CHAPTER 15

APPROXIMATE SOLUTION METHODS FOR ONE-DIMENSIONAL MEDIA

Because of their importance and their relative simplicity, the cases of gray (or spectral), plane-parallel, or other one-dimensional media bounded by isothermal, gray, diffusely emitting and reflecting walls have been studied extensively. Even for the simplest case the exact solution can only be cast implicitly in the form of an integral equation, thus prompting the development of numerous approximate solution techniques. These approximate methods may roughly be classified into those applicable for limiting conditions (cold medium approximation, optically thin approximation, optically thick approximation) and those making approximations for the directional distribution of intensity (two-flux approximation, moment method). In the following we shall discuss a few of these methods as applied to the one-dimensional case of a medium contained between isothermal gray surfaces. In principle, all of these methods could also be applied to more complicated geometries, although such an extension is not obvious for all of them (and may, indeed, be very tedious). We shall assume that the medium is gray and, therefore, not carry along any spectral subscripts on intensity and other quantities; however, all of the methods discussed here are equally valid on a spectral basis.

15.1 THE OPTICALLY THIN APPROXIMATION

The exact integral equations describing incident radiation G and radiative heat flux q for a gray medium (or on a spectral basis) confined between two isothermal, gray-diffuse, and parallel plates were developed with equations (14.21) and (14.22) as

$$G(\tau) = 2J_1E_2(\tau) + 2J_2E_2(\tau_L - \tau) + 2\pi \int_0^\tau S(\tau')E_1(\tau - \tau')\,d\tau' + 2\pi \int_\tau^{\tau_L} S(\tau')E_1(\tau' - \tau)\,d\tau', \quad (15.1)$$

$$q(\tau) = 2J_1E_3(\tau) - 2J_2E_3(\tau_L - \tau) + 2\pi \int_0^\tau S(\tau')E_2(\tau - \tau')\,d\tau' - 2\pi \int_\tau^{\tau_L} S(\tau')E_2(\tau' - \tau)\,d\tau', \quad (15.2)$$

where J_1 and J_2 are the radiosities at the two surfaces and the radiative source $S(\tau)$ has been assumed to be independent of direction [limiting us to isotropic scattering, see equation (14.5)].

We shall now assume that the medium is optically thin, i.e., $\tau_L \ll 1$. If we want to evaluate q accurately up to $\mathcal{O}(\tau)$ [i.e., neglecting terms of $\mathcal{O}(\tau^2)$ or smaller], we must evaluate the E_3 in equation (15.2) accurate up to $\mathcal{O}(\tau)$, while it is sufficient to evaluate E_2 (and S) accurate up to $\mathcal{O}(1)$ [since the integration itself is $\mathcal{O}(\tau)$]. Thus, with

$$E_2(x) = 1 + \mathcal{O}(x), \qquad E_3(x) = \tfrac{1}{2} - x + \mathcal{O}(x^2),$$

we get

$$q(\tau) \simeq J_1(1-2\tau) - J_2(1-2\tau_L+2\tau) + 2\pi\left[\int_0^\tau S(\tau')\,d\tau' - \int_\tau^{\tau_L} S(\tau')\,d\tau'\right]. \tag{15.3}$$

Evaluating the radiative source, equation (14.5),

$$S = (1-\omega)I_b(\tau) + \frac{\omega}{4\pi}G(\tau), \tag{15.4}$$

implies evaluating $G(\tau)$ accurate up to $\mathcal{O}(1)$, i.e., from equation (15.1),

$$G(\tau) = 2J_1 + 2J_2 + \mathcal{O}(\tau) \tag{15.5}$$

[keeping in mind that, while $\lim_{x\to 0} E_1(x) \to \infty$, $\lim_{x\to 0} xE_1(x) \to 0$]. Either the blackbody intensity, $I_b(\tau)$, is "known" (by considering other modes of heat transfer), or we have $S = I_b = G/4\pi = (J_1 + J_2)/2\pi$ (radiative equilibrium in an almost transparent gray medium). Thus, the radiative heat flux for an optically thin slab is:

$I_b(\tau)$ specified:

$$q(\tau) = J_1[1-2(1-\omega)\tau - \omega\tau_L] - J_2[1+2(1-\omega)\tau - (2-\omega)\tau_L] + 2\pi(1-\omega)\left[\int_0^\tau I_b(\tau')\,d\tau' - \int_\tau^{\tau_L} I_b(\tau')\,d\tau'\right]; \tag{15.6}$$

