + (8/3) \times 0.6} = \frac{1.6 - 1.6}{3.2} = 0,$$

$$r_\perp = \frac{2 \times 0.6 - (8/3) \times 0.8}{2 \times 0.6 + (8/3) \times 0.8} = \frac{3.6 - 6.4}{10.0} = -0.28,$$

and the respective reflectivities follow as

$$\rho_\parallel = 0 \text{ and } \rho_\perp = (-0.28)^2 = 0.0784.$$

For the present wave and interface, the wave impinges on the surface at Brewster's angle, i.e., the component of the wave that is linearly polarized in the plane of incidence is totally transmitted.

In general, to calculate the overall reflectivity, the wave must be decomposed into two linear polarized components, vibrating within the plane of incidence and perpendicular to it. Fortunately, this was already done in Example 2.3. From equation (2.95), together with the values of $E_{i\parallel} = [5(2+i)/\sqrt{154}]E_0$ and $E_{i\perp} = [(2-5i)/\sqrt{154}]E_0$ from the previous example, we obtain

$$\rho = \frac{E_{i\parallel}E^*_{i\parallel}\rho_\parallel + E_{i\perp}E^*_{i\perp}\rho_\perp}{E_{i\parallel}E^*_{i\parallel} + E_{i\perp}E^*_{i\perp}} = \frac{125 \times 0 + 29 \times 0.0784}{154} = 0.0148,$$

and the overall transmissivity τ follows as

$$\tau = 1 - \rho = 0.9852.$$

To determine the polarization of the reflected beam, we first need to determine the reflected electric field amplitude vector. From the definition of the reflection coefficient we have

$$E_{r\parallel} = r_\parallel E_{i\parallel} = 0, \quad E_{r\perp} = r_\perp E_{i\perp} = -0.28 \times \frac{2-5i}{\sqrt{154}}E_0$$

and, from equations (2.50) through (2.53),

$$I = -Q = E_{r\perp}E^*_{r\perp} = \frac{0.28^2}{154} 29\, E_0^2 = 0.01476\, E_0^2,$$
$$U = V = 0.$$

Therefore, the wave remains 100% polarized, but the polarization is not completely linear. Indeed, any polarized radiation reflecting off a surface at Brewster's angle will become linearly polarized with only a perpendicular component.

The Interface between a Perfect Dielectric and an Absorbing Medium

The analysis of reflection and transmission at the interface between two perfect dielectrics is relatively straightforward, since an incident plane homogeneous wave remains plane and homogeneous after reflection and transmission. However, if a plane homogeneous wave is incident upon an absorbing medium, then the transmitted wave is, in general, inhomogeneous. If a beam travels from one absorbing medium into another absorbing medium, then the wave is usually inhomogeneous in both, making the analysis somewhat cumbersome. Fortunately, the interface between two absorbers is rarely important: A wave traveling through an absorbing medium is usually strongly attenuated, if not totally absorbed, before hitting a second absorber. In this section we shall consider a plane homogeneous light wave incident from a perfect dielectric on an absorbing medium.

The incident, reflected, and transmitted waves are again described by equations (2.73) through (2.76), except that the wave vector for transmission, $\mathbf{w}_t$, may be complex. Thus using equations (2.67) and (2.68), the interface condition may be written as

$$\mathbf{E}_{0i} \times \hat{\mathbf{n}}\, e^{-2\pi i \mathbf{w}'_i \cdot \mathbf{r}} + \mathbf{E}_{0r} \times \hat{\mathbf{n}}\, e^{-2\pi i \mathbf{w}'_r \cdot \mathbf{r}} = \mathbf{E}_{0t} \times \hat{\mathbf{n}}\, e^{-2\pi i (\mathbf{w}'_t \cdot \mathbf{r} - i \mathbf{w}''_t \cdot \mathbf{r})}, \tag{2.101}$$
$$\mathbf{H}_{0i} \times \hat{\mathbf{n}}\, e^{-2\pi i \mathbf{w}'_i \cdot \mathbf{r}} + \mathbf{H}_{0r} \times \hat{\mathbf{n}}\, e^{-2\pi i \mathbf{w}'_r \cdot \mathbf{r}} = \mathbf{H}_{0t} \times \hat{\mathbf{n}}\, e^{-2\pi i (\mathbf{w}'_t \cdot \mathbf{r} - i \mathbf{w}''_t \cdot \mathbf{r})}, \tag{2.102}$$

where $\mathbf{r}$ is left arbitrary here in order to derive formally the generalized form of Snell's law although, for convenience, we still assume that the coordinate origin lies on the interface. We note that none of the amplitude vectors, $\mathbf{E}_{0i}$, $\mathbf{H}_{0i}$, etc., depends on location, and that $\mathbf{r}$ is a vector to an arbitrary point on the interface, which may be varied independently. Thus, in order for equations (2.101) and (2.102) to hold at any point on the interface, we must have

$$\mathbf{w}'_i \cdot \mathbf{r} = \mathbf{w}'_r \cdot \mathbf{r} = \mathbf{w}'_t \cdot \mathbf{r}, \tag{2.103}$$
$$0 = \mathbf{w}''_t \cdot \mathbf{r}, \tag{2.104}$$

that is, since $\mathbf{r}$ is tangential to the interface, the tangential components of the wave vector $\mathbf{w}'$ must be continuous across the interface, while the tangential component of the attenuation vector $\mathbf{w}''_t$ must be zero, or $\mathbf{w}''_t = w''_t \hat{\mathbf{n}}$. Thus, within the absorbing medium, planes of equal amplitude are parallel to the interface, as indicated in Fig. 2-10. Since $\mathbf{w}'_r$ has the same tangential component as $\mathbf{w}'_i$ as well as the same magnitude [cf. equation (2.31)], it follows again that the reflection must be specular, or $\theta_r = \theta_i$.

