
CHAPTER
18

THE ZONAL METHOD

18.1 INTRODUCTION

The zonal method for the determination of radiative heat transfer rates within an absorbing, emitting, and isotropically scattering medium is an extension of the net radiation method developed in Chapter 5 for surface exchange (i.e., for enclosures without a participating medium). In this method the enclosure is subdivided into a finite number of isothermal volume and surface area zones. An energy balance is then performed for the radiative exchange between any two zones, employing precalculated "exchange areas." This process leads to a set of simultaneous equations for the unknown temperatures or heat fluxes. The method was first developed by Hottel and Cohen [1] for an absorbing, emitting, nonscattering gray gas with constant absorption coefficient. Hottel and Sarofim [2] extended it to deal with nonconstant and nongray absorption coefficients as well as with isotropically scattering media. The discussion by Hottel and Sarofim [2] is extensive but is limited to three-dimensional problems (i.e., only volume zones finite in all three dimensions are discussed). The method was extended by Walther and coworkers [3] to allow for linear variation of emissive power through a one-dimensional slab zone and by Einstein to deal with two-dimensional zones in Cartesian [4] and cylindrical [5] coordinate systems. A slight variation of the method has been given by Larsen and Howell [6], who expressed the energy balances in terms of "exchange factors," which are physically measurable quantities. Such measurements were carried out by Liu and Howell [7]. Another derivative model is the REM^2 method by Maruyama [8–10]. Yuen and Takara [11] (and Ma [12] for a one-dimensional slab) have shown that the zonal method can be applied to anisotropically scattering media, but only at a great increase in required computer time.

18.2 SURFACE EXCHANGE — NO PARTICIPATING MEDIUM

We shall start by rederiving the relations for radiative exchange between surfaces in the zonal form, since doing so clearly demonstrates the similarities between the standard net radiation method and the zonal method. As in the net radiation approach we break up the surface of the enclosure into N isothermal subsurfaces, or zones, as shown in Fig. 18-1.

Black Surfaces — Direct Exchange Areas

If the enclosure consists of only black surfaces then the net exchange of radiative energy between any two surfaces is, from equation (5.9),

$$Q_{i\leftrightarrow j} = -Q_{j\leftrightarrow i} = \overline{s_i s_j}\,(E_{bi} - E_{bj}), \qquad i, j = 1, 2, \ldots, N, \tag{18.1}$$

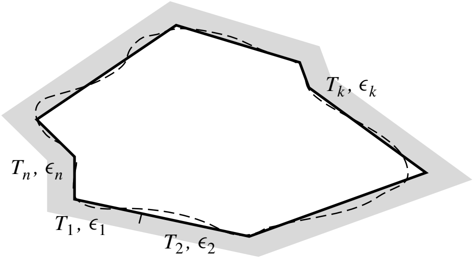

FIGURE 18-1
General enclosure broken into N isothermal surface zones.

where $Q_{i\leftrightarrow j}$ is the net energy exchange between zones i and j (positive if zone i is losing heat as a result of the exchange) and the

$$\overline{s_is_j} = \overline{s_js_i} = A_iF_{i\to j} = A_jF_{j\to i} = \int_{A_i}\int_{A_j} \frac{\cos\theta_i\cos\theta_j}{\pi S_{ij}^2}\,dA_j\,dA_i \tag{18.2}$$

are known as *direct exchange areas*. Unlike the view factors, $F_{i\to j}$, the $\overline{s_is_j}$ are not nondimensional but have the dimensions of area. While this is a minor inconvenience the formulation of equation (18.2) has the advantage that the principle of reciprocity is more easily applied (eliminating a common source of error). Although the use of *direct exchange factors* (reducing to view factors in the absence of a participating medium) would be just as acceptable, the use of exchange areas is accepted practice. Summing equation (18.1) over all zones yields the net heat flux at zone i as

$$Q_i = A_iq_i = \sum_{j=1}^{N} \overline{s_is_j}\,(E_{bi} - E_{bj}) = A_iE_{bi} - \sum_{j=1}^{N} \overline{s_is_j}\,E_{bj}, \quad i = 1,2,\ldots,N. \tag{18.3}$$

Since the total heat flux leaving surface i is A_iE_{bi}, we find that

$$\sum_{j=1}^{N} \overline{s_is_j} = A_i, \tag{18.4}$$

which is equivalent to equation (4.18), or $\sum_{j=1}^{N} F_{i\to j} = 1$.

Gray Diffuse Surfaces — Total Exchange Areas

If the surfaces are not black but are partially reflective, then the energy exchange between zones is not only by direct travel, but also may include contributions due to single and multiple reflections from any number of surface zones. If the reflective behavior of the surfaces is gray and diffuse, then the reflected radiation leaving a surface cannot be distinguished from emitted radiation and equation (18.1) still holds after replacing emissive power E_b by radiosity J (as described in detail in Chapter 5). Thus,

$$Q_{i\leftrightarrow j} = -Q_{j\leftrightarrow i} = \overline{s_is_j}\,(J_i - J_j), \quad i,j = 1,2,\ldots,N, \tag{18.5}$$

and, performing an energy balance on A_i,

$$Q_i = A_iq_i = A_i\,(J_i - H_i) = A_iJ_i - \sum_{j=1}^{N} \overline{s_is_j}\,J_j, \quad i = 1,2,\ldots,N. \tag{18.6}$$

Thus, similar to the net radiation method, we must now assume that the surface zones are small enough so that their *radiosities* do not vary appreciably across them.

We may eliminate the radiosities from equation (18.6) by using equation (5.26),

$$q_i = \frac{\epsilon_i}{1-\epsilon_i}\left(E_{bi} - J_i\right), \quad i = 1, 2, \ldots, N. \tag{18.7}$$

This leads to

$$\sum_{j=1}^{N}\left(\frac{A_j\delta_{ij}}{\epsilon_j} - \frac{1-\epsilon_j}{\epsilon_j}\,\overline{s_is_j}\right)q_j = \sum_{j=1}^{N}\left(A_j\delta_{ij} - \overline{s_is_j}\right)E_{bj}, \quad i = 1, 2, \ldots, N, \tag{18.8}$$

which is equivalent to equation (5.37). Thus, if all temperatures are known, the unknown wall heat fluxes may be determined by matrix inversion. If the zonal temperatures are not known, but must be determined iteratively by considering conduction and/or convection as well, then one matrix inversion must be performed for every iteration. To avoid such unnecessary matrix inversions (since the exchange areas do not depend on temperature), Hottel and Cohen [1] introduced the concept of *total exchange areas*, defined by

$$Q_{i\leftrightarrow j} = -Q_{j\leftrightarrow i} = \overline{S_iS_j}\left(E_{bi} - E_{bj}\right), \quad i, j = 1, 2, \ldots, N, \tag{18.9}$$

which include energy exchange by direct travel as well as by paths with one or more surface reflections. This concept implies that reciprocity holds also for total exchange areas, that is,

$$\overline{S_iS_j} = \overline{S_jS_i}, \quad \text{all}\, i, j. \tag{18.10}$$

Once the $\overline{S_iS_j}$ have been determined, the heat flux for each zone is found immediately and without matrix inversion from

$$\begin{aligned} Q_i = A_iq_i &= \sum_{j=1}^{N}\overline{S_iS_j}\left(E_{bi} - E_{bj}\right) \\ &= \epsilon_iA_iE_{bi} - \sum_{j=1}^{N}\overline{S_iS_j}\,E_{bj}, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{18.11}$$

where the last part of equation (18.11) follows from the fact that total emission from zone i is $\epsilon_iA_iE_{bi}$. We conclude that, for a nonparticipating medium,

$$\sum_{j=1}^{N}\overline{S_iS_j} = \epsilon_iA_i, \quad i = 1, 2, \ldots, N. \tag{18.12}$$

There are many ways of expressing the total exchange areas $\overline{S_iS_j}$ in terms of direct exchange areas $\overline{s_is_j}$. Hottel and Cohen [1] and Hottel and Sarofim [2] achieved this by setting the emissive power of all zones to zero except for zone k, for which the emissive power is set to unity. From equation (18.11) it follows then that, for this case,

$$Q_i = -\overline{S_iS_k} = A_i\,{}_kJ_i - \sum_{j=1}^{N}\overline{s_is_j}\,{}_kJ_j, \quad i = 1, 2, \ldots, N, \tag{18.13}$$

where the presubscript k for the J_i implies that these artificial radiosities are for a single emitting zone with unit emissive power. The Q_i can be eliminated using equation (18.7), resulting in N simultaneous equations for the unknown ${}_kJ_i$ ($i = 1, 2, \ldots, N$). After their determination the $\overline{S_iS_k}$ may be determined from equation (18.13). Utilizing a different approach Noble [13] cast the governing equations in matrix form and used elegant matrix manipulation to evaluate the total exchange areas. We shall follow here an approach similar to Noble's but shall use a somewhat more conventional notation to accommodate the reader who is not very familiar with matrix manipulation.

Making a simple heat balance for zone i,

$$Q_i = \epsilon_i A_i E_{bi} - \epsilon_i A_i H_i \tag{18.14}$$

(stating that the net heat flux leaving zone i is the difference between emitted and absorbed irradiation), we find from equation (18.6) that the direct exchange areas are related to the absorbed irradiation, $\epsilon_i A_i H_i$, where H_i is total irradiation per unit area on zone i, or

$$A_i H_i = \sum_{j=1}^{N} \overline{s_i s_j} J_j, \quad i = 1, 2, \ldots, N. \tag{18.15}$$

We now eliminate the radiosities from equation (18.15) using its definition, equation (5.18),

$$J_j = \epsilon_j E_{bj} + \rho_j H_j, \tag{18.16}$$

which leads to

$$\sum_{j=1}^{N} \left(\frac{\delta_{ij}}{\epsilon_j} - \frac{\rho_j \overline{s_i s_j}}{\epsilon_j A_j} \right) \epsilon_j A_j H_j = \sum_{j=1}^{N} \overline{s_i s_j}\, \epsilon_j E_{bj}, \quad i = 1, 2, \ldots, N. \tag{18.17}$$

Equation (18.17) is a set of N linear equations in the N unknown H_j. The general solution to this equation is most easily found by casting it into matrix form. We define two $N \times N$ matrices (or *second rank tensors*),

$$\mathbf{T} = \begin{pmatrix} \dfrac{1}{\epsilon_1} - \dfrac{\rho_1 \overline{s_1 s_1}}{\epsilon_1 A_1} & -\dfrac{\rho_2 \overline{s_1 s_2}}{\epsilon_2 A_2} & \cdots & -\dfrac{\rho_N \overline{s_1 s_N}}{\epsilon_N A_N} \\ -\dfrac{\rho_1 \overline{s_2 s_1}}{\epsilon_1 A_1} & \dfrac{1}{\epsilon_2} - \dfrac{\rho_2 \overline{s_2 s_2}}{\epsilon_2 A_2} & \cdots & -\dfrac{\rho_N \overline{s_2 s_N}}{\epsilon_N A_N} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{\rho_1 \overline{s_N s_1}}{\epsilon_1 A_1} & -\dfrac{\rho_2 \overline{s_N s_2}}{\epsilon_2 A_2} & \cdots & \dfrac{1}{\epsilon_N} - \dfrac{\rho_N \overline{s_N s_N}}{\epsilon_N A_N} \end{pmatrix} \tag{18.18}$$

and

$$\mathbf{S} = \begin{pmatrix} \overline{s_1 s_1}\,\epsilon_1 & \overline{s_1 s_2}\,\epsilon_2 & \cdots & \overline{s_1 s_N}\,\epsilon_N \\ \overline{s_2 s_1}\,\epsilon_1 & \overline{s_2 s_2}\,\epsilon_2 & \cdots & \overline{s_2 s_N}\,\epsilon_N \\ \vdots & \vdots & \ddots & \vdots \\ \overline{s_N s_1}\,\epsilon_1 & \overline{s_N s_2}\,\epsilon_2 & \cdots & \overline{s_N s_N}\,\epsilon_N \end{pmatrix}, \tag{18.19}$$

as well as two vectors $\mathbf{h}$ and $\mathbf{e_b}$,[1]

$$\mathbf{h} = \begin{pmatrix} \epsilon_1 A_1 H_1 \\ \epsilon_2 A_2 H_2 \\ \vdots \\ \epsilon_N A_N H_N \end{pmatrix}, \quad \mathbf{e_b} = \begin{pmatrix} E_{b1} \\ E_{b2} \\ \vdots \\ E_{bN} \end{pmatrix}. \tag{18.20}$$

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[1] For easy readability of matrix manipulations we shall follow here the convention that a two-dimensional matrix is denoted by a bold capitalized letter, while a vector is written as a bold lowercase letter.