Radiative equilibrium:

$$q = (J_1 - J_2)(1-\tau_L) = \text{const.} \tag{15.7}$$

If the temperature of the medium is specified, we are usually interested in the divergence of the radiative heat flux, or dq/dz (the radiative source within the overall energy equation). From equation (14.24)

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G). \tag{15.8}$$

In order to predict dq/dz accurate to $\mathcal{O}(\tau)$, we must specify G accurate to $\mathcal{O}(1)$ [since we are using τ rather than z as the independent variable in equation (15.8)]. Thus, from equation (15.5)

$$\frac{dq}{d\tau} = 2(1-\omega)(2\pi I_b - J_1 - J_2). \tag{15.9}$$

Equation (15.6) may be interpreted physically: Intensity leaving the surfaces (radiosities J_1 and J_2) is attenuated by absorption and scattering, but the attenuation is linear since the strength of the radiosities is diminished very little (i.e., every point within the medium has essentially the same incident radiation and, therefore, attenuation rate). The integral terms describe emission within the medium, traveling up (+) and down (−). Emission is virtually unattenuated by self-absorption since the extinction coefficient is too low. Equation (15.7) shows no dependence on scattering albedo ω: In a gray medium at radiative equilibrium it is not possible to distinguish between absorption and isotropic scattering. If a photon is absorbed at a certain location, the same amount of energy must immediately be reemitted isotropically. Since the (different) wavelength of emission cannot be detected for a gray medium, this process is equivalent to isotropic scattering.

15.2 THE OPTICALLY THICK APPROXIMATION (DIFFUSION APPROXIMATION)

A particularly simple expression for the radiative heat flux can be obtained if the slab is optically thick, or $\tau_L \gg 1$. We note from equation (15.2) that the radiative source S is accompanied by an exponential integral that acts as a weight function. If the medium is optically thick, this exponential integral decays very rapidly over a short (geometrical) distance away from $\tau' = \tau$. To exploit this fact we first rewrite equation (15.2) by changing the integration variable τ' to $\tau'' = |\tau - \tau'|$:

$$q(\tau) = 2J_1E_3(\tau) - 2J_2E_3(\tau_L - \tau) + 2\pi \int_0^{\tau} S(\tau-\tau'')E_2(\tau'')\,d\tau'' - 2\pi \int_0^{\tau_L-\tau} S(\tau+\tau'')E_2(\tau'')\,d\tau''. \quad (15.10)$$

We shall now assume that we are a large optical distance away from either of the surfaces, i.e., $\tau \gg 1$ and $\tau_L - \tau \gg 1$. Under these conditions the influence of the boundaries (J_1 and J_2) becomes negligible and the integration limit may be replaced by infinity [since $E_2(\tau'') \simeq 0$ beyond the actual limits]. Thus,

$$q(\tau) \simeq 2\pi \int_0^{\infty} S(\tau-\tau'')E_2(\tau'')\,d\tau'' - 2\pi \int_0^{\infty} S(\tau+\tau'')E_2(\tau'')\,d\tau'', \quad (15.11)$$

where the arguments in q and S simply denote a physical location between the two plates. Since $E_2(\tau'')$ is expected to vanish a very short geometrical distance away from $\tau'' = 0$ (or $\tau' = \tau$), the radiative source can vary only slightly over this distance. Therefore, we may expand S into a Taylor series as

$$S(\tau \pm \tau'') = S(\tau) \pm \tau'' \left(\frac{dS}{d\tau}\right)_\tau + \frac{(\tau'')^2}{2}\left(\frac{d^2S}{d\tau^2}\right)_\tau \pm \cdots .$$

Substituting this expression into equation (15.11) leads to

$$\begin{aligned}
\frac{q(\tau)}{2\pi} &= S(\tau)\int_0^\infty E_2(\tau'')\,d\tau'' - \frac{dS}{d\tau}\int_0^\infty \tau''E_2(\tau'')\,d\tau'' + \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2E_2(\tau'')\,d\tau'' - + \cdots \\
&\quad - S(\tau)\int_0^\infty E_2(\tau'')\,d\tau'' - \frac{dS}{d\tau}\int_0^\infty \tau''E_2(\tau'')\,d\tau'' - \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2E_2(\tau'')\,d\tau'' - - \cdots \\
&= -2\frac{dS}{d\tau}\int_0^\infty xE_2(x)\,dx + \mathcal{O}\left(\frac{1}{\tau^3}\right).
\end{aligned}$$