The continuity of the tangential component for the transmitted wave vector indicates that

$$w'_i \sin\theta_1 = \eta_0 n_1 \sin\theta_1 = w'_t \sin\theta_2. \tag{2.105}$$

The wave vector for transmission, w'_t, may be eliminated from equation (2.105) by using equation (2.31):

$$\mathbf{w}_t \cdot \mathbf{w}_t = w'^2_t - w''^2_t - 2i \mathbf{w}'_t \cdot \mathbf{w}''_t = \eta_0^2 m_2^2 = \eta_0^2 (n_2^2 - k_2^2 - 2i n_2 k_2), \tag{2.106a}$$

or

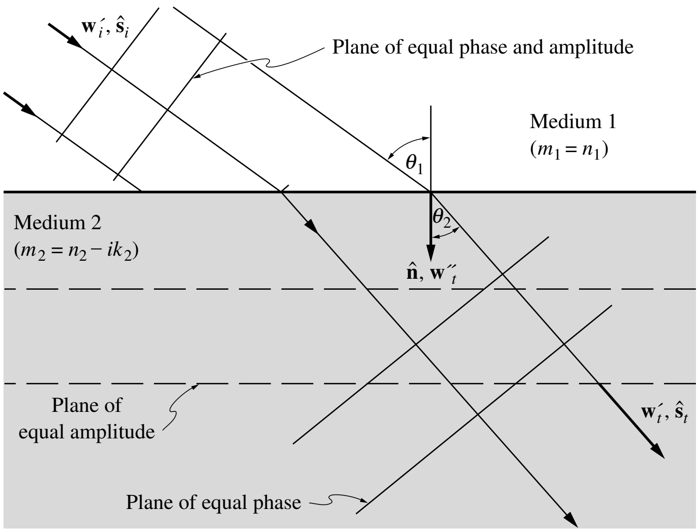

FIGURE 2-10
Transmission and reflection at the interface between a dielectric and an absorbing medium.

$$w_t'^2 - w_t''^2 = \eta_0^2(n_2^2 - k_2^2), \tag{2.106b}$$

$$\mathbf{w}_t' \cdot \mathbf{w}_t'' = w_t' w_t'' \cos\theta_2 = \eta_0^2 n_2 k_2. \tag{2.106c}$$

Thus, equations (2.105) and (2.106) constitute three equations in the three unknowns θ_2, w_t', and w_t''. This system of equations may be solved to yield

$$p^2 = \left(\frac{w_t' \cos\theta_2}{\eta_0}\right)^2 = \frac{1}{2}\left[\sqrt{(n_2^2 - k_2^2 - n_1^2\sin^2\theta_1)^2 + 4n_2^2k_2^2} + (n_2^2 - k_2^2 - n_1^2\sin^2\theta_1)\right], \tag{2.107a}$$

$$q^2 = \left(\frac{w_t''}{\eta_0}\right)^2 = \frac{1}{2}\left[\sqrt{(n_2^2 - k_2^2 - n_1^2\sin^2\theta_1)^2 + 4n_2^2k_2^2} - (n_2^2 - k_2^2 - n_1^2\sin^2\theta_1)\right], \tag{2.107b}$$

and the refraction angle θ_2 may be calculated from equation (2.105) as

$$p\tan\theta_2 = n_1\sin\theta_1. \tag{2.108}$$

Equation (2.108) together with equations (2.107) is known as the *generalized Snell's law*.

The *reflection coefficients* are calculated in the same fashion as was done for two dielectrics (left as an exercise). This leads to

$$\widetilde{r}_\parallel = \frac{E_{r\parallel}}{E_{i\parallel}} = \frac{n_1^2(w_t'\cos\theta_2 - iw_t'') - m_2^2 w_i'\cos\theta_1}{n_1^2(w_t'\cos\theta_2 - iw_t'') + m_2^2 w_i'\cos\theta_1}, \tag{2.109a}$$

$$\widetilde{r}_\perp = \frac{E_{r\perp}}{E_{i\perp}} = \frac{w_i'\cos\theta_1 - (w_t'\cos\theta_2 - iw_t'')}{w_i'\cos\theta_1 + (w_t'\cos\theta_2 - iw_t'')}, \tag{2.109b}$$

where the tilde has been added to indicate that the reflection coefficients are now *complex*. From equations (2.106) through (2.107) we find

$$m_2^2 = \frac{p^2}{\cos^2\theta_2} - q^2 - 2ipq = p^2(1 + \tan^2\theta_2) - q^2 - 2ipq = p^2 - q^2 + n_1^2\sin^2\theta_1 - 2ipq. \tag{2.110}$$

Eliminating the wave vectors, the reflection coefficients may be written as

$$\widetilde{r}_\parallel = \frac{n_1(p - iq) - (p^2 - q^2 + n_1^2\sin^2\theta_1 - 2ipq)\cos\theta_1}{n_1(p - iq) + (p^2 - q^2 + n_1^2\sin^2\theta_1 - 2ipq)\cos\theta_1}, \tag{2.111a}$$

$$\widetilde{r}_\perp = \frac{n_1\cos\theta_1 - p + iq}{n_1\cos\theta_1 + p - iq}. \tag{2.111b}$$

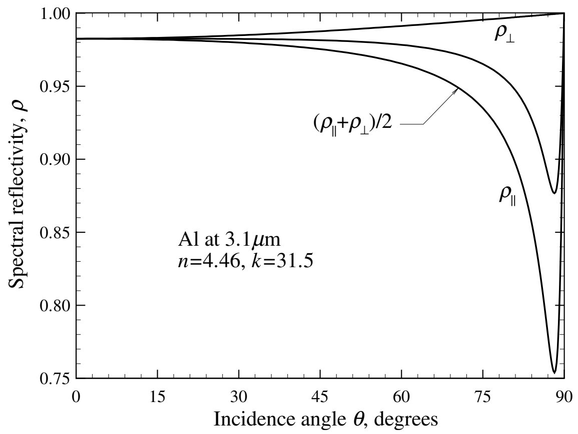

FIGURE 2-11
Directional reflectivity for a metal (aluminum at 3.1 μm with $n_2 = 4.46$, $k_2 = 31.5$) in contact with air ($n_1 = 1$).