Equation (18.17) may then be rewritten[2] as

$$\mathbf{T}\cdot\mathbf{h} = \mathbf{S}\cdot\mathbf{e_b}. \tag{18.21}$$

If we form the dot product of equation (18.21) with the *inverse* of **T**, or $\mathbf{T^{-1}}$, we get

$$\mathbf{T^{-1}}\cdot\mathbf{T}\cdot\mathbf{h} = \mathbf{h} = \mathbf{T^{-1}}\cdot\mathbf{S}\cdot\mathbf{e_b}, \tag{18.22}$$

where we have made use of the fact that $\mathbf{T^{-1}}\cdot\mathbf{T} = \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the *unit tensor* with elements δ_{ij}, and $\boldsymbol{\delta}\cdot\mathbf{h} = \left[\sum_j \delta_{ij}h_j\right] = [h_i] = \mathbf{h}$. Thus, we may write, in series notation,

$$\epsilon_i A_i H_i = \sum_{k=1}^{N}\sum_{j=1}^{N}(T^{-1})_{ik}S_{kj}E_{bj}. \tag{18.23}$$

By comparing this expression with equations (18.11) and (18.14) we find the *total exchange areas* as

$$\overline{S_iS_j} = \sum_{k=1}^{N}(T^{-1})_{ik}S_{kj}, \tag{18.24}$$

or, in matrix notation

$$\overline{\mathbf{SS}} = \mathbf{T^{-1}}\cdot\mathbf{S}. \tag{18.25}$$

We note in passing that, for every black zone, an entire column of **T** has elements that are zero (with the exception of the diagonal term), simplifying the inversion of **T**.

Example 18.1. Evaluate the radiative heat flux between two infinitely long concentric cylinders separated by a nonparticipating medium. Both cylinders are isothermal and are covered with a gray, diffusely emitting and reflecting material. The inner cylinder is of radius R_1 with temperature T_1 and emittance ϵ_1. The outer cylinder has the corresponding values of R_2, T_2, and ϵ_2. Use the zonal method and employ the concept of total exchange areas.

Solution

Letting the inner cylinder be Zone 1 and the outer cylinder Zone 2 we have

$$\overline{s_1s_2} = \overline{s_2s_1} = A_1F_{1\to2} = A_1, \quad \overline{s_1s_1} = 0, \quad \overline{s_2s_2} = A_2F_{2\to2} = A_2 - A_1.$$

Therefore, we get

$$\mathbf{T} = \begin{pmatrix} \dfrac{1}{\epsilon_1} & -\dfrac{\rho_2}{\epsilon_2}\dfrac{A_1}{A_2} \\ -\dfrac{\rho_1}{\epsilon_1} & 1+\dfrac{\rho_2}{\epsilon_2}\dfrac{A_1}{A_2} \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} 0 & A_1\epsilon_2 \\ A_1\epsilon_1 & (A_2-A_1)\epsilon_2 \end{pmatrix}.$$

The inverse of **T** follows immediately as

$$\mathbf{T^{-1}} = \frac{1}{T}\begin{pmatrix} 1+\dfrac{\rho_2}{\epsilon_2}\dfrac{A_1}{A_2} & \dfrac{\rho_2}{\epsilon_2}\dfrac{A_1}{A_2} \\ \dfrac{\rho_1}{\epsilon_1} & \dfrac{1}{\epsilon_1} \end{pmatrix},$$

where

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[2] Again, for easy readability of matrix manipulations we adopt the convention that a dot product denotes summation over the closest indices on both sides of the dot, e.g., $\mathbf{T}\cdot\mathbf{h} = \left[\sum_j T_{ij}h_j\right]$, $\mathbf{D}\cdot\mathbf{T}\cdot\mathbf{h} = \left[\sum_i\sum_j D_{ki}T_{ij}h_j\right]$, etc. Thus, a tensor dotted with a vector gives a vector, a tensor dotted with a tensor results in another tensor, and so forth.

$$T = |\mathbf{T}| = \frac{1}{\epsilon_1} + \frac{\rho_2}{\epsilon_2}\frac{A_1}{A_2}.$$

The total exchange areas are calculated from

$$\overline{S_iS_j} = \sum_{k=1}^{2} (T^{-1})_{ik} S_{kj},$$

or

$$\begin{aligned}
\overline{S_1S_1} &= T^{-1}_{11}S_{11} + T^{-1}_{12}S_{21} = \frac{1}{T}\frac{\rho_2}{\epsilon_2}\frac{A_1}{A_2}A_1\epsilon_1,\\
\overline{S_1S_2} &= T^{-1}_{11}S_{12} + T^{-1}_{12}S_{22} = \overline{S_2S_1}\\
&= \frac{1}{T}\left[\left(1 + \frac{\rho_2}{\epsilon_2}\frac{A_1}{A_2}\right)A_1\epsilon_2 + \frac{\rho_2}{\epsilon_2}\frac{A_1}{A_2}(A_2 - A_1)\,\epsilon_2\right] = \frac{A_1}{T},\\
\overline{S_2S_2} &= T^{-1}_{21}S_{12} + T^{-1}_{22}S_{22} = \frac{1}{T}\left[\frac{\rho_1}{\epsilon_1}A_1\epsilon_2 + \frac{\epsilon_2}{\epsilon_1}(A_2 - A_1)\right].
\end{aligned}$$

The net heat flux at Surface 1 follows as

$$Q_1 = A_1q_1 = \overline{S_1S_2}\,(E_{b1} - E_{b2})$$

or

$$\Psi_1 = \frac{q_1}{\sigma(T_1^4 - T_2^4)} = \frac{1}{T} = \frac{1}{\dfrac{1}{\epsilon_1} + \dfrac{A_1}{A_2}\left(\dfrac{1}{\epsilon_2} - 1\right)}.$$

This is, of course, somewhat of a hard way to arrive at the well-known result. The method will save considerable computer time, however, if many zones and iterative determination of the temperature field are involved.

18.3 RADIATIVE EXCHANGE IN GRAY ABSORBING/EMITTING MEDIA

We shall now consider radiative transfer through a gray absorbing, emitting, but nonscattering medium with constant absorption coefficient. The medium is confined by an enclosure with gray, diffusely emitting and reflecting surfaces. Again, the surfaces are broken into N isothermal zones (with weakly varying radiosities), while the medium is broken into K isothermal volume zones. There will now be radiative transfer between surface zones and other surface zones, from surface zones to volume zones and vice versa, and from volume zones to volume zones.

Direct Exchange Areas

Surface–Surface Exchange

The direct exchange area between two surface zones is defined, as for pure surface exchange, by

$$Q_{i\to j} = \overline{s_is_j}\,J_i,$$

where $Q_{i\to j}$ is the total heat flux coming from zone i that travels directly (without reflections) to zone j. The exchange areas are the same as in the last section, except that only a fraction of the energy leaving i toward j will arrive at j (while the rest will be absorbed by the medium). Thus, similar to the development of standard view factors developed in Chapter 4, we find that the heat flux leaving zone i arriving at j = intensity leaving dA_i into the direction of dA_j × area normal to ray × solid angle subtended by dA_j as seen from dA_i × fraction transmitted, or

$$\left(\frac{J_1}{\pi}\right) \times (dA_i\cos\theta_i) \times \left(\frac{dA_j\cos\theta_j}{S^2}\right) \times e^{-\kappa S},$$

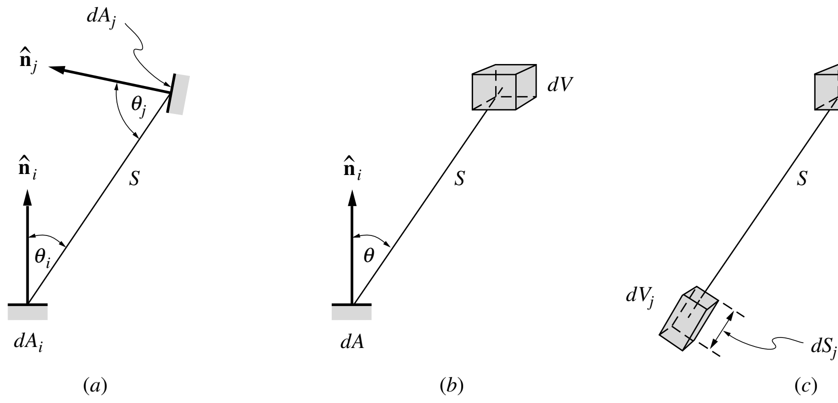

FIGURE 18-2
Radiative exchange between (*a*) two differential surface elements, (*b*) differential volume and surface elements, (*c*) two differential volume elements.

as shown in Fig. 18-2*a*. The *surface-to-surface direct exchange area* may, therefore, be determined from

$$\overline{s_i s_j} = \int_{A_i} \int_{A_j} e^{-\kappa S} \frac{\cos\theta_i \cos\theta_j}{\pi S^2} \, dA_j \, dA_i. \tag{18.26}$$

Equation (18.26) is identical to equation (18.2) except for the transmission factor $e^{-\kappa S}$.