Evaluating the integral we get, using the relations for exponential integrals in Appendix E,

$$\int_0^\infty xE_2(x)\,dx = -xE_3(x)\Big|_0^\infty + \int_0^\infty E_3(x)\,dx = -E_4(x)\Big|_0^\infty = \frac{1}{3},$$

and

$$q(\tau) = -\frac{4\pi}{3}\frac{dS}{d\tau}. \quad (15.12)$$

For a nonscattering medium, or a gray medium at radiative equilibrium, $S = I_b$, and equation (15.12) reduces to

$$q(\tau) = -\frac{4\pi}{3}\frac{dI_b}{d\tau}. \quad (15.13)$$

For the general case of an isotropically scattering medium the radiative source must be determined from equation (15.4) by first determining the incident radiation $G(\tau)$ from equation (15.1). Following the same procedure as for $q(\tau)$, we get

$$\begin{aligned}\frac{G(\tau)}{2\pi} &= S(\tau)\int_0^\infty E_1(\tau'')\,d\tau'' - \frac{dS}{d\tau}\int_0^\infty \tau'' E_1(\tau'')\,d\tau'' + \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2 E_1(\tau'')\,d\tau'' - + \cdots\\ &\quad + S(\tau)\int_0^\infty E_1(\tau'')\,d\tau'' + \frac{dS}{d\tau}\int_0^\infty \tau'' E_1(\tau'')\,d\tau'' + \frac{1}{2}\frac{d^2S}{d\tau^2}\int_0^\infty (\tau'')^2 E_1(\tau'')\,d\tau'' + + \cdots\\ &= 2\,S(\tau)\int_0^\infty E_1(\tau'')\,d\tau'' + \mathcal{O}\left(\frac{1}{\tau^2}\right) = 2\,S(\tau), \end{aligned} \tag{15.14}$$

or

$$\frac{G}{4\pi} = S = (1-\omega)I_b + \omega\frac{G}{4\pi}$$

and

$$S(\tau) = \frac{G}{4\pi}(\tau) = I_b(\tau).$$

Thus, for an optically thick, isotropically scattering medium equation (15.13) holds, whether the medium is at radiative equilibrium or not, and the heat flux, on a spectral basis, is determined from

$$q_\eta = -\frac{4\pi}{3\beta_\eta}\frac{dI_{b\eta}}{dz}, \tag{15.15}$$

or, after integration over wavenumbers, the total heat flux from

$$q = -\frac{4\sigma}{3\beta_R}\frac{d(n^2T^4)}{dz}, \tag{15.16}$$

where β_R is the *Rosseland-mean extinction coefficient* as defined in equation (11.188). Equations (15.15) and (15.16) are commonly known as the *Rosseland approximation*, since they were originally derived by Rosseland [1], or the *diffusion approximation*, since equation (15.16) is of the same type as Fourier's law of heat diffusion and Fick's law of mass diffusion.

The diffusion approximation is extremely convenient to use. One may even define a "radiative conductivity"

$$k_R = \frac{16n^2\sigma T^3}{3\beta_R}, \tag{15.17}$$

so that

$$q = -k_R\frac{dT}{dz}, \tag{15.18}$$

and the radiation problem reduces to a simple conduction problem with strongly temperature-dependent conductivity. Similar to Fourier's law, equation (15.15) may be extended to three-dimensional geometries by writing

$$\mathbf{q}_\eta = -\frac{4\pi}{3\beta_\eta}\nabla I_{b\eta}, \tag{15.19}$$

and

$$\mathbf{q} = -\frac{4\sigma}{3\beta_R}\nabla(n^2T^4) = -k_R\nabla T. \tag{15.20}$$

However, it is important to keep in mind that the diffusion approximation is not valid near a boundary, where it often fails quite miserably. In practice, the method is useful only in optically extremely thick situations (for example, heat transfer through hot glass and other semitransparent materials).

Example 15.1. Consider a gray, isothermal medium at temperature T, confined between two parallel, black, isothermal plates both at temperature T_w. Determine the radiative heat flux as a function of distance across the layer, using the diffusion approximation.

Solution
From equation (15.16) we find $q \equiv 0$ everywhere inside the medium, while $q \to \infty$ at both surfaces (because of the temperature jump there). The diffusion approximation is clearly inadequate in the optical vicinity of walls, in particular if temperature discontinuities are present.