The expression for $\widetilde{r}_\|$ may be simplified by dividing the numerator (and denominator) of $\widetilde{r}_\|$ by $\cos\theta_1$ times the numerator (or denominator) of $\widetilde{r}_\perp$. This operation leads to

$$\widetilde{r}_\| = \frac{p - n_1 \sin\theta_1 \tan\theta_1 - iq}{p + n_1 \sin\theta_1 \tan\theta_1 - iq}\widetilde{r}_\perp. \tag{2.112}$$

Finally, the reflectivities are again calculated as

$$\rho_\| = \widetilde{r}_\| \widetilde{r}_\|^* = \frac{(p - n_1 \sin\theta_1 \tan\theta_1)^2 + q^2}{(p + n_1 \sin\theta_1 \tan\theta_1)^2 + q^2}\rho_\perp, \tag{2.113a}$$

$$\rho_\perp = \widetilde{r}_\perp \widetilde{r}_\perp^* = \frac{(n_1 \cos\theta_1 - p)^2 + q^2}{(n_1 \cos\theta_1 + p)^2 + q^2}. \tag{2.113b}$$

Subroutine `fresnel` in Appendix F calculates $\rho_\|$, $\rho_\perp$, and $\rho = (\rho_\| + \rho_\perp)/2$ from this generalized version of Fresnel's relation for an interface between a perfect dielectric and an absorbing medium, where `n` $= n_2/n_1$, `k` $= k_2/n_1$, and `th` $= \theta_1$.

We note that for normal incidence $\theta_1 = \theta_2 = 0$, resulting in $p = n_2$, $q = k_2$ and

$$\rho_\| = \rho_\perp = \frac{(n_1 - n_2)^2 + k_2^2}{(n_1 + n_2)^2 + k_2^2}. \tag{2.114}$$

The directional behavior of the reflectivity for a typical metal with $n_2 = 4.46$ and $k_2 = 31.5$ (corresponding to the experimental values for aluminum at 3.1 μm [6]) exposed to air ($n_1 = 1$) is shown in Fig. 2-11.

Example 2.5. Redo Example 2.4 for a metallic interface, i.e., the plane homogeneous wave of the previous examples encounters the flat interface with a metal ($n_2 = k_2 = 90$), which again is described by the equation $z = 0$. Calculate the incidence, reflection, and refraction angles. What fraction of energy of the wave is reflected, and how much is transmitted?

Solution
If n_2 and k_2 are much larger than n_1 it follows from equations (2.107) that $p \approx n_2$ and $q \approx k_2$ and, from equation (2.105),

$$n_1 \sin\theta_1 \approx n_2 \tan\theta_2 \approx n_2 \sin\theta_2$$

(i.e., as long as $n_2 \gg n_1$, Snell's law between dielectrics holds) and it follows that $\theta_2 = 1.02°$. With $n_2 = k_2$

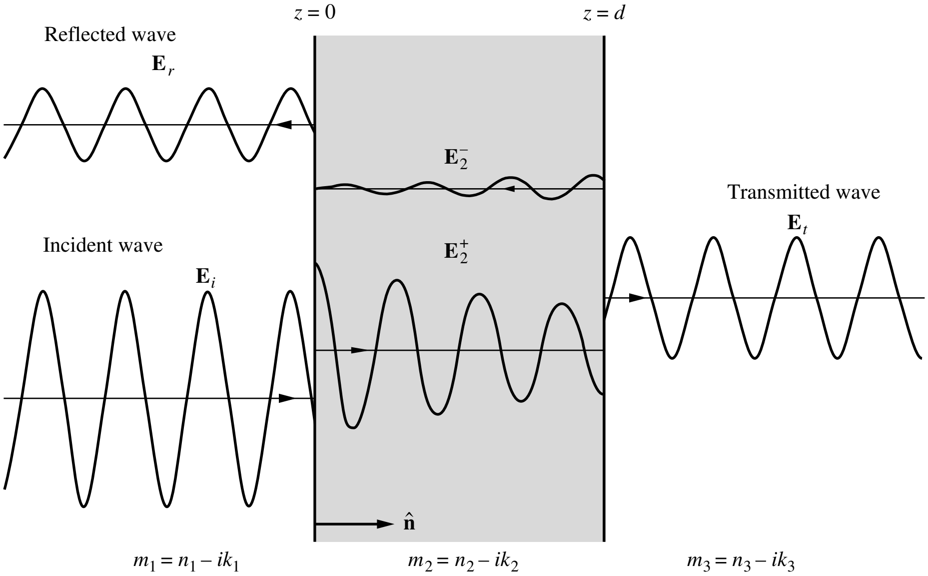

FIGURE 2-12
Reflection and transmission by a slab.

equations (2.113) reduce to

$$\rho_\perp = \frac{(n_1 \cos\theta_1 - n_2)^2 + n_2^2}{(n_1 \cos\theta_1 + n_2)^2 + n_2^2} = \frac{(1.2 - 90)^2 + 90^2}{(1.2 + 90)^2 + 90^2} = 0.9737,$$

$$\rho_\parallel = \frac{(n_2 - n_1 \sin\theta_1 \tan\theta_1)^2 + n_2^2}{(n_2 + n_1 \sin\theta_1 \tan\theta_1)^2 + n_2^2}\rho_\perp = \frac{(90 - 2\times 0.8^2/0.6)^2 + 90^2}{(90 + 2\times 0.8^2/0.6)^2 + 90^2}\times 0.9737 = 0.9286,$$

and the total reflectivity is again evaluated from equation (2.95) as

$$\rho = \frac{E_{i\parallel}E_{i\parallel}^*\rho_\parallel + E_{i\perp}E_{i\perp}^*\rho_\perp}{E_{i\parallel}E_{i\parallel}^* + E_{i\perp}E_{i\perp}^*} = \frac{125\times 0.9286 + 29\times 0.9737}{154} = 0.9371.$$