Volume–Surface Exchange

The volume-to-surface direct exchange area is defined similarly through

$$Q_{i\to j} = \overline{g_i s_j} \, E_{bi}.$$

Inspecting Fig. 18-2*b*, we find that the heat flux emitted from volume zone i arriving at surface zone j = energy emitted from dV_i into all 4π directions as given by equation (10.54) × fraction leaving toward dA_j × fraction transmitted, or

$$(4\kappa E_{bi} \, dV_i) \times \left(\frac{dA_j \cos\theta_j}{4\pi S^2} \right) \times e^{-\kappa S},$$

leading to a *volume-to-surface direct exchange area* of

$$\overline{g_i s_j} = \int_{V_i} \int_{A_j} e^{-\kappa S} \frac{\cos\theta_j}{\pi S^2} \, \kappa \, dA_j \, dV_i. \tag{18.27}$$

Volume–Volume Exchange

The volume-to-volume direct exchange area is related to heat flux leaving one zone and absorbed by another after direct travel (without reflections) through

$$Q_{i\to j} = \overline{g_i g_j} \, E_{bi}.$$

The magnitude of the direct exchange area is most easily derived by orienting the receiving volume, $dV_j = dA_j \times dS_j$, as shown in Fig. 18-2*c*: The heat flux from i intercepted by j is equal to energy emitted from dV_i into all 4π directions × fraction leaving toward dA_j × fraction transmitted × fraction absorbed over dS_j, or

$$(4\kappa E_{bi} \, dV_i) \times \left(\frac{dA_j}{4\pi S^2} \right) \times (e^{-\kappa S}) \times (\kappa \, dS_j),$$

leading to a *volume-to-volume direct exchange area* of

$$\overline{g_i g_j} = \int_{V_i}\int_{V_j} e^{-\kappa S}\frac{\kappa^2}{\pi S^2}\,dV_j\,dV_i. \tag{18.28}$$

Making the same arguments in the opposite direction, or by inspecting equations (18.26) through (18.28), we find that reciprocity for direct exchange areas holds, that is,

$$\overline{s_i s_j} = \overline{s_j s_i}, \quad \overline{g_i s_j} = \overline{s_j g_i}, \quad \overline{g_i g_j} = \overline{g_j g_i}. \tag{18.29}$$

Making an energy balance on surface zone i, we have

$$\begin{aligned} Q_{si} &= A_i q_i = A_i(J_i - H_{si}) = A_i\epsilon_i(E_{bsi} - H_{si}) \\ &= \sum_{j=1}^{N}\overline{s_j s_i}\,(J_i - J_j) + \sum_{k=1}^{K}\overline{g_k s_i}\,(J_i - E_{bgk}) \\ &= \epsilon_i\left(A_i E_{bsi} - \sum_{j=1}^{N}\overline{s_j s_i}\,J_j - \sum_{k=1}^{K}\overline{g_k s_i}\,E_{bgk}\right), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{18.30}$$

where we have added the subscripts s and g to distinguish between emissive powers and irradiation of surface and volume zones, respectively.

The first version of equation (18.30) uses the concept that the net heat flux at zone i is equal to the sum of the net exchange between any two zones, and the second states that net heat flux is equal to total emission minus the absorbed fraction of total irradiation. Realizing that the exchange areas do not depend on temperatures and that, for an isothermal enclosure, all heat fluxes go to zero (as well as $J_j = E_{bj}$), we get the relationship

$$\sum_{j=1}^{N}\overline{s_j s_i} + \sum_{k=1}^{K}\overline{g_k s_i} = A_i, \quad i = 1, 2, \ldots, N. \tag{18.31}$$

Similarly, making an energy balance over a volume zone, we find[3]

$$\begin{aligned} Q_{gi} &= \kappa V_i(4E_{bgi} - G_i) = \sum_{j=1}^{N}\overline{s_j g_i}\,(E_{bgi} - J_j) + \sum_{k=1}^{K}\overline{g_k g_i}\,(E_{bgi} - E_{bgk}) \\ &= 4\kappa V_i E_{bgi} - \sum_{j=1}^{N}\overline{s_j g_i}\,J_j - \sum_{k=1}^{K}\overline{g_k g_i}\,E_{bgk}, \quad i = 1, 2, \ldots, K. \end{aligned} \tag{18.32}$$

Comparing, or looking at an isothermal enclosure, we find

$$\sum_{j=1}^{N}\overline{s_j g_i} + \sum_{k=1}^{K}\overline{g_k g_i} = 4\kappa V_i, \qquad i = 1, 2, \ldots, K. \tag{18.33}$$

For a volume zone the Q_{gi} represents the net radiative source within volume V_i and is, therefore,

$$Q_{gi} = \int_{V_i}\nabla\cdot\mathbf{q}\,dV_i. \tag{18.34}$$

If the Q_{si} for surface zones are eliminated through equation (18.7), and if the Q_{gi} for volume zones are known (by assuming radiative equilibrium) or the gas zone emissive powers are known (through connection with an overall energy equation), then equations (18.30) and (18.32) form a system of $N + K$ equations in the unknowns J_i ($i = 1, 2, \ldots, N$) and Q_{gk} or E_{bgk} ($k = 1, 2, \ldots, K$).

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[3] Here $\kappa V_i G_i = \int_{V_i}\int_{4\pi}\kappa I\,d\Omega\,dV_i$ is the total incident radiation absorbed by V_i.

Total Exchange Areas

Again, we would like to eliminate the radiosities, J_j, from equations (18.30) and (18.32) to avoid repeated matrix inversions in case the zonal temperatures must be determined by iteration. The total exchange areas are defined by

$$Q_{si} = \epsilon_i A_i E_{bsi} - \sum_{j=1}^{N} \overline{S_i S_j} E_{bsj} - \sum_{k=1}^{K} \overline{S_i G_k} E_{bgk}, \qquad i = 1, 2, \dots, N, \tag{18.35}$$

$$Q_{gi} = 4\kappa V_i E_{bgi} - \sum_{j=1}^{N} \overline{G_i S_j} E_{bsj} - \sum_{k=1}^{K} \overline{G_i G_k} E_{bgk}, \qquad i = 1, 2, \dots, K. \tag{18.36}$$

The law of reciprocity follows again from the definition of total exchange areas, that is,

$$\overline{S_j S_i} = \overline{S_i S_j}, \qquad \overline{G_k S_i} = \overline{S_i G_k}, \qquad \overline{G_k G_i} = \overline{G_i G_k}, \tag{18.37}$$

and, setting all emissive powers equal in equations (18.35) and (18.36) (resulting in zero heat fluxes everywhere) gives the relations

$$\sum_{j=1}^{N} \overline{S_i S_j} + \sum_{k=1}^{K} \overline{S_i G_k} = \epsilon_i A_i, \qquad i = 1, 2, \dots, N, \tag{18.38}$$

$$\sum_{j=1}^{N} \overline{G_i S_j} + \sum_{k=1}^{K} \overline{G_i G_k} = 4\kappa V_i, \qquad i = 1, 2, \dots, K. \tag{18.39}$$

To determine the total exchange factors, we extract the surface irradiation from equation (18.30), giving

$$A_i H_{si} = \sum_{j=1}^{N} \overline{s_i s_j}\, J_j + \sum_{k=1}^{K} \overline{s_i g_k}\, E_{bgk}, \qquad i = 1, 2, \dots, N, \tag{18.40}$$

or, after using equation (18.16),

$$\sum_{j=1}^{N} \left(\frac{\delta_{ij}}{\epsilon_j} - \frac{\rho_j\, \overline{s_i s_j}}{\epsilon_j A_j} \right) \epsilon_j A_j H_{sj} = \sum_{j=1}^{N} \overline{s_i s_j}\, \epsilon_j E_{bsj} + \sum_{k=1}^{K} \overline{s_i g_k}\, E_{bgk}, \quad i = 1, 2, \dots, N. \tag{18.41}$$

In vector notation we may write[4]

$$\mathbf{T} \cdot \mathbf{h_s} = \mathbf{S} \cdot \mathbf{e_{bs}} + \overline{\mathbf{sg}} \cdot \mathbf{e_{bg}}, \tag{18.42}$$

where $\mathbf{T}, \mathbf{h_s}, \mathbf{S}, \mathbf{e_{bs}}$ are defined as in the previous section (although the direct exchange areas in them are different because of the transmission factor), while the other two are defined as

$$\overline{\mathbf{sg}} = \begin{pmatrix} \overline{s_1 g_1} & \overline{s_1 g_2} & \cdots & \overline{s_1 g_K} \\ \overline{s_2 g_1} & \overline{s_2 g_2} & \cdots & \overline{s_2 g_K} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{s_N g_1} & \overline{s_N g_2} & \cdots & \overline{s_N g_K} \end{pmatrix}, \quad \mathbf{e_{bg}} = \begin{pmatrix} E_{bg1} \\ E_{bg2} \\ \vdots \\ E_{bgK} \end{pmatrix}. \tag{18.43}$$

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[4] We augment our vector notation rules here somewhat by stating that a bolded and barred two-letter name may be used for a matrix (besides a bold capital letter).

Note that $\mathbf{T}$ and $\mathbf{S}$ are $N \times N$ matrices and $\mathbf{h_s}$ and $\mathbf{e_{bs}}$ are N-vectors, while $\overline{\mathbf{sg}}$ is an $N \times K$ matrix and $\mathbf{e_{bg}}$ is a K-vector. Inverting $\mathbf{T}$ gives

$$\mathbf{h_s} = \mathbf{T}^{-1} \cdot \mathbf{S} \cdot \mathbf{e_{bs}} + \mathbf{T}^{-1} \cdot \overline{\mathbf{sg}} \cdot \mathbf{e_{bg}}, \tag{18.44}$$

and, after comparing with equation (18.35), the total exchange areas can be identified as

$$\overline{\mathbf{SS}} = \mathbf{T}^{-1} \cdot \mathbf{S}, \quad \text{or} \quad \overline{S_iS_j} = \sum_{l=1}^{N} (T^{-1})_{il}\, S_{lj}, \tag{18.45}$$

$$\overline{\mathbf{SG}} = \mathbf{T}^{-1} \cdot \overline{\mathbf{sg}}, \quad \text{or} \quad \overline{S_iG_k} = \sum_{l=1}^{N} (T^{-1})_{il}\, \overline{s_lg_k}. \tag{18.46}$$

A similar procedure for the volume zones gives, from equation (18.32),

$$\begin{aligned} \kappa V_i G_i &= \sum_{j=1}^{N} \overline{g_is_j}\, J_j + \sum_{k=1}^{K} \overline{g_ig_k}\, E_{bgk}, \\ &= \sum_{j=1}^{N} \overline{g_is_j}\, (\epsilon_j E_{bsj} + \rho_j H_{sj}) + \sum_{k=1}^{K} \overline{g_ig_k}\, E_{bgk}, \quad i = 1, 2, \ldots, K. \end{aligned} \tag{18.47}$$

Remembering that $h_{sj} = \epsilon_j A_j H_{sj}$ and using equations (18.44) and (18.45), we find that equation (18.47) is transformed to

$$\begin{aligned} \kappa V_i G_i &= \sum_{j=1}^{N} \overline{g_is_j}\, \epsilon_j\, E_{bsj} + \sum_{j=1}^{N} \frac{\overline{g_is_j}}{A_j} \frac{\rho_j}{\epsilon_j} \sum_{l=1}^{N} \overline{S_jS_l}\, E_{bsl} \\ &\quad + \sum_{j=1}^{N} \frac{\overline{g_is_j}}{A_j} \frac{\rho_j}{\epsilon_j} \sum_{k=1}^{K} \overline{S_jG_k}\, E_{bgk} + \sum_{k=1}^{K} \overline{g_ig_k}\, E_{bgk}, \quad i = 1, 2, \ldots, K. \end{aligned} \tag{18.48}$$