Example 15.2. Now consider a gray medium contained between two black, isothermal cylinders. The inner cylinder has a radius of R_1 and is at temperature T_1. The outer cylinder has a radius of R_2 and is at temperature T_2. The medium absorbs and emits but does not scatter radiation, and radiative equilibrium prevails. Determine temperature profile and heat flux across the medium.

Solution
From the condition of radiative equilibrium,

$$\nabla \cdot \mathbf{q} = \frac{1}{r}\frac{d}{dr}(rq) = 0,$$

we find $rq = C_1' = \text{const}$, or $q = C_1'/r = C_1/\tau$ if the optical coordinate $\tau = \kappa r$ is used. Thus, from equation (15.13)

$$q = \frac{C_1}{\tau} = -\frac{4\sigma}{3}\frac{dT^4}{d\tau}$$

(assuming a refractive index of unity). We may integrate this expression to find

$$\sigma T^4 = -\frac{3}{4}C_1 \ln \tau + C_2, \tag{15.21}$$

or, using the boundary conditions $T = T_1$ at $r = R_1$ and $T = T_2$ at $r = R_2$,

$$\frac{T^4 - T_1^4}{T_2^4 - T_1^4} = \frac{\ln(\tau/\tau_1)}{\ln(\tau_2/\tau_1)},$$

and

$$\Psi = \frac{q}{\sigma(T_1^4 - T_2^4)} = \frac{4}{3\tau \ln(\tau_2/\tau_1)}. \tag{15.22}$$

As seen by comparison with the exact and other approximate solutions in Fig. 15-1, equation (15.22) does reasonably well for optically thick situations, but fails for the optically thin limit as $\kappa \to 0$, where $\Psi \to \infty$ [as opposed to the correct limit, $\Psi(R_1) = 1$].

Deissler's Jump Boundary Conditions

When we derived equation (15.13) we assumed that the medium was optically thick ($\tau_L \gg 1$) and that we were far removed from a boundary ($\tau \gg 0$, $\tau_L - \tau \gg 0$). But in the examples we assumed that equation (15.13) holds also at the walls and found the temperature profile by using the boundary temperatures (Example 15.2). The examples showed that this assumption is not very good. Deissler [2] argued that, while flux must be conserved [and, therefore, equation (15.13) must hold at the surface if it holds inside the medium adjacent to it], no *radiative* principle states that the temperature of the surface and adjacent medium must be continuous (as already known

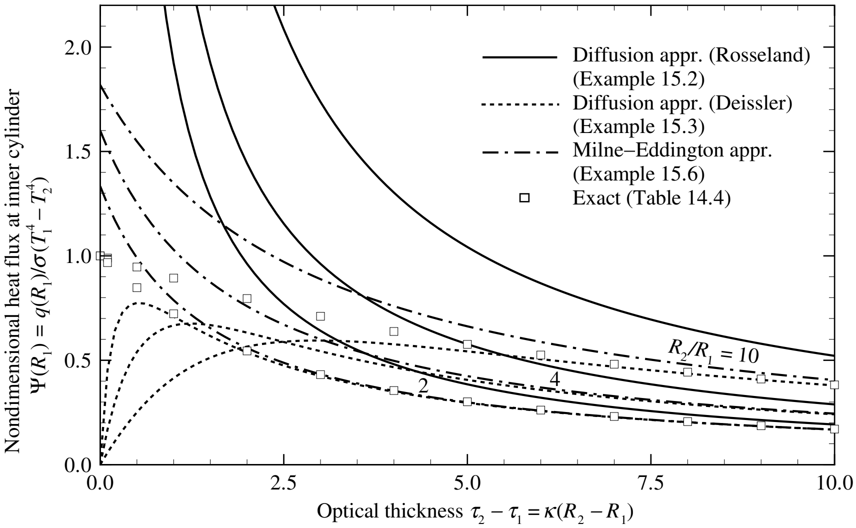

FIGURE 15-1
Nondimensional radiative heat flux in a medium at radiative equilibrium, confined between isothermal black cylinders.