Thus, nearly 94% of the radiation is being reflected (and even more would have been reflected if the metal was surrounded by air with $n \approx 1$), and only 6% is transmitted into the metal, where it undergoes total attenuation after a very short distance because of the large value of k_2: Equation (2.42) shows that the transmission reaches its $1/e$ value at

$$4\pi\eta_0 k_2 z = 1, \quad \text{or} \quad z = 1/(4\pi\times 2500\times 90) = 3.5\times 10^{-7}\,\text{cm} = 0.0035\,\mu\text{m}.$$

Reflection and Transmission by a Thin Film or Slab

As a final topic we shall briefly consider the reflection and transmission by a thin film or slab of thickness d and complex index of refraction $m_2 = n_2 - ik_2$, embedded between two media with indices of refraction m_1 and m_3, as illustrated in Fig. 2-12. While the theory presented in this section is valid for slabs of arbitrary thickness, it is most appropriate for the study of *interference wave effects* in *thin films* or *coatings*. When an electromagnetic wave is reflected by a thin film, the waves reflected from both interfaces have different phases and *interfere* with one another (i.e., they may augment each other for small phase differences, or cancel each other for phase differences of 180°). For thick slabs, such as window panes, *geometric optics* provides a much simpler vehicle to determine overall reflectivity and transmissivity. However, for an antireflective coating on a window, *thin film optics* should be considered.

Normal Incidence

Since the computations become rather cumbersome, we shall limit ourselves to the simpler case of normal incidence ($\theta = 0$). For more detailed discussions, including oblique incidence angles, the reader is referred to books on the subject such as the one by Knittl [7] or to the very readable monograph by Anders [8].

Consider the slab shown in Fig. 2-12: The wave incident at the left interface is partially reflected, and partially transmitted toward the second interface. At the second interface, again, the wave is partially reflected and partially transmitted into Medium 3. The reflected part travels back to the first interface where a part is reflected back toward the second interface, and a part is transmitted into Medium 1, i.e., it is added to the reflected wave, etc. Therefore, the reflected wave $\mathbf{E}_r$ and the transmitted wave $\mathbf{E}_t$ consist of many contributions, and inside Medium 2 there are two waves $\mathbf{E}_2^+$ and $\mathbf{E}_2^-$ traveling into the directions $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$, respectively. Thus, the boundary conditions, equations (2.67) and (2.68), may be written for the first interface, similar to equations (2.82) through (2.85), as

$$z = \mathbf{r}\cdot\hat{\mathbf{n}} = 0: \qquad E_i + E_r = E_2^+ + E_2^-, \tag{2.115}$$

$$H_i + H_r = H_2^+ + H_2^-, \tag{2.116}$$

where polarization of the beam does not appear since at normal incidence $E_\parallel = E_\perp$. The magnetic field may again be eliminated using equation (2.25), as well as $\mathbf{w}_i = -\mathbf{w}_r = \eta_0 m_1 \hat{\mathbf{n}}$ and $\mathbf{w}^+ = -\mathbf{w}^- = \eta_0 m_2 \hat{\mathbf{n}}$ [from equation (2.31)], or

$$(E_i - E_r)m_1 = (E_2^+ - E_2^-)m_2. \tag{2.117}$$

The boundary condition at the second interface follows [similar to equations (2.101) and (2.102)] as

$$z = \mathbf{r}\cdot\hat{\mathbf{n}} = d: \qquad E_2^+\,e^{-2\pi i\eta_0 m_2 d} + E_2^-\,e^{+2\pi i\eta_0 m_2 d} = E_t\,e^{-2\pi i\eta_0 m_3 d} \tag{2.118}$$

$$(E_2^+\,e^{-2\pi i\eta_0 m_2 d} - E_2^-\,e^{+2\pi i\eta_0 m_2 d})m_2 = E_t\,e^{-2\pi i\eta_0 m_3 d} m_3. \tag{2.119}$$

Equations (2.115), (2.117), (2.118), and (2.119) are four equations in the unknowns E_r, E_2^+, E_2^-, and E_t, which may be solved for the *reflection and transmission coefficients of a thin film*. After some algebra one obtains

$$\widetilde{r}_{\text{film}} = \frac{E_r}{E_i} = \frac{\widetilde{r}_{12} + \widetilde{r}_{23}\,e^{-4\pi i\eta_0 d m_2}}{1 + \widetilde{r}_{12}\widetilde{r}_{23}\,e^{-4\pi i\eta_0 d m_2}}, \tag{2.120}$$

$$\widetilde{t}_{\text{film}} = \frac{E_t\,e^{-2\pi i\eta_0 d m_3}}{E_i} = \frac{\widetilde{t}_{12}\widetilde{t}_{23}\,e^{-2\pi i\eta_0 d m_2}}{1 + \widetilde{r}_{12}\widetilde{r}_{23}\,e^{-4\pi i\eta_0 d m_2}}, \tag{2.121}$$

where $\widetilde{r}_{ij}$ and $\widetilde{t}_{ij}$ are the complex reflection and transmission coefficients of the two interfaces,

$$\widetilde{r}_{12} = \frac{m_1 - m_2}{m_1 + m_2}, \qquad \widetilde{r}_{23} = \frac{m_2 - m_3}{m_2 + m_3}; \tag{2.122a}$$

$$\widetilde{t}_{12} = \frac{2m_1}{m_1 + m_2}, \qquad \widetilde{t}_{23} = \frac{2m_2}{m_2 + m_3}. \tag{2.122b}$$