Switching the dummy counters j and l in the second term on the right-hand side and rearranging, we have

$$\begin{aligned} \kappa V_i G_i &= \sum_{j=1}^{N} \left(\overline{g_is_j}\, \epsilon_j + \sum_{l=1}^{N} \frac{\overline{g_is_l}}{A_l} \frac{\rho_l}{\epsilon_l} \overline{S_lS_j} \right) E_{bsj} \\ &\quad + \sum_{k=1}^{K} \left(\overline{g_ig_k} + \sum_{l=1}^{N} \frac{\overline{g_is_l}}{A_l} \frac{\rho_l}{\epsilon_l} \overline{S_lG_k} \right) E_{bgk}, \qquad i = 1, 2, \ldots, K. \end{aligned} \tag{18.49}$$

By comparison with equation (18.36), we find

$$\overline{G_iS_j} = \overline{g_is_j}\, \epsilon_j + \sum_{l=1}^{N} \frac{\overline{g_is_l}}{A_l} \frac{\rho_l}{\epsilon_l} \overline{S_lS_j}, \quad i = 1, 2, \ldots, K, \tag{18.50}$$

$$\overline{G_iG_k} = \overline{g_ig_k} + \sum_{l=1}^{N} \frac{\overline{g_is_l}}{A_l} \frac{\rho_l}{\epsilon_l} \overline{S_lG_k}, \quad i = 1, 2, \ldots, K, \tag{18.51}$$

or, in matrix notation

$$\overline{\mathbf{GS}} = \mathbf{R} + \mathbf{Q} \cdot \overline{\mathbf{SS}}, \tag{18.52}$$

$$\overline{\mathbf{GG}} = \overline{\mathbf{gg}} + \mathbf{Q} \cdot \overline{\mathbf{SG}}, \tag{18.53}$$

where

$$\mathbf{R} = \begin{pmatrix} \overline{g_1s_1}\,\epsilon_1 & \overline{g_1s_2}\,\epsilon_2 & \cdots & \overline{g_1s_N}\,\epsilon_N \\ \overline{g_2s_1}\,\epsilon_1 & \overline{g_2s_2}\,\epsilon_2 & \cdots & \overline{g_2s_N}\,\epsilon_N \\ \vdots & \vdots & \ddots & \vdots \\ \overline{g_Ks_1}\,\epsilon_1 & \overline{g_Ks_2}\,\epsilon_2 & \cdots & \overline{g_Ks_N}\,\epsilon_N \end{pmatrix}, \tag{18.54}$$

$$\mathbf{Q} = \begin{pmatrix} \frac{\overline{g_1s_1}}{A_1}\frac{\rho_1}{\epsilon_1} & \frac{\overline{g_1s_2}}{A_2}\frac{\rho_2}{\epsilon_2} & \cdots & \frac{\overline{g_1s_N}}{A_N}\frac{\rho_N}{\epsilon_N} \\ \frac{\overline{g_2s_1}}{A_1}\frac{\rho_1}{\epsilon_1} & \frac{\overline{g_2s_2}}{A_2}\frac{\rho_2}{\epsilon_2} & \cdots & \frac{\overline{g_2s_N}}{A_N}\frac{\rho_N}{\epsilon_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\overline{g_Ks_1}}{A_1}\frac{\rho_1}{\epsilon_1} & \frac{\overline{g_Ks_2}}{A_2}\frac{\rho_2}{\epsilon_2} & \cdots & \frac{\overline{g_Ks_N}}{A_N}\frac{\rho_N}{\epsilon_N} \end{pmatrix}. \tag{18.55}$$

Because of reciprocity $\overline{G_kS_j} = \overline{S_jG_k}$ or $\overline{\mathbf{GS}} = \overline{\mathbf{SG}}^T$ (where the superscript T denotes the *transpose* of a matrix) and

$$\mathbf{R} + \mathbf{Q}\cdot\overline{\mathbf{SS}} = \overline{\mathbf{gs}}\cdot(\mathbf{T}^{-1})^T. \tag{18.56}$$

Obviously, the $\overline{G_kS_j}$ may be evaluated using either of equations (18.46) and (18.50) (or both, using one as a checking mechanism).

Example 18.2. Consider a gray, absorbing–emitting (but not scattering) isothermal medium confined between two parallel, isothermal, gray and diffuse plates. The temperature of the medium is T_m, its absorption coefficient is κ, and the distance between the plates is L. Both plates have the same temperature, T_w, and emittance, ϵ. Determine the net radiative heat flux at the plates.

Solution

We begin by evaluating the direct exchange areas. Since the problem is one-dimensional with infinitely large plates, all direct and total exchange areas will become infinitely large. Thus, we shall evaluate all exchange areas per unit area, using a tilde instead of a bar in the notation. From equation (18.26)

$$\widetilde{s_1s_2} = \lim_{A_1\to\infty}\frac{1}{A_1}\int_{A_1}\int_{A_2} e^{-\kappa S}\frac{\cos\theta_1\cos\theta_2}{\pi S^2}\,dA_2\,dA_1 = \int_{A_2} e^{-\kappa S}\frac{\cos\theta_1\cos\theta_2}{\pi S^2}\,dA_2,$$

since the exchange factor between any point on Surface 1 and all of Surface 2 is the same everywhere. Thus, we get, with $\cos\theta_1 = \cos\theta_2 = L/S$, $S^2 = r^2 + L^2$ and $\tau_L = \kappa L$,

$$\widetilde{s_1s_2} = \int_0^\infty e^{-\kappa S}\frac{L^2}{\pi S^4}2\pi r\,dr = 2\int_1^\infty e^{-\tau_L\gamma}\frac{d\gamma}{\gamma^3} = 2E_3(\tau_L),$$

where $\gamma = S/L = \sqrt{(r/L)^2+1}$. We also have $\widetilde{s_1s_1} = \widetilde{s_2s_2} = 0$ and $\widetilde{s_2s_1} = \widetilde{s_1s_2}$. From equation (18.31), it follows immediately that

$$\widetilde{gs_1} = 1 - \widetilde{s_1s_2} = \widetilde{gs_2} = \widetilde{gs} = \widetilde{sg},$$

and, from equation (18.33),

$$\widetilde{gg} = 4\tau_L - 2\widetilde{sg}.$$

To determine the total exchange factors, we first establish the relevant matrices,

$$\mathbf{T} = \begin{pmatrix} \dfrac{1}{\epsilon} & -\dfrac{\rho}{\epsilon}\,\widetilde{s_1s_2} \\ -\dfrac{\rho}{\epsilon}\,\widetilde{s_1s_2} & \dfrac{1}{\epsilon} \end{pmatrix}, \quad T = \frac{1}{\epsilon^2}\left[1 - \rho^2\,(\widetilde{s_1s_2})^2\right],$$

$$\mathbf{T}^{-1} = \frac{1}{T}\begin{pmatrix} \dfrac{1}{\epsilon} & \dfrac{\rho}{\epsilon}\,\widetilde{s_1s_2} \\ \dfrac{\rho}{\epsilon}\,\widetilde{s_1s_2} & \dfrac{1}{\epsilon} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & \epsilon\,\widetilde{s_1s_2} \\ \epsilon\,\widetilde{s_1s_2} & 0 \end{pmatrix},$$

where $\mathbf{S}$ is also evaluated per unit area. Now, from equations (18.45) and (18.46)

$$\begin{aligned} \widetilde{S_1S_1} &= \frac{1}{T}\rho\,(\widetilde{s_1s_2})^2 = \widetilde{S_2S_2}, \quad \widetilde{S_1S_2} = \frac{1}{T}\,\widetilde{s_1s_2} = \widetilde{S_2S_1}, \\ \widetilde{S_1G} &= \frac{1}{T}\left(\frac{1}{\epsilon} + \frac{\rho}{\epsilon}\,\widetilde{s_1s_2}\right)\widetilde{sg} = \widetilde{S_2G} = \widetilde{SG} = \widetilde{GS}. \end{aligned}$$

The last result may be verified by comparing with the result from equation (18.50). Finally with

$$\mathbf{Q} = \left(\frac{\rho}{\epsilon}\,\widetilde{sg} \quad \frac{\rho}{\epsilon}\,\widetilde{sg}\right),$$

$$\widetilde{GG} = \widetilde{gg} + 2\frac{\rho}{\epsilon}\,\widetilde{sg}\,\widetilde{SG} = \widetilde{gg} + \frac{2\rho}{\epsilon^2 T}\left(1 + \rho\widetilde{S_1S_2}\right)(\widetilde{sg})^2.$$

These results are readily (and should be) verified by substituting into equations (18.38) and (18.39).

The net heat flux to a plate is calculated from equation (18.35) as

$$q_w = \left(\epsilon - \widetilde{S_1S_1} - \widetilde{S_1S_2}\right)E_{bw} - \widetilde{SG}\,E_{bm} = \widetilde{SG}\,(E_{bw} - E_{bm}),$$

or

$$\Psi = \frac{q_w}{\sigma\left(T_w^4 - T_m^4\right)} = \widetilde{SG} = \frac{\epsilon\,[1 - 2E_3(\tau_L)]}{1 - 2\rho E_3(\tau_L)},$$

which is, of course, the exact result.

A number of researchers have applied the zonal method to gray media without scattering. For example, Einstein calculated combined convective/radiative heat fluxes for two-dimensional duct flow for a gray gas in a black-walled duct [4, 5] and Modest investigated radiative equilibrium in a rectangular enclosure with a single nonblack wall [14], and for the space between concentric, gray cylinders [15].

18.4 RADIATIVE EXCHANGE IN GRAY MEDIA WITH ISOTROPIC SCATTERING

We shall now extend the zonal method to the most general case to which it can be applied by including the effects of isotropic scattering and spatially varying absorption and scattering coefficients. As before, we break up the surfaces into N isothermal zones (with weakly varying radiosities) and the medium into K isothermal volume zones (which, as we shall see, should have weakly varying radiative source terms). We shall rederive the expressions for direct exchange areas by making allowance for property variations as well as for isotropic scattering.

Direct Exchange Areas

Surface–Surface Exchange

The direct exchange area between two surface zones applies now to total heat flux coming from zone i traveling to j directly without reflections and without being scattered. Therefore, equation (18.26) must be rewritten as

$$\overline{s_i s_j} = \int_{A_i}\int_{A_j} e^{-\beta_{ij}S}\,\frac{\cos\theta_i\cos\theta_j}{\pi S^2}\,dA_j\,dA_i, \tag{18.57}$$

where

$$\beta_{ij} = \frac{1}{S}\int_0^S \beta\,dS \tag{18.58}$$

is the *average extinction coefficient* between zones i and j. In general, of course, the value for β_{ij} would be different for any two zones.

Volume–Surface Exchange

Radiative energy leaves an infinitesimal volume not only in the form of isotropic emission ($= 4\pi\kappa_i I_{bgi} = 4\kappa_i E_{bgi}$) but also as isotropic out-scattering ($= \sigma_{si}G_i$, where σ_s is the scattering coefficient).[5] Defining a *volume zone radiosity* as

$$J_g = (1-\omega)E_{bg} + \frac{\omega}{4}G, \tag{18.59}$$

we may express the direct exchange of energy between volume and surface zones as

$$Q_{i\leftrightarrow j} = \overline{g_i s_j}\,(J_{gi} - J_{sj}), \tag{18.60}$$

and equation (18.27) becomes

$$\overline{g_i s_j} = \int_{V_i}\int_{A_j} e^{-\beta_{ij}S}\,\frac{\cos\theta_j}{\pi S^2}\beta_i\,dA_j\,dV_i. \tag{18.61}$$

Therefore, with this definition we demand that we may assume the volume zone radiosity, J_g, to be constant throughout the zone.