from the exact solution).[1] To develop a boundary condition, he used the same principles that were applied to equation (15.10) and the development following it, and applied them to a point at the boundary. For $\tau = 0$ equation (15.10) becomes (with $\tau_L \gg 1$)

$$\begin{aligned} q(0) &= J_1 - 2\pi \int_0^\infty S(\tau'')E_2(\tau'')\,d\tau'' \\ &= J_1 - 2\pi \left[S(0)\int_0^\infty E_2(\tau'')\,d\tau'' + \frac{dS}{d\tau}(0)\int_0^\infty \tau'' E_2(\tau'')\,d\tau'' \right. \\ &\quad \left. + \frac{1}{2}\frac{d^2S}{d\tau^2}(0)\int_0^\infty (\tau'')^2 E_2(\tau'')\,d\tau'' + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right]. \end{aligned}$$

Using equation (15.15) this expression becomes

$$q(0) = J_1 - \pi I_b(0) - \frac{2\pi}{3}\frac{dI_b}{d\tau}(0) - \frac{\pi}{2}\frac{d^2I_b}{d\tau^2}(0) + \mathcal{O}\left(\frac{1}{\tau^3}\right). \tag{15.23}$$

Deissler truncated the series after the second derivative since doing so gives the same level of approximation as equation (15.13). Substituting equation (15.13) into equation (15.23) leads to

$$J_1 = \pi I_b(0) - \frac{2\pi}{3}\frac{dI_b}{d\tau}(0) + \frac{\pi}{2}\frac{d^2I_b}{d\tau^2}(0). \tag{15.24}$$

For radiative equilibrium of a one-dimensional slab this further simplifies to

$$J_1 - \pi I_b(0) = -\frac{2\pi}{3}\frac{dI_b}{d\tau}(0) = \frac{1}{2}q(0) = \frac{1}{2}q, \tag{15.25}$$

since q = const and, therefore, $d^2I_b/d\tau^2 = 0$.

The jump boundary condition may be generalized to multidimensional geometries [2]:

$$J_w(\mathbf{r}_w) = \pi I_b(\mathbf{r}_w) - \frac{2\pi}{3}\frac{\partial I_b}{\partial \tau_z}(\mathbf{r}_w) + \frac{\pi}{4}\left(2\frac{\partial^2 I_b}{\partial \tau_z^2} + \frac{\partial^2 I_b}{\partial \tau_x^2} + \frac{\partial^2 I_b}{\partial \tau_y^2}\right)(\mathbf{r}_w), \tag{15.26}$$

[1] Of course, in the presence of conduction and/or convection, temperature continuity is forced by those other heat transfer modes.

where τ_z is, as before, an optical coordinate measured in the direction of the outward surface normal, and τ_x and τ_y are optical coordinates tangential to the surface.

Example 15.3. Repeat Example 15.2 using Deissler's jump boundary conditions.

Solution
For a cylindrical coordinate system, equation (15.26) becomes[2]

$$\tau = \tau_1: \quad \sigma T_1^4 = \sigma T^4 - \frac{2}{3}\sigma\frac{dT^4}{d\tau} + \frac{\sigma}{4}\left(\frac{1}{\tau}\frac{dT^4}{d\tau} + 2\frac{d^2T^4}{d\tau^2}\right),$$
$$\tau = \tau_2: \quad \sigma T_2^4 = \sigma T^4 + \frac{2}{3}\sigma\frac{dT^4}{d\tau} + \frac{\sigma}{4}\left(\frac{1}{\tau}\frac{dT^4}{d\tau} + 2\frac{d^2T^4}{d\tau^2}\right)$$

[the change of sign in the second boundary condition is due to the fact that in equation (15.24) τ is measured *away* from the surface, while at τ_2 it is measured *toward* the surface]. Utilizing the general diffusion solution for a one-dimensional cylinder at radiative equilibrium, equation (15.21), we find

$$\tau = \tau_1: \quad \sigma T_1^4 = C_2 - \frac{3}{4}C_1 \ln\tau_1 + \frac{C_1}{2\tau_1} + \frac{3C_1}{16\tau_1^2},$$
$$\tau = \tau_2: \quad \sigma T_2^4 = C_2 - \frac{3}{4}C_1 \ln\tau_2 - \frac{C_1}{2\tau_2} + \frac{3C_1}{16\tau_2^2}.$$

From these two equations we obtain

$$C_1 = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{3}{4}\ln\dfrac{\tau_2}{\tau_1} + \dfrac{1}{2}\left(\dfrac{1}{\tau_1} + \dfrac{1}{\tau_2}\right) + \dfrac{3}{16}\left(\dfrac{1}{\tau_1^2} - \dfrac{1}{\tau_2^2}\right)},$$

and, with $q_1 = C_1/\tau_1$,

$$\Psi = \frac{q_1}{\sigma\left(T_1^4 - T_2^4\right)} = 1\Bigg/\left[\frac{3\tau_1}{4}\ln\frac{\tau_2}{\tau_1} + \frac{1}{2}\left(1+\frac{\tau_1}{\tau_2}\right) + \frac{3}{16\tau_1}\left(1-\frac{\tau_1^2}{\tau_2^2}\right)\right].$$

A plot of this nondimensional flux is also included in Fig. 15-1.