To evaluate the thin film reflectivity and transmissivity from the complex coefficients, it is advantageous to write the coefficients in polar notation (cf., for example, Wylie [3]),

$$\widetilde{r}_{ij} = r_{ij}\,e^{i\delta_{ij}}, \qquad r_{ij} = |\widetilde{r}_{ij}|, \qquad \tan\delta_{ij} = \frac{\Im(\widetilde{r}_{ij})}{\Re(\widetilde{r}_{ij})}, \tag{2.123a}$$

$$\widetilde{t}_{ij} = t_{ij}\,e^{i\epsilon_{ij}}, \qquad t_{ij} = |\widetilde{t}_{ij}|, \qquad \tan\epsilon_{ij} = \frac{\Im(\widetilde{t}_{ij})}{\Re(\widetilde{t}_{ij})}, \tag{2.123b}$$

where r_{ij} and t_{ij} are the absolute values, and δ_{ij} and ϵ_{ij} the phase angles of the coefficients. Care must be taken in the evaluation of phase angles, since the tangent has a period of π, rather than 2π: The correct quadrant for δ_{ij} and ϵ_{ij} is found by inspecting the signs of the real and imaginary parts of $\widetilde{r}_{ij}$ and $\widetilde{t}_{ij}$, respectively. This calculation leads, after more algebra, to the reflectivity, $R_{\rm film}$, and transmissivity, $T_{\rm film}$, of the thin film as

$$R_{\rm film} = \widetilde{r}\widetilde{r}^* = \frac{r_{12}^2 + 2r_{12}r_{23}\,e^{-\kappa_2 d}\cos(\delta_{12} - \delta_{23} + \zeta_2) + r_{23}^2\,e^{-2\kappa_2 d}}{1 + 2r_{12}r_{23}\,e^{-\kappa_2 d}\cos(\delta_{12} + \delta_{23} - \zeta_2) + r_{12}^2 r_{23}^2\,e^{-2\kappa_2 d}}, \tag{2.124}$$

$$T_{\rm film} = \frac{n_3}{n_1}\widetilde{t}\,\widetilde{t}^* = \frac{\tau_{12}\tau_{23}\,e^{-\kappa_2 d}}{1 + 2r_{12}r_{23}\,e^{-\kappa_2 d}\cos(\delta_{12} + \delta_{23} - \zeta_2) + r_{12}^2 r_{23}^2\,e^{-2\kappa_2 d}}, \tag{2.125}$$

where

$$r_{ij}^2 = \rho_{ij} = \frac{(n_i - n_j)^2 + (k_i - k_j)^2}{(n_i + n_j)^2 + (k_i + k_j)^2}, \tag{2.126a}$$

$$\frac{n_j}{n_i}t_{ij}^2 = \tau_{ij} = \frac{n_i}{n_j}\frac{4(n_i^2 + k_i^2)}{(n_i + n_j)^2 + (k_i + k_j)^2}, \tag{2.126b}$$

$$\tan\delta_{ij} = \frac{2(n_i k_j - n_j k_i)}{n_i^2 + k_i^2 - (n_j^2 + k_j^2)}, \tag{2.126c}$$

$$\kappa_i = 4\pi\eta_0 k_i, \quad \zeta_i = 4\pi\eta_0 n_i d. \tag{2.126d}$$

The correct quadrant for δ_{ij} is found by checking the sign of both the numerator and denominator in equation (2.126*c*) (which, while different from the real and imaginary parts of $\widetilde{r}_{ij}$, carry their signs). If both adjacent media, i and j, are dielectrics then $\widetilde{r}_{ij} = r_{ij}$ is real. In that case we set $\delta_{ij} = 0$ and let r_{ij} carry a sign. The definition of the thin film transmissivity includes the factor (n_3/n_1), since it is the magnitude of the transmitted and incoming *Poynting vector*, equation (2.42), that must be compared.

Example 2.6. Determine the reflectivity and transmissivity of a 5 μm thick manganese sulfide (MnS) crystal ($n = 2.68$, $k \ll 1$), suspended in air, for the wavelength range between 1 μm and 1.25 μm.

Solution

Assuming $n_1 = n_3 = 1$, $k_1 = k_2 = k_3 = 0$, and $n_2 = 2.68$ and substituting these into equations (2.126) leads to

$$r_{12} = r_{23} = \frac{n_2 - 1}{n_2 + 1}; \quad t_{12} = \frac{2}{n_2 + 1}, \quad t_{23} = \frac{2n_2}{n_2 + 1};$$

$$\tan\delta_{12} = \frac{0}{1 - n_2^2} = 0; \quad \tan\delta_{23} = \frac{0}{n_2^2 - 1} = 0.$$

Since the real part of $\widetilde{r}_{12}$ is negative, i.e., $1 - n_2^2 < 0$, it follows that $\delta_{12} = \pi$. By similar reasoning $\delta_{23} = 0$. Alternatively, since all media are dielectrics, we could have set $\delta_{12} = \delta_{23} = 0$ and $r_{12} = -r_{23}$. Thus, with $\kappa_2 = 0$, the reflectivity and transmissivity of a dielectric thin film follow as

$$R_{\rm film} = \frac{2\rho_{12}(1 - \cos\zeta_2)}{1 - 2\rho_{12}\cos\zeta_2 + \rho_{12}^2}, \tag{2.127}$$

$$T_{\rm film} = \frac{\tau_{12}^2}{1 - 2\rho_{12}\cos\zeta_2 + \rho_{12}^2}. \tag{2.128}$$

It is a simple matter to show that $\tau_{12} = \tau_{23} = 1 - \rho_{12}$ and, therefore, $R_{\rm film} + T_{\rm film} = 1$ for a dielectric medium. Substituting numbers for MnS gives $\rho_{12} = 0.2084$ and

$$R_{\rm film} = \frac{0.3995(1 - \cos\zeta_2)}{1 - 0.3995\cos\zeta_2}, \quad T_{\rm film} = \frac{0.6005}{1 - 0.3995\cos\zeta_2},$$

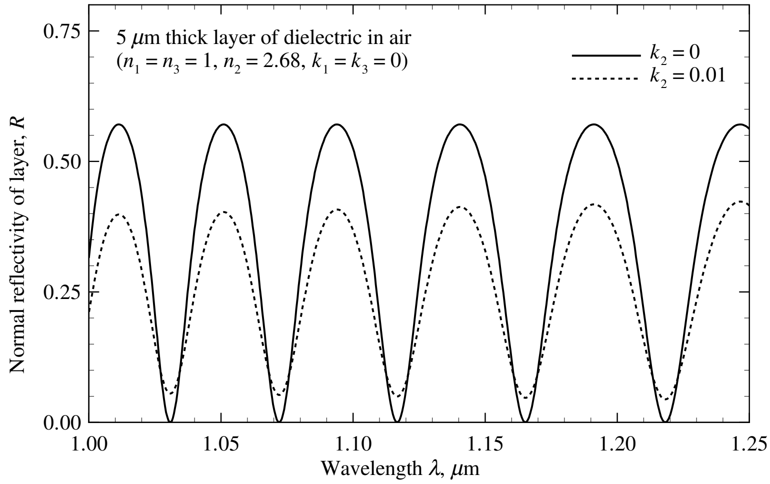

FIGURE 2-13
Normal reflectivity of a thin film with interference effects.