Volume–Volume Exchange

By the same reasoning we need to redefine the volume-to-volume direct exchange areas as

$$\overline{g_i g_j} = \int_{V_i}\int_{V_j} e^{-\beta_{ij}S}\,\frac{\beta_i\beta_j}{\pi S^2}\,dV_j\,dV_i, \tag{18.62}$$

where the β_i comes from averaging the source term over dV_i, and the β_j is associated with the extinction in dV_j.

Rewriting the energy balance for surface zone i, equation (18.30), we get

$$\begin{aligned} Q_{si} &= A_i q_i = A_i(J_{si} - H_{si}) = A_i\epsilon_i(E_{bsi} - H_{si}) \\ &= \sum_{j=1}^{N}\overline{s_i s_j}\,(J_{si} - J_{sj}) + \sum_{k=1}^{K}\overline{s_i g_k}\,(J_{si} - J_{gk}) \\ &= \epsilon_i\left(A_i E_{bsi} - \sum_{j=1}^{N}\overline{s_i s_j}\,J_{sj} - \sum_{k=1}^{K}\overline{s_i g_k}\,J_{gk}\right), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{18.63}$$

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[5] See also the development for the radiative source function for isotropic scattering, $S = (1-\omega)I_b + \omega G/4\pi$, in equation (10.25).

Similarly, for a volume zone, equation (18.32) is rewritten as

$$\begin{aligned} Q_{gi} &= \kappa_i V_i (4E_{bgi} - G_i) = \beta_i V_i (4J_{gi} - G_i) \\ &= \sum_{j=1}^{N} \overline{g_i s_j}\,(J_{gi} - J_{sj}) + \sum_{k=1}^{K} \overline{g_i g_k}\,(J_{gi} - J_{gk}) \\ &= 4\kappa_i V_i E_{bgi} - (1-\omega_i)\left(\sum_{j=1}^{N} \overline{g_i s_j} J_{sj} + \sum_{k=1}^{K} \overline{g_i g_k} J_{gk}\right), \quad i = 1, 2, \ldots, K. \end{aligned} \tag{18.64}$$

Examining an isothermal enclosure, we find that equations (18.31) and (18.33) continue to hold, if the absorption coefficient is replaced by the extinction coefficient everywhere, and if the extinction coefficient (and, therefore, the direct exchange areas) does not depend on temperature:

$$\sum_{j=1}^{N} \overline{s_j s_i} + \sum_{k=1}^{K} \overline{g_k s_i} = A_i, \qquad i = 1, 2, \ldots, N, \tag{18.65}$$

$$\sum_{j=1}^{N} \overline{s_j g_i} + \sum_{k=1}^{K} \overline{g_k g_i} = 4\beta_i V_i, \qquad i = 1, 2, \ldots, K. \tag{18.66}$$

Eliminating the radiosities from equations (18.63) and (18.64) through equations (18.16) and (18.59), we obtain, in a similar fashion as for equations (18.41) and (18.47),

$$\begin{aligned} &\sum_{j=1}^{N} \left(\frac{\delta_{ij}}{\epsilon_j} - \frac{\rho_j\, \overline{s_i s_j}}{\epsilon_j A_j}\right) h_{sj} - \sum_{k=1}^{K} \frac{\overline{s_i g_k}\, \omega_k}{4\kappa_k V_k} h_{gk} \\ &\qquad = \sum_{j=1}^{N} \overline{s_i s_j}\, \epsilon_j E_{bsj} + \sum_{k=1}^{K} \overline{s_i g_k}\,(1-\omega_k) E_{bgk}, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{18.67}$$

$$\begin{aligned} &-\sum_{j=1}^{N} \frac{\overline{g_i s_j}\, \rho_j}{\epsilon_j A_j} h_{sj} + \sum_{k=1}^{K} \left(\frac{\delta_{ik}}{1-\omega_k} - \frac{\overline{g_i g_k}\, \omega_k}{4\kappa_k V_k}\right) h_{gk} \\ &\qquad = \sum_{j=1}^{N} \overline{g_i s_j}\, \epsilon_j E_{bsj} + \sum_{k=1}^{K} \overline{g_i g_k}\,(1-\omega_k) E_{bgk}, \quad i = 1, 2, \ldots, K, \end{aligned} \tag{18.68}$$

where $h_{gk} = \kappa_k V_k G_k$. Equations (18.67) and (18.68) are a system of $N+K$ equations in the $N+K$ unknown h_{sj} and h_{gk}.[6] A solution may be found by standard techniques.

Total Exchange Areas

If the temperature field is to be determined iteratively, determination of *total exchange areas* is again desirable so long as radiative properties and, therefore, direct and total exchange areas do not vary with temperature (although they may vary with location). We may rewrite equations (18.67) and (18.68) in matrix notation as

$$\mathbf{T} \cdot \mathbf{h_s} - \mathbf{U} \cdot \mathbf{h_g} = \mathbf{S} \cdot \mathbf{e_{bs}} + \mathbf{V} \cdot \mathbf{e_{bg}}, \tag{18.69}$$

$$-\mathbf{Q} \cdot \mathbf{h_s} + \mathbf{W} \cdot \mathbf{h_g} = \mathbf{R} \cdot \mathbf{e_{bs}} + \mathbf{X} \cdot \mathbf{e_{bg}}, \tag{18.70}$$

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[6] This is valid if the medium temperature field is "known." In the case of radiative equilibrium we have, from equation (18.64), $G_k = 4E_{bk}$ or $h_{gk} = 4\kappa_k V_k E_{bgk}$ as the unknowns for the volume zones, and the second series on the right-hand side of equation (18.68) is eliminated.

where $\mathbf{T}, \mathbf{S}, \mathbf{Q}, \mathbf{R}, \mathbf{e_{bs}}, \mathbf{e_{bg}}$, and $\mathbf{h_s}$ have been defined in the previous section and

$$\mathbf{U} = \begin{pmatrix} \dfrac{\overline{s_1g_1}\,\omega_1}{4\kappa_1V_1} & \dfrac{\overline{s_1g_2}\,\omega_2}{4\kappa_2V_2} & \cdots & \dfrac{\overline{s_1g_K}\,\omega_K}{4\kappa_KV_K} \\ \dfrac{\overline{s_2g_1}\,\omega_1}{4\kappa_1V_1} & \dfrac{\overline{s_2g_2}\,\omega_2}{4\kappa_2V_2} & \cdots & \dfrac{\overline{s_2g_K}\,\omega_K}{4\kappa_KV_K} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\overline{s_Ng_1}\,\omega_1}{4\kappa_1V_1} & \dfrac{\overline{s_Ng_2}\,\omega_2}{4\kappa_2V_2} & \cdots & \dfrac{\overline{s_Ng_K}\,\omega_K}{4\kappa_KV_K} \end{pmatrix}, \tag{18.71}$$

$$\mathbf{W} = \begin{pmatrix} \dfrac{1}{1-\omega_1} - \dfrac{\overline{g_1g_1}\,\omega_1}{4\kappa_1V_1} & -\dfrac{\overline{g_1g_2}\,\omega_2}{4\kappa_2V_2} & \cdots & -\dfrac{\overline{g_1g_K}\,\omega_K}{4\kappa_KV_K} \\ -\dfrac{\overline{g_2g_1}\,\omega_1}{4\kappa_1V_1} & \dfrac{1}{1-\omega_2} - \dfrac{\overline{g_2g_2}\,\omega_2}{4\kappa_2V_2} & \cdots & -\dfrac{\overline{g_2g_K}\,\omega_K}{4\kappa_KV_K} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{\overline{g_Kg_1}\,\omega_1}{4\kappa_1V_1} & -\dfrac{\overline{g_Kg_2}\,\omega_2}{4\kappa_2V_2} & \cdots & \dfrac{1}{1-\omega_K} - \dfrac{\overline{g_Kg_K}\,\omega_K}{4\kappa_KV_K} \end{pmatrix}, \tag{18.72}$$

$$\mathbf{V} = \begin{pmatrix} \overline{s_1g_1}\,(1-\omega_1) & \overline{s_1g_2}\,(1-\omega_2) & \cdots & \overline{s_1g_K}\,(1-\omega_K) \\ \overline{s_2g_1}\,(1-\omega_1) & \overline{s_2g_2}\,(1-\omega_2) & \cdots & \overline{s_2g_K}\,(1-\omega_K) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{s_Ng_1}\,(1-\omega_1) & \overline{s_Ng_2}\,(1-\omega_2) & \cdots & \overline{s_Ng_K}\,(1-\omega_K) \end{pmatrix}, \tag{18.73}$$

$$\mathbf{X} = \begin{pmatrix} \overline{g_1g_1}\,(1-\omega_1) & \overline{g_1g_2}\,(1-\omega_2) & \cdots & \overline{g_1g_K}\,(1-\omega_K) \\ \overline{g_2g_1}\,(1-\omega_1) & \overline{g_2g_2}\,(1-\omega_2) & \cdots & \overline{g_2g_K}\,(1-\omega_K) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{g_Kg_1}\,(1-\omega_1) & \overline{g_Kg_2}\,(1-\omega_2) & \cdots & \overline{g_Kg_K}\,(1-\omega_K) \end{pmatrix}, \tag{18.74}$$

$$\mathbf{h_g} = \begin{pmatrix} \kappa_1V_1G_1 \\ \kappa_2V_2G_2 \\ \vdots \\ \kappa_KV_KG_K \end{pmatrix}. \tag{18.75}$$

Inverting $\mathbf{W}$, we can solve equation (18.70) for $\mathbf{h_g}$ as

$$\mathbf{h_g} = \mathbf{W}^{-1}\cdot\mathbf{Q}\cdot\mathbf{h_s} + \mathbf{W}^{-1}\cdot\mathbf{R}\cdot\mathbf{e_{bs}} + \mathbf{W}^{-1}\cdot\mathbf{X}\cdot\mathbf{e_{bg}}, \tag{18.76}$$

and substituting this expression into equation (18.69), we find

$$\mathbf{h_s} = \mathbf{P}^{-1}\cdot\mathbf{C}\cdot\mathbf{e_{bs}} + \mathbf{P}^{-1}\cdot\mathbf{D}\cdot\mathbf{e_{bg}}, \tag{18.77}$$

$$\mathbf{h_g} = \mathbf{W}^{-1}\cdot(\mathbf{Q}\cdot\mathbf{P}^{-1}\cdot\mathbf{C}+\mathbf{R})\cdot\mathbf{e_{bs}} + \mathbf{W}^{-1}\cdot(\mathbf{Q}\cdot\mathbf{P}^{-1}\cdot\mathbf{D}+\mathbf{X})\cdot\mathbf{e_{bg}}, \tag{18.78}$$

where

$$\mathbf{C} = \mathbf{S} + \mathbf{U}\cdot\mathbf{W}^{-1}\cdot\mathbf{R}, \qquad \mathbf{D} = \mathbf{V} + \mathbf{U}\cdot\mathbf{W}^{-1}\cdot\mathbf{X}, \tag{18.79a}$$

$$\mathbf{P} = \mathbf{T} - \mathbf{U}\cdot\mathbf{W}^{-1}\cdot\mathbf{Q}. \tag{18.79b}$$

In terms of total exchange areas, equations (18.63) and (18.64) are rewritten as

$$Q_{si} = \epsilon_i A_i E_{bsi} - \sum_{j=1}^{N} \overline{S_iS_j}\, E_{bsj} - \sum_{k=1}^{K} \overline{S_iG_k}\, E_{bgk}, \quad i = 1,2,\ldots,N, \tag{18.80}$$

$$Q_{gi} = 4\kappa_i V_i E_{bgi} - \sum_{j=1}^{N} \overline{G_iS_j}\, E_{bsj} - \sum_{k=1}^{K} \overline{G_iG_k}\, E_{bgk}, \quad i = 1,2,\ldots,K, \tag{18.81}$$

which are the same as equations (18.35) and (18.36) (but with modified total exchange areas).