with $\zeta_2 = 4\pi n_2 d\eta_0 = 168.4\,\mu\text{m}\,\eta_0 = 168.4\,\mu\text{m}/\lambda_0$. R_{film} and T_{film} are periodic with a period of $\Delta\eta_0 = 2\pi/168.4\,\mu\text{m} = 0.0373\,\mu\text{m}^{-1}$. At $\lambda_0 = 1\,\mu$m this fact implies $\Delta\lambda_0 = \lambda_0^2\Delta\eta_0 = 0.0373\,\mu$m. The reflectivity of the dielectric film in Fig. 2-13 shows a periodic reflectivity with maxima of 0.5709 (at $\zeta_2 = \pi, 3\pi, \ldots$). For values of $\zeta_2 = 2\pi, 4\pi, \ldots$, the reflectivity of the layer vanishes altogether. Also shown is the case of a slightly absorbing film, with $k_2 = 0.01$. Maximum and minimum reflectivity (as well as transmissivity) decrease and increase somewhat, respectively. This effect is less pronounced at larger wavelengths, i.e., wherever the absorption coefficient κ_2 is smaller [cf. equation (2.126*d*)].

While equations (2.124) through (2.126) are valid for arbitrary slab thicknesses, their application to thick slabs becomes problematic as well as unnecessary. Problematic because *(i)* for $d \gg \lambda_0$ the period of reflectivity oscillations corresponds to smaller values of $\Delta\lambda_0$ between extrema than can be measured, and *(ii)* for $d \gg \lambda_0$ it becomes rather unlikely that the distance d remains constant within a fraction of λ_0 over an extended area. Thick slab reflectivities and transmissivities may be obtained by averaging equations (2.124) and (2.125) over a period through integration, which results in

$$R_{\text{slab}} = \rho_{12} + \frac{\rho_{23}(1-\rho_{12})^2\,e^{-2\kappa_2 d}}{1-\rho_{12}\rho_{23}\,e^{-2\kappa_2 d}}, \tag{2.129}$$

$$T_{\text{slab}} = \frac{(1-\rho_{12})(1-\rho_{23})\,e^{-\kappa_2 d}}{1-\rho_{12}\rho_{23}\,e^{-2\kappa_2 d}}, \tag{2.130}$$

where for T_{slab} use has been made of the fact that k_1 and k_2 must be very small, if an appreciable amount of energy is to reach Medium 3. The same relations for thick sheets without wave interference will be developed in the following chapter through geometric optics.

Oblique Incidence

Knittl [7] has shown that equations (2.124) and (2.125) remain valid for each polarization for oblique incidence if the interface reflectivities, ρ_{ij}, and transmissivities, τ_{ij}, are replaced by their directional values; see, for example, equations (2.113). We will state the final result here, mostly following the development of Zhang [9]. The field reflection and transmission coefficients are then expressed as

$$\widetilde{r} = \widetilde{r}_{12} + \frac{\widetilde{t}_{12}\widetilde{t}_{21}\widetilde{r}_{23}e^{-2i\beta}}{1-\widetilde{r}_{21}\widetilde{r}_{23}e^{-2i\beta}}, \tag{2.131a}$$

$$\widetilde{t} = \frac{\widetilde{t}_{12}\widetilde{t}_{23}e^{-i\beta}}{1-\widetilde{r}_{21}\widetilde{r}_{23}e^{-2i\beta}}, \tag{2.131b}$$

which are known as *Airy's formulae*. Here the interface reflectivity and transmissivity coefficients are given by equations (2.89) through (2.92) for dielectrics, and by equations (2.111) and (2.112) for absorbing media, and the phase shift in Medium 2 is, for a dielectric film, calculated from

$$\beta = 2\pi\eta_0 n_i d\cos\theta_2. \tag{2.131c}$$

The overall reflectivity of the film follows from

$$R_{\text{film}} = \widetilde{r}\widetilde{r}^* = \left| \widetilde{r}_{12} + \frac{\widetilde{t}_{12}\widetilde{t}_{21}\widetilde{r}_{23}e^{-2i\beta}}{1-\widetilde{r}_{21}\widetilde{r}_{23}e^{-2i\beta}} \right|^2, \tag{2.132}$$

and, if Media 1 and 3 are dielectrics, the film transmissivity is evaluated as

$$T_{\text{film}} = \frac{n_3\cos\theta_3}{n_1\cos\theta_1}\widetilde{t}\,\widetilde{t}^* = \frac{n_3\cos\theta_3}{n_1\cos\theta_1}\left| \frac{\widetilde{t}_{12}\widetilde{t}_{23}e^{-i\beta}}{1-\widetilde{r}_{21}\widetilde{r}_{23}e^{-2i\beta}} \right|. \tag{2.133}$$

As for single interfaces, for random polarization equations (2.132) and (2.133) are evaluated independently for parallel and perpendicular polarizations, followed by averaging.

2.6 THEORIES FOR OPTICAL CONSTANTS

If the radiative properties of a surface—absorptivity, emissivity, and reflectivity—are to be theoretically evaluated from electromagnetic wave theory, the complex index of refraction, m, must be known over the spectral range of interest. A number of classical and quantum mechanical *dispersion theories* have been developed to predict the phenomenological coefficients ϵ (electrical permittivity) and σ_e (electrical conductivity) as functions of the frequency (or wavelength) of incident electromagnetic waves for a number of different interaction phenomena and types of surfaces. While the complex index of refraction, $m = n - ik$, is most convenient for the treatment of wave propagation, the *complex dielectric function* (or *relative permittivity*), $\varepsilon = \varepsilon' - i\varepsilon''$, is more appropriate when the microscopic mechanisms are considered that determine the magnitude of the phenomenological coefficients. The two sets of parameters are related by the expression

$$\varepsilon = \varepsilon' - i\varepsilon'' = \frac{\epsilon}{\epsilon_0} - i\frac{\sigma_e}{2\pi\nu\epsilon_0} = m^2 \tag{2.134}$$

[compare equations (2.31) through (2.35)] and, therefore,

$$\varepsilon' = \frac{\epsilon}{\epsilon_0} = n^2 - k^2, \tag{2.135a}$$

$$\varepsilon'' = \frac{\sigma_e}{2\pi\nu\epsilon_0} = 2nk, \tag{2.135b}$$

$$n^2 = \frac{1}{2}\left(\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}\right), \tag{2.136a}$$

$$k^2 = \frac{1}{2}\left(-\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}\right), \tag{2.136b}$$

where we have again assumed the medium to be nonmagnetic ($\mu = \mu_0$).

Any material may absorb or emit radiative energy at many different wavelengths as a result of impurities (presence of foreign atoms) and imperfections in the ionic crystal lattice. However, a number of phenomena tend to dominate the optical behavior of a substance. In the frequency

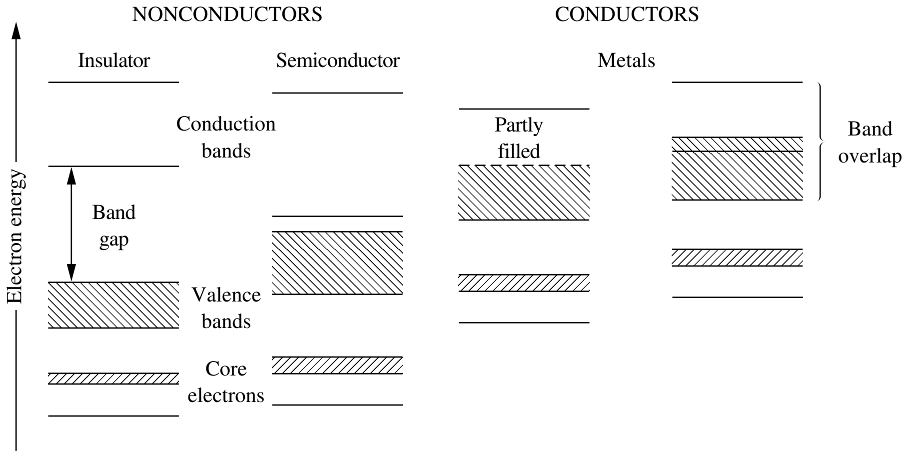

FIGURE 2-14
Electron energy bands and band gaps in a solid (shading indicates amount of electrons filling the bands) [2].

range of interest to the heat transfer engineer (ultraviolet to midinfrared), electromagnetic waves are primarily absorbed by *free* and *bound electrons* or by change in the energy level of *lattice vibration* (converting a *photon* into a *phonon*, i.e., a quantum of lattice vibration). Since electricity is conducted by free electrons, and since free electrons are a major contributor to a solid's ability to absorb radiative energy, there are distinct optical differences between *conductors* and *nonconductors* of electricity. Every solid has a large number of electrons, resulting in a near-continuum of possible energy states (and, therefore, a near-continuum of photon frequencies that can be absorbed). However, these allowed energy states occur in *bands.* Between the bands of allowed energy states may be *band gaps*, i.e., energy states that the solid cannot attain. This is schematically shown in Fig. 2-14. If a material has a band gap between completely filled and completely empty energy bands, the material is a *nonconductor*, i.e., an *insulator* (wide band gap), or a *semiconductor* (narrow band gap). If a band of electron energy states is incompletely filled or overlaps another, empty band, electrons can be excited into adjacent energy states resulting in an electric current, and the material is called a *conductor*. Electronic absorption by nonconductors is likely only for photons with energies greater than the band gap, although sometimes two or more photons may combine to bridge the band gap. An *intraband transition* occurs when an electron changes its energy level, but stays within the same band (which can only occur in a conductor); if an electron moves into a different band (i.e., overcomes the band gap) the movement is termed an *interband transition* (and can occur in both conductors and nonconductors). This difference between conductors and nonconductors causes substantially different optical behavior: Insulators tend to be transparent and weakly reflecting for photons with energies less than the band gap, while metals tend to be highly absorbing and reflecting between the visible and infrared wavelengths [2].

During the beginning of the century Lorentz [10]* developed a classical theory for the evaluation of the dielectric function by assuming electrons and ions are *harmonic oscillators* (i.e., springs) subjected to forces from interacting electromagnetic waves. His result was equivalent to

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*__Hendrik Anton Lorentz (1853–1928)__
Dutch physicist. Lorentz studied at Leiden University, where he subsequently served as professor of mathematical physics for the rest of his life. His major work lay in refining the electromagnetic theory of Maxwell. For his theory that the oscillations of charged particles inside atoms were the source of light, he and his student Pieter Zeeman received the 1902 Nobel Prize in Physics. Lorentz is also famous for his Lorentz transformations, which describe the increase of mass of a moving body. These laid the foundation for Einstein's special theory of relativity.