Thus, by comparison we find

$$\overline{\mathbf{SS}} = \mathbf{P}^{-1}\cdot\mathbf{C} \quad \text{or} \quad \overline{S_iS_j} = \sum_{l=1}^{N} P_{il}^{-1} C_{lj}, \tag{18.82}$$

$$\overline{\mathbf{SG}} = \mathbf{P}^{-1}\cdot\mathbf{D} \quad \text{or} \quad \overline{S_iG_k} = \sum_{l=1}^{N} P_{il}^{-1} D_{lk}, \tag{18.83}$$

$$\begin{aligned}\overline{\mathbf{GS}} &= \mathbf{W}^{-1}\cdot(\mathbf{Q}\cdot\mathbf{P}^{-1}\cdot\mathbf{C}+\mathbf{R}) = \overline{\mathbf{SG}}^{T},\\ &\text{or} \quad \overline{G_kS_i} = \sum_{l=1}^{K} W_{kl}^{-1}\left(\sum_{m=1}^{N}\sum_{n=1}^{N} Q_{lm}P_{mn}^{-1}C_{ni} + R_{li}\right),\end{aligned} \tag{18.84}$$

$$\begin{aligned}\overline{\mathbf{GG}} &= \mathbf{W}^{-1}\cdot(\mathbf{Q}\cdot\mathbf{P}^{-1}\cdot\mathbf{D}+\mathbf{X}),\\ &= \text{or} \quad \overline{G_kG_i} = \sum_{l=1}^{K} W_{kl}^{-1}\left(\sum_{m=1}^{N}\sum_{n=1}^{N} Q_{lm}P_{mn}^{-1}D_{ni} + X_{li}\right).\end{aligned} \tag{18.85}$$

Setting all emissive powers equal in equations (18.80) and (18.81) shows that equations (18.38) and (18.39) still hold, *without* replacing absorption coefficient κ by extinction coefficient β on the right-hand side of equation (18.39),

$$\sum_{j=1}^{N} \overline{S_iS_j} + \sum_{k=1}^{K} \overline{S_iG_k} = \epsilon_i A_i, \qquad i = 1,2,\ldots,N, \tag{18.86}$$

$$\sum_{j=1}^{N} \overline{G_iS_j} + \sum_{k=1}^{K} \overline{G_iG_k} = 4\kappa_i V_i, \qquad i = 1,2,\ldots,K. \tag{18.87}$$

Temperature-Dependent Properties

If the absorption and extinction coefficients depend on temperature, then all direct exchange areas also depend on temperature. Thus, if the temperature field is found iteratively, then the direct exchange areas must be recalculated after every iteration. This requirement, in turn, eliminates the advantages of using total exchange areas. Therefore, for these problems one would, in general, solve equations (18.67) and (18.68) (or some variation) directly. There is, however, one important case for which a set of total exchange areas can be calculated beforehand (and which do not enter the iteration).

For this we shall make the following two assumptions: (*i*) the medium does not scatter, and (*ii*) the temperature dependence of the average absorption coefficient between two zones, κ_{ij}, may be neglected (or is only updated after a given number of iterations for the temperature field). With these assumptions we may write

$$\overline{s_i s_j} = (\overline{s_i s_j})_{\mathrm{av}}, \quad \overline{s_i g_j} = \overline{g_j s_i} = (\overline{s_i g_j})_{\mathrm{av}} \frac{\kappa_j}{\kappa_{\mathrm{av}}}, \quad \overline{g_i g_j} = (\overline{g_i g_j})_{\mathrm{av}} \frac{\kappa_i \kappa_j}{\kappa_{\mathrm{av}}^2}, \tag{18.88}$$

where $(\overline{s_i s_j})_{\mathrm{av}}$, etc., are direct exchange areas evaluated from equations (18.57), (18.61), and (18.62), with β_i and β_j (or κ_i and κ_j for a nonscattering medium) replaced by an average absorption coefficient κ_{av}. The assumption of a temperature-independent κ_{ij} thus implies that the average direct exchange areas do not depend on temperature. Substituting equation (18.88) into (18.45), and realizing that neither **T** nor **S** depends on temperature, we find

$$\overline{S_i S_j} = \sum_{l=1}^{N} (T^{-1})_{il} S_{lj} = \left(\overline{S_i S_j}\right)_{\mathrm{av}}, \tag{18.89}$$

that is, the $\overline{S_i S_j}$ do not depend on temperature either. From equation (18.46)

$$\overline{S_i G_k} = \sum_{l=1}^{N} (T^{-1})_{il} \, (\overline{s_l g_k})_{\mathrm{av}} \, \frac{\kappa_k}{\kappa_{\mathrm{av}}} = \left(\overline{S_i G_k}\right)_{\mathrm{av}} \frac{\kappa_k}{\kappa_{\mathrm{av}}}, \tag{18.90}$$

that is, the $(\overline{\mathbf{SG}})_{\mathrm{av}}$ may be calculated once and for all, and the actual $\overline{\mathbf{SG}}$ are found by a simple multiplication. Similarly, from equation (18.51),

$$\overline{G_i G_k} = \left[(\overline{g_i g_k})_{\mathrm{av}} + \sum_{l=1}^{N} \frac{(\overline{g_i s_l})_{\mathrm{av}}}{A_l} \frac{\rho_l}{\epsilon_l} \left(\overline{S_l G_k}\right)_{\mathrm{av}} \right] \frac{\kappa_i \kappa_k}{\kappa_{\mathrm{av}}^2} = \left(\overline{G_i G_k}\right)_{\mathrm{av}} \frac{\kappa_i \kappa_k}{\kappa_{\mathrm{av}}^2}. \tag{18.91}$$

Equations (18.35) and (18.36) may then be rewritten in terms of average total exchange areas.

Example 18.3. Repeat Example 18.2 for a medium that not only emits and absorbs, but that also scatters isotropically.

Solution

We shall still limit ourselves to a single gas zone even though this will no longer render the analysis exact: For that, the radiative source term, $4(1-\omega)E_b + \omega G$, must be constant across the volume zone.

It follows immediately that the direct exchange areas remain unchanged (except that $\tau_L = \beta L$ is now based on the extinction coefficient). From equations (18.80) and (18.86) it follows that the nondimensional heat flux at the surface is still evaluated as

$$\Psi = \frac{q_w}{\sigma(T_w^4 - T_m^4)} = \widetilde{SG},$$

and that only the evaluation of $\widetilde{SG}$ from the direct exchange areas is different from the previous example. While the tensors **T** and **Q** remain unchanged, we still need to evaluate **U**, **V**, **W**, and **X** for the evaluation of $\widetilde{SG}$. Thus,

$$\mathbf{U} = \frac{\widetilde{sg}\,\omega}{4\tau_L(1-\omega)} \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix}; \qquad \mathbf{V} = \widetilde{sg}\,(1-\omega) \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix};$$

$$\mathbf{W} = \left(\frac{4\tau_L - \omega\,\widetilde{gg}}{4\tau_L(1-\omega)} \right); \qquad \mathbf{X} = \left(\widetilde{gg}(1-\omega) \right);$$

where **W** and **X** are single-term 1 × 1 matrices. Thus,

$$\mathbf{W}^{-1} = (1/W) \quad \text{and} \quad D_{ij} = V_{ij} + \sum_{k=1}^{1}\sum_{l=1}^{1} U_{ik} W_{kl}^{-1} X_{lj},$$

or

$$\mathbf{D} = \left[\widetilde{sg}(1-\omega) + \frac{\widetilde{sg}\,\omega}{4\tau_L(1-\omega)} \times \frac{4\tau_L(1-\omega)}{4\tau_L - \omega\,\widetilde{gg}} \times \widetilde{gg}\,(1-\omega)\right]\begin{pmatrix}1\\ \\1\end{pmatrix} = \frac{4\tau_L(1-\omega)\widetilde{sg}}{4\tau_L-\omega\,\widetilde{gg}}\begin{pmatrix}1\\ \\1\end{pmatrix}.$$

Similarly,

$$\begin{aligned} P_{ij} &= T_{ij} - \frac{U_{i1}Q_{1j}}{W} = T_{ij} - \frac{\widetilde{sg}\,\omega}{4\tau_L(1-\omega)} \times \frac{4\tau_L(1-\omega)}{4\tau_L - \omega\,\widetilde{gg}} \times \frac{\rho}{\epsilon}\widetilde{sg} \\ &= T_{ij} - \frac{\rho}{\epsilon}\frac{\omega\,(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}, \end{aligned}$$

or

$$\mathbf{P} = \begin{pmatrix} \dfrac{1}{\epsilon} - \dfrac{\rho}{\epsilon}\dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}} & -\dfrac{\rho}{\epsilon}\left[\widetilde{s_1s_2} + \dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}\right] \\ -\dfrac{\rho}{\epsilon}\left[\widetilde{s_1s_2} + \dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}\right] & \dfrac{1}{\epsilon} - \dfrac{\rho}{\epsilon}\dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}} \end{pmatrix},$$

$$\mathbf{P}^{-1} = \frac{1}{P}\begin{pmatrix} \dfrac{1}{\epsilon} - \dfrac{\rho}{\epsilon}\dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}} & \dfrac{\rho}{\epsilon}\left[\widetilde{s_1s_2} + \dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}\right] \\ \dfrac{\rho}{\epsilon}\left[\widetilde{s_1s_2} + \dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}\right] & \dfrac{1}{\epsilon} - \dfrac{\rho}{\epsilon}\dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}} \end{pmatrix}$$

and

$$\begin{aligned} P &= P_{11}^2 - P_{12}^2 = \frac{1}{\epsilon^2}\left[1 - 2\rho\frac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}} - \rho^2(\widetilde{s_1s_2})^2 - 2\rho^2\widetilde{s_1s_2}\frac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}\right] \\ &= \frac{1}{\epsilon^2}\left[1 - (\rho\widetilde{s_1s_2})^2 - 2\rho\,(1 + \rho\widetilde{s_1s_2})\,\frac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}\right]. \end{aligned}$$

Finally,

$$\begin{aligned} \widetilde{\mathbf{SG}} &= \mathbf{P}^{-1}\cdot\mathbf{D} = \begin{pmatrix} P_{11}^{-1}D_{11} + P_{12}^{-1}D_{21} \\ \\ P_{21}^{-1}D_{11} + P_{22}^{-1}D_{21} \end{pmatrix} \\ &= \epsilon\frac{(1 + \rho\widetilde{s_1s_2})\,4\tau_L(1-\omega)\widetilde{sg}/\,(4\tau_L - \omega\,\widetilde{gg})}{1 - (\rho\widetilde{s_1s_2})^2 - 2\rho\,(1 + \rho\widetilde{s_1s_2})\,\dfrac{\omega(\widetilde{sg})^2}{4\tau_L - \omega\,\widetilde{gg}}}\begin{pmatrix}1\\ \\1\end{pmatrix} \\ &= \frac{4\tau_L(1-\omega)\epsilon\widetilde{sg}}{(1 - \rho\widetilde{s_1s_2})\,(4\tau_L - \omega\,\widetilde{gg}) - 2\rho\omega(\widetilde{sg})^2}\begin{pmatrix}1\\ \\1\end{pmatrix}. \end{aligned}$$

Substituting for $\widetilde{s_1s_2}$, $\widetilde{sg}$, and $\widetilde{gg}$, which have already been evaluated in Example 18.2 (although τ_L is now based on the extinction coefficient), this reduces to

$$\Psi = \frac{q_w}{\sigma(T_1^4 - T_2^4)} = \widetilde{SG} = \frac{\epsilon[1 - 2E_3(\tau_L)]}{1 - 2\rho E_3(\tau_L) + \dfrac{2\epsilon\omega}{4\tau_L(1-\omega)}[1 - 2E_3(\tau_L)]}.$$

The zonal method has been applied to gray, isotropically scattering media by a number of researchers. For example, Naraghi and Kassemi [16] treated the same problem of a rectangular medium at radiative equilibrium as Modest [14], but included isotropic scattering. They also applied the method to combined free convection–radiation problems [17,18].