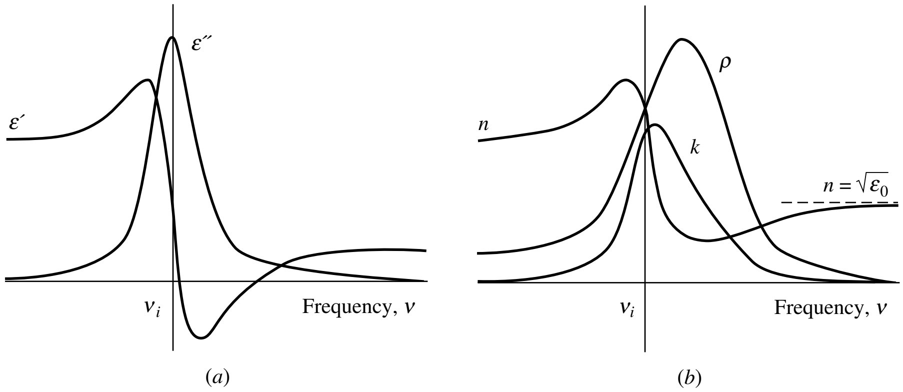

FIGURE 2-15
Lorentz model for *(a)* the dielectric function, *(b)* the index of refraction, and normal, spectral reflectivity.

the subsequent quantum mechanical development, and may be stated, as described by Bohren and Huffman [2], as

$$\varepsilon(\nu) = 1 + \sum_j \frac{\nu_{pj}^2}{\nu_j^2 - \nu^2 + i\gamma_j\nu}, \tag{2.137}$$

where the summation is over different types of oscillators, ν_{pj} is known as the *plasma frequency* (and ν_{pj}^2 is proportional to the number of oscillators of type j), ν_j is the *resonance frequency*, and γ_j is the damping factor of the oscillators. Thus, the dielectric function may have a number of bands centered at ν_j, which may or may not overlap one another. Inspecting equation (2.137), we see that for $\nu \gg \nu_j$ the contribution of band j to ε vanishes, while for $\nu \ll \nu_j$ it goes to the constant value of $(\nu_{pj}/\nu_j)^2$. Therefore, for any nonoverlapping band i, we may rewrite equation (2.137) as

$$\varepsilon(\nu) = \varepsilon_0 + \frac{\nu_{pi}^2}{\nu_i^2 - \nu^2 + i\gamma_i\nu}, \tag{2.138}$$

where ε_0 incorporates the contributions from all bands with $\nu_j > \nu_i$. Equation (2.138) may be separated into its real and imaginary components, or

$$\varepsilon' = \varepsilon_0 + \frac{\nu_{pi}^2(\nu_i^2 - \nu^2)}{(\nu_i^2 - \nu^2)^2 + \gamma_i^2\nu^2}, \tag{2.139a}$$

$$\varepsilon'' = \frac{\nu_{pi}^2\gamma_i\nu}{(\nu_i^2 - \nu^2)^2 + \gamma_i^2\nu^2}. \tag{2.139b}$$

The frequency dependence of the real and imaginary parts of the dielectric function for a single oscillating band is shown qualitatively in Fig. 2-15; also shown are the corresponding curves for the real and imaginary parts of the complex index of refraction as evaluated from equation (2.136), along with the qualitative behavior of the normal, spectral reflectivity of a surface from equation (2.114). A strong band with $k \gg 0$ results in a region with strong absorption around the resonance frequency and an associated region of high reflection: Incoming photons are mostly reflected, and those few that penetrate into the medium are rapidly attenuated. On either side outside the band the refractive index n increases with increasing frequency (or decreasing wavelength); this is called *normal dispersion*. However, close to the resonance frequency, n decreases with increasing frequency; this decrease is known as *anomalous dispersion*. Note that ε' may become negative, resulting in spectral regions with $n < 1$.

All solids and liquids may absorb photons whose energy content matches the energy difference between filled and empty electron energy levels on separate bands. Since such transitions require a substantial amount of energy, they generally occur in the ultraviolet (i.e., at high frequency). A near-continuum of electron energy levels results in an extensive region of strong absorption (and often many overlapping bands). It takes considerably less energy to excite the vibrational modes of a crystal lattice, resulting in absorption bands in the midinfrared (around 10 μm). Since generally few different vibrational modes exist in an isotropic lattice, such transitions can often be modeled by equation (2.137) with a single band. In the case of electrical conductors photons may also be absorbed to raise the energy levels of free electrons and of bound electrons within partially filled or partially overlapping electron bands. The former, because of the nearly arbitrary energy levels that a free electron may assume, results in a single large band in the far infrared; the latter causes narrower bands in the ultraviolet to infrared.

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Problems

2.1 Show that for an electromagnetic wave traveling through a dielectric ($m_1 = n_1$), impinging on the interface with another, optically less dense dense dielectric ($n_2 < n_1$), light of any polarization is totally reflected for incidence angles larger than $\theta_c = \sin^{-1}(n_2/n_1)$.
Hint: Use equations (2.105) and (2.106) with $k_2 = 0$.

2.2 Derive equations (2.109) using the same approach as in the development of equations (2.89) through (2.92).
Hint: Remember that within the absorbing medium, $\mathbf{w} = \mathbf{w}' - i\mathbf{w}'' = w'\hat{\mathbf{s}} - iw''\hat{\mathbf{n}}$; this implies that $\mathbf{E}_0$ is *not* a vector normal to $\hat{\mathbf{s}}$. It is best to assume $\mathbf{E}_0 = E_\parallel\hat{\mathbf{e}}_\parallel + E_\perp\hat{\mathbf{e}}_\perp + E_s\hat{\mathbf{s}}$.

2.3 Find the normal spectral reflectivity at the interface between two absorbing media.
Hint: Use an approach similar to the one that led to equations (2.89) and (2.90), keeping in mind that all wave vectors will be complex, but that the wave will be homogeneous in both media, i.e., all components of the wave vectors are colinear with the surface normal.

2.4 A circularly polarized wave in air is incident upon a smooth dielectric surface ($n = 1.5$) with a direction of 45° off normal. What are the normalized Stokes' parameters before and after the reflection, and what are the degrees of polarization?

2.5 A circularly polarized wave in air traveling along the z-axis is incident upon a dielectric surface ($n = 1.5$). How must the dielectric–air interface be oriented so that the reflected wave is a linearly polarized wave in the y-z-plane?

2.6 A polished platinum surface is coated with a 1 μm thick layer of MgO.

(a) Determine the material's reflectivity in the vicinity of $\lambda = 2\,\mu$m (for platinum at 2 μm $m_{\text{Pt}} = 5.29 - 6.71\,i$, for MgO $m_{\text{MgO}} = 1.65 - 0.0001\,i$).

(b) Estimate the thickness of MgO required to reduce the average reflectivity in the vicinity of 2 μm to 0.4. What happens to the interference effects for this case?