A different approach than described here for the evaluation of total exchange areas was taken by Vercammen and Froment [19], who used the Monte Carlo method with subsequent data smoothing (to eliminate statistical scatter).[7] A similar approach was also taken by Naraghi and Chung [20]. Naraghi and Chung also presented what they called a "unified matrix approach" by combining equations (18.69) and (18.70) into a single matrix equation,

$$\begin{pmatrix} \mathbf{T} & -\mathbf{U} \\ -\mathbf{Q} & \mathbf{W} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{h}_s \\ \mathbf{h}_g \end{pmatrix} = \begin{pmatrix} \mathbf{S} & \mathbf{V} \\ \mathbf{R} & \mathbf{X} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e_{bs}} \\ \mathbf{e_{bg}} \end{pmatrix}, \tag{18.92}$$

which may be inverted to give

$$\begin{pmatrix} \overline{\mathbf{SS}} & \overline{\mathbf{SG}} \\ \overline{\mathbf{GS}} & \overline{\mathbf{GG}} \end{pmatrix} = \begin{pmatrix} \mathbf{T} & -\mathbf{U} \\ -\mathbf{Q} & \mathbf{W} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{S} & \mathbf{V} \\ \mathbf{R} & \mathbf{X} \end{pmatrix}. \tag{18.93}$$

It is not clear whether this approach will result in increased computer efficiency, since here a single $(N + K) \times (N + K)$ matrix must be inverted while, in the standard method, two matrices must be inverted (one $N \times N$ and the other $K \times K$).

In principle, the zonal method can also be applied to media that scatter anisotropically (and/or have nondiffuse bounding surfaces). This was shown by Yuen and Takara [11] (and Ma [12] for a one-dimensional slab). However, this requires converting (surface and/or volume) radiosities into vectors [i.e., exchange between any two zones, leading to $(N + K)^2$ unknowns and simultaneous equations and, thus, an unacceptable increase in computer storage and time].

18.5 RADIATIVE EXCHANGE THROUGH A NONGRAY MEDIUM

In the foregoing discussion we have assumed that the medium and the surfaces are gray, although all equations in this chapter up to this point are equally valid, on a spectral basis, for a nongray enclosure with a nongray medium. Unfortunately, the zonal method—like many other solution methods for radiative transfer problems—is not well suited for nongray media. However, Hottel and coworkers [1,2,21] have demonstrated that the zonal method may be applied to a nongray medium in one very special, but very important, case: an absorbing, emitting (but not scattering) medium confined in a black-walled enclosure. This important situation is often encountered in furnace applications where mixtures of combustion gases and soot are confined by furnace refractories blackened by soot.

In this method, known as the *weighted-sum-of-gray-gases* method, the medium is assumed to consist of different fractions of gray gases with different (but gray) absorption coefficients.[8] The "gas" can also be a semitransparent liquid: Yin and Jaluria [22,23] modeled the absorption coefficient of glass to consist of one transparent and two gray bands.

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[7] For a detailed description of the Monte Carlo method, see Chapter 21.

[8] Further development of the weighted-sum-of-gray-gases method has shown that it can also be used in the presence of gray walls and scattering. For a more detailed discussion see Section 20.6.

Surface–Surface Exchange

From equation (18.26) it follows that the radiative heat flux leaving dA_i and arriving at dA_j is

$$Q_{i\to j} = \int_0^\infty \int_{A_i}\int_{A_j} e^{-\kappa_\nu S}\,\frac{\cos\theta_i\cos\theta_j}{\pi S^2}\,dA_j\,dA_i\,J_{\nu i}\,d\nu, \tag{18.94}$$

where, for simplicity, it is assumed that the spectral absorption coefficient, κ_ν, does not depend on temperature or location. For a black enclosure $J_\nu = E_{b\nu}$, and equation (18.94) may be integrated over the entire spectrum by using the definition for *medium emissivity*, equation (11.171),

$$\epsilon_m(T,S) = \frac{1}{\sigma T^4}\int_0^\infty (1-e^{-\kappa_\nu S})\,E_{b\nu}(T)\,d\nu, \tag{18.95}$$

leading to

$$Q_{i\to j} = \int_{A_i}\int_{A_j}[1-\epsilon_m(T_i,S)]\,\frac{\cos\theta_i\cos\theta_j}{\pi S^2}\,dA_j\,dA_i\,E_{bi} = \overrightarrow{s_is_j}E_{bi}. \tag{18.96}$$

Similarly, we have

$$Q_{j\to i} = \int_{A_i}\int_{A_j}\left[1-\epsilon_m(T_j,S)\right]\frac{\cos\theta_i\cos\theta_j}{\pi S^2}\,dA_j\,dA_i\,E_{bj} = \overleftarrow{s_is_j}E_{bj}. \tag{18.97}$$

Both exchange areas now have arrows, rather than bars, since they are no longer equal, but depend on the temperature of the emitting surface, or

$$Q_{i\leftrightarrow j} = \overrightarrow{s_is_j}E_{bi} - \overleftarrow{s_is_j}E_{bj}. \tag{18.98}$$

Hottel [21] has shown that the emissivity of the medium may be written in the form

$$\epsilon_m(T,S) \simeq \sum_{l=0}^{L} a_l(T)\left(1-e^{-\kappa_l S}\right), \qquad \sum_{l=0}^{L} a_l = 1, \tag{18.99}$$

where $l = 0$ with $\kappa_0 = 0$ has been included for the case in which the medium consists entirely of molecular gases with large windows between bands. Experience has shown that $L = 3$ usually gives sufficiently accurate results, and even $L = 1$ (gray plus clear gas) often suffices [2]. Using this "weighted-sum-of-gray-gases" we have

$$Q_{i\leftrightarrow j} \simeq \sum_{l=0}^{L}(\overline{s_is_j})_l\left[a_l(T_i)E_{bi} - a_l(T_j)E_{bj}\right], \tag{18.100}$$

where $(\overline{s_is_j})_l$ is $\overline{s_is_j}$ evaluated for an absorption coefficient κ_l. Thus, the solution for the nongray medium is identical to that for a gray medium, except that emissive power is replaced by a weighted emissive power, and the calculations have to be carried out for a number of gray gases.

Surface–Volume Exchange

Similarly, from equation (18.27) it follows that, for a black enclosure,

$$\begin{aligned} Q_{i\to j} &= \int_0^\infty\int_{V_i}\int_{A_j}\frac{\cos\theta_j}{\pi S^2}\kappa_\nu\,e^{-\kappa_\nu S}\,dA_j\,dV_i\,E_{b\nu i}\,d\nu \\ &= \int_{V_i}\int_{A_j}\frac{\cos\theta_j}{\pi S^2}\frac{d}{dS}\left[\int_0^\infty\left(1-e^{-\kappa_\nu S}\right)E_{b\nu i}\,d\nu\right]dA_j\,dV_i \\ &= \int_{V_i}\int_{A_j}\frac{d\epsilon_m}{dS}(T_i,S)\frac{\cos\theta_j}{\pi S^2}\,dA_j\,dV_i\,E_{bi} = \overrightarrow{g_is_j}E_{bi}. \end{aligned} \tag{18.101}$$

Note that E_{bi} can be pulled inside the operator d/dS, since it is assumed constant over V_i. Therefore, using equation (18.99), we find

$$Q_{i\leftrightarrow j} = \overrightarrow{g_i s_j} E_{bi} - \overleftarrow{g_i s_j} E_{bj} \simeq \sum_{l=1}^{L} (\overline{g_i s_j})_l \left[a_l(T_i)E_{bi} - a_l(T_j)E_{bj}\right], \tag{18.102}$$

where the summation starts with $l = 1$ since there is no emission within the medium for $\kappa_0 = 0$.

Volume–Volume Exchange

By the same reasoning equation (18.28) becomes[9]

$$\begin{aligned} Q_{i\to j} &= \int_0^\infty \int_{V_i} \int_{V_j} \frac{1}{\pi S^2} \kappa_\nu^2 \, e^{-\kappa_\nu S} \, dV_j \, dV_i \, E_{b\nu i} \, d\nu \\ &= -\int_{V_i} \int_{V_j} \frac{d^2\epsilon_m}{dS^2}(T_i, S) \frac{1}{\pi S^2} \, dV_j \, dV_i \, E_{bi} = \overrightarrow{g_i g_j} E_{bi}, \end{aligned} \tag{18.103}$$

and

$$Q_{i\leftrightarrow j} = \overrightarrow{g_i g_j} E_{bi} - \overleftarrow{g_i g_j} E_{bj} \simeq \sum_{l=1}^{L} \left(\overline{g_i g_j}\right)_l \left[a_l(T_i)E_{bi} - a_l(T_j)E_{bj}\right]. \tag{18.104}$$

Therefore, the net heat flow between any two surface or volume zones is calculated by determining the heat fluxes for a number of enclosures filled with a gray gas and with weighted emissive powers, followed by adding these heat fluxes.

Note that the approximation of equation (18.99) is not necessary: The directional direct exchange areas may be evaluated in terms of the total emissivity $\epsilon_m(T, S)$, leading to more accurate results. The approximation is made for convenience, reducing the problem to the simpler gray case with somewhat less involved evaluations of the direct exchange areas. The major advantage of the weighted-sum-of-gray-gases method is that the direct exchange areas $(\overline{s_i s_j})_l$, $(\overline{s_i g_k})_l$, and $(\overline{g_i g_k})_l$ do not depend on temperature and, therefore, (*i*) may be looked up in tables and graphs such as the ones given by [2] (this fact was of major importance during the method's early days, when fast computers were not yet available), and (*ii*) do not have to be recalculated, if the temperature field is found by iteration.

While equations (18.100) or (18.98), (18.102), and (18.104) are strictly valid only for black enclosures without scattering, a case can be made to extend this to reflecting surfaces and to scattering media: The difference in direct and total exchange areas is due to the average total *vs.* direct distance traveled between zones i and j. Since S does not appear directly in these equations[10] we may assume that, approximately, they are also valid for total exchange areas,

$$\overrightarrow{S_i S_j} E_{bi} - \overleftarrow{S_i S_j} E_{bj} \simeq \sum_{l=0}^{L} \left(\overline{S_i S_j}\right)_l \left[a_l(T_i)E_{bi} - a_l(T_j)E_{bj}\right], \tag{18.105}$$

$$\overrightarrow{G_i S_j} E_{bi} - \overleftarrow{G_i S_j} E_{bj} \simeq \sum_{l=1}^{L} (\overline{G_i S_j})_l \left[a_l(T_i)E_{bi} - a_l(T_j)E_{bj}\right], \tag{18.106}$$

$$\overrightarrow{G_i G_j} E_{bi} - \overleftarrow{G_i G_j} E_{bj} \simeq \sum_{l=1}^{L} (\overline{G_i G_j})_l \left[a_l(T_i)E_{bi} - a_l(T_j)E_{bj}\right], \tag{18.107}$$

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[9] In the light of property variations, equation (18.62), a more correct representation of $\kappa^2 e^{-\kappa S}$ would be $d^2\epsilon_m/dS_{i\to j}\, dS_{j\to i}$, where $S = S_{i\to j} = -S_{j\to i}$, i.e., measured from different points of origin.

[10] The total average distance between i and j, including reflections and scattering events, does of course to some degree depend on absorption coefficient.

and equations (18.80) and (18.81) may be used to determine net heat fluxes.

Considerable work has appeared in the literature dealing with the nongray zonal method. Nelson [24] pointed out that the use of equation (18.99) is not necessary, except for numerical efficiency for iterative calculations. He also proposed an approximation to apply the zonal method to reflecting boundaries. Correlations of the type of equation (18.99) have been determined for CO_2–H_2O gas mixtures by Smith and coworkers [25]. They later applied the weighted-sum-of-gray-gases model to combined radiative and convective heat transfer in a pipe [26]. A similar problem was addressed by Sistino [27].

18.6 DETERMINATION OF DIRECT EXCHANGE AREAS

The use of equations (18.63) and (18.64) together with the definition of direct exchange areas, or equations (18.80) and (18.81) together with the definition of total exchange areas, demands that the variation of the radiosity must be neglected over each surface zone, and that the variation of the radiation source term must be neglected over each volume zone. Hottel and Sarofim [2] state that, if 5% accuracy in the calculation of heat fluxes is to be assured, the zone sizes should be such that no zone has an optical depth (extinction coefficient × largest dimension of zone over which radiosity or source varies) exceeding the value of 0.4.

Accuracy of the zonal method may be improved in two ways: (*i*) increasing the number of zones, and (*ii*) increasing the accuracy of the numerical quadrature used to determine individual direct exchange areas. Increasing the number of zones allows the use of simple efficient techniques for the evaluation of exchange areas, but requires the time-consuming inversion of a large matrix. Modest [14, 15] found that, for sufficiently many zones, all exchange areas but one may be calculated by evaluating the integrand between zonal centers, multiplied by the applicable zonal areas and/or volumes. The error made for the closest zones is then offset by applying equations (18.65) and (18.66) for the last one ($\overline{g_i g_i}$ for volume–volume areas, common-face $\overline{g_i s_j}$ for volume–surface areas, and common-boundary $\overline{s_i s_j}$ for a corner zone). This finite-difference version of the zonal method has been generalized and refined by Naraghi and coworkers [28, 29].

Using relatively few zones with accurately evaluated exchange areas, on the other hand, requires only the inversion of a small matrix. But the numerical quadrature for the evaluation of exchange areas becomes time-consuming since many internal nodal points will be required. Still, if the temperature field is to be determined by iteration, considerable computer time may be saved since the evaluation of exchange areas does not enter the iteration. Accuracy for the zonal method with relatively few zones may be increased by allowing the emissive power to vary across a zone. Walther and coworkers [3] have shown for one-dimensional plane-parallel slab zones that a linear emissive power variation across the slab can be incorporated in a rigorous fashion. For two-dimensional rectangular volume zones Einstein [4, 5] included linear emissive power variations in a semirigorous manner (for two-dimensional Cartesian and cylindrical geometries).

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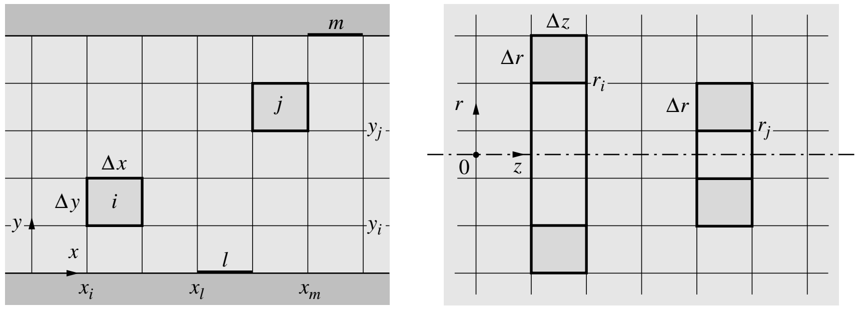
Problems

18.1 For a one-dimensional problem of a gas medium confined between two parallel plates, determine the direct exchange factor $\widetilde{g_i g_j}$ (i.e., per unit area), where medium zone i is an infinitely large slab of thickness Δz (located between $z_i \leq z \leq z_i + \Delta z$), and j is another medium zone (located at $z_j \leq z \leq z_j + \Delta z$).

18.2 For a two-dimensional problem determine the direct exchange factor $\overline{g_i g_j}'$ (i.e., per unit length) between two infinitely long volume zones of constant rectangular cross-section, as shown in Fig. 18-3*a* (the zones are infinitely long in the direction normal to the figure). You may assume that the zones



(*a*) (*b*)

FIGURE 18-3
Two-dimensional zones for Cartesian and cylindrical systems.

are sufficiently far apart so that $S \simeq$ const across the cross-section of the zones. In a similar fashion determine $\overline{g_i s_l}'$ and $\overline{s_m s_l}'$.

18.3 Consider a two-dimensional medium confined in a cylinder (with variation in the radial and axial directions). Determine the direct exchange area $\overline{g_i g_j}$ between two toroidal volume zones of rectangular cross-section, as shown in Fig. 18-3*b*.

18.4 For Problem 18.1 determine the total exchange area $\widetilde{G_i G_j}$ (per unit area) for the case that the bottom surface ($z = 0$) is a diffuse reflector (emittance ϵ), while the top surface ($z = L$) is black.

18.5 Do Example 18.2 for a molecular gas with one vibration–rotation band in the infrared at wavenumber η_0, assuming that the total emissivity of the gas may be approximated as

$$\epsilon(T, S) = \frac{\omega E_{b\eta 0}}{E_b}(T)\left(1 - e^{-\kappa_0 S}\right),$$

where ω is the band width parameter, and $E_{b\eta 0}$ is the spectral emissive power at the band center.

The following problems are considerably more involved and require the development of a small computer program.

18.6 Consider a rectangular enclosure of width $W = 1\,\mathrm{m}$ and height $H = 20\,\mathrm{cm}$ (and infinite length, without any dependence on that direction). The bottom surface has a temperature of $T_1 = 1000\,\mathrm{K}$ and emittance $\epsilon_1 = 0.8$. The other three walls are at $T_2 = T_3 = T_4 = 400\,\mathrm{K}$, $\epsilon_2 = \epsilon_3 = \epsilon_4 = 0.5$. There is no participating medium in the enclosure. Break up the enclosure surfaces into zones of 4 cm width, and calculate the local heat transfer rates by the zonal method with total exchange areas.

18.7 Do Problem 18.6 for an enclosure filled with an absorbing/emitting gray medium with absorption coefficient $\kappa = 5\,\mathrm{m}^{-1}$. The enclosure is in radiative equilibrium. Considering 4 cm × 4 cm volume zones, calculate surface heat fluxes for the case of black walls.

18.8 Do Problem 18.7 for an isothermal medium at $T_m = 400\,\mathrm{K}$.

18.9 Do Problem 18.7 for a gray-walled enclosure with emittance values taken from Problem 18.6.

18.10 Do Problem 18.9 for a medium that also scatters isotropically (with $\sigma_s = 3\,\mathrm{m}^{-1}$).

18.11 Consider a one-dimensional problem such as Problem 18.1. The plates are a distance $L = 1\,\mathrm{m}$ apart, the bottom plate is at $T_1 = 1000\,\mathrm{K}$ with emittance $\epsilon_1 = 0.5$, and the top plate is at $T_2 = 300\,\mathrm{K}$ and $\epsilon_2 = 0.5$. The medium between the plates is isothermal at $T_m = 1000\,\mathrm{K}$ and has an absorption coefficient of

$$\kappa_\eta = \kappa_0\, e^{-2|\eta - \eta_0|/\omega}, \quad \kappa_0 = 10\,\mathrm{cm}^{-1}, \quad \eta_0 = 3000\,\mathrm{cm}^{-1}, \quad \omega = 200\,\mathrm{cm}^{-1}$$

(that is, it has a single molecular gas band modeled by the exponential wide band approximation). Calculate the heat gain of the top plate using

(*a*) spectral direct exchange areas,
(*b*) the weighted-sum-of-gray-gases approach.

18.12 Repeat Problem 18.11 for a medium at radiative equilibrium.

18.13 Consider a sphere of very hot dissociated gas of radius 5 cm. The gas may be approximated as a gray, isotropically scattering medium with $\kappa = 0.1\,\text{cm}^{-1}$, $\sigma_s = 0.2\,\text{cm}^{-1}$. The gas is suspended magnetically in vacuum within a large cold container and is initially at a uniform temperature $T_g = 10{,}000\,\text{K}$. Using the zonal method and neglecting conduction and convection, specify the total heat loss per unit time from the entire sphere at $t = 0$. Outline the solution for times $t > 0$.

18.14 Consider a sphere of nonscattering gas, initially at uniform $T = 3000\,\text{K}$, with a single vibration–rotation band at $\eta_0 = 3000\,\text{cm}^{-1}$. The gas, with a radius of 20 cm, is suspended magnetically in vacuum within a large cold container. For this gas it is known that band strength $\rho_a\alpha(T) = 500\,\text{cm}^{-2}$, bandwidth $\omega = 100\sqrt{T/100\,\text{K}}\,\text{cm}^{-1}$, and line overlap $\beta \gg 1$. These properties imply that the absorption coefficient may be determined from

$$\kappa_\eta = \kappa_0\, e^{-2|\eta-\eta_0|/\omega}, \quad \kappa_0 = \frac{\rho_a\alpha}{\omega};$$

and the band absorptance from

$$A(s) = \omega A^* = \omega\left[E_1(\kappa_0 s) + \ln(\kappa_0 s) + \gamma_E\right].$$

Neglecting conduction and convection, specify the total heat loss per unit time from the entire sphere at time $t = 0$, using the zonal method. Outline the solution procedure for times $t > 0$.