15.3 THE SCHUSTER–SCHWARZSCHILD APPROXIMATION

A very simple solution method for a one-dimensional, plane-parallel slab was given independently by Schuster [3] and Schwarzschild [4]. While they limited their derivation to nonscattering media, the method is readily extended to include isotropic scattering, which we will consider here. The equation of transfer for a one-dimensional, plane-parallel, isotropically scattering, gray medium (or on a spectral basis) is, setting $\Phi \equiv 1$ in equation (14.19),

$$\mu\frac{dI}{d\tau} = (1-\omega)I_b - I + \frac{\omega}{2}\int_{-1}^{+1} I\,d\mu, \quad -1 < \mu < +1. \tag{15.27}$$

Schuster and Schwarzschild assumed the radiative intensity to be isotropic, but different, over the upper and lower hemisphere, that is,

$$I(\tau,\mu) = \begin{cases} I^-(\tau), & -1 < \mu < 0, \\ I^+(\tau), & 0 < \mu < +1. \end{cases} \tag{15.28}$$

[2]The development of these boundary conditions is left as an exercise; see Problem 15.1.

Substituting this expression into equation (15.27) leads to

$$\mu \frac{dI}{d\tau} = (1-\omega)I_b - I + \frac{\omega}{2}(I^- + I^+). \tag{15.29}$$

Because of the approximation made for I, equation (15.29) can, of course, only be solved in an approximate way. Since intensity has been reduced to two unknown functions of space only, equation (15.29) must be reduced to two space-dependent equations. Integrating equation (15.29) over the upper and lower hemispheres, respectively, achieves this goal and results in

$$\frac{1}{2}\frac{dI^+}{d\tau} = (1-\omega)I_b - I^+ + \frac{\omega}{2}(I^- + I^+), \tag{15.30a}$$

$$-\frac{1}{2}\frac{dI^-}{d\tau} = (1-\omega)I_b - I^- + \frac{\omega}{2}(I^- + I^+), \tag{15.30b}$$

subject to the boundary conditions

$$\tau = 0: \quad I^+ = J_1/\pi, \tag{15.31a}$$

$$\tau = \tau_L: \quad I^- = J_2/\pi, \tag{15.31b}$$

where J_1 and J_2 are the radiosities of the bounding plates. From the definitions for incident radiation and radiative heat flux we find

$$G = 2\pi \int_{-1}^{1} I\, d\mu = 2\pi(I^+ + I^-), \tag{15.32}$$

and

$$q = 2\pi \int_{-1}^{1} I\mu\, d\mu = \pi(I^+ - I^-). \tag{15.33}$$

I^+ and I^- are easily eliminated from equations (15.30) and their boundary conditions (by adding and subtracting the equations), leading to

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G), \tag{15.34}$$

$$\frac{dG}{d\tau} = -4q, \tag{15.35}$$

$$\tau = 0: \quad G + 2q = 4J_1, \tag{15.36a}$$

$$\tau = \tau_L: \quad G - 2q = 4J_2. \tag{15.36b}$$

Example 15.4. Find an expression for the heat flux within a gray, nonscattering isothermal medium (temperature T) confined between two isothermal, parallel, black plates at (the same) temperature T_w. Use the Schuster–Schwarzschild approximation.

Solution
Differentiating equation (15.34) and using equation (15.35) gives

$$\frac{d^2q}{d\tau^2} = -\frac{dG}{d\tau} = 4q,$$

or

$$q = C_1 e^{2\tau} + C_2 e^{-2\tau},$$

subject to the boundary conditions

$$\tau = 0: \quad 4\sigma T^4 - \frac{dq}{d\tau} + 2q = 4\sigma T_w^4,$$

$$\tau = \tau_L: \quad 4\sigma T^4 - \frac{dq}{d\tau} - 2q = 4\sigma T_w^4,$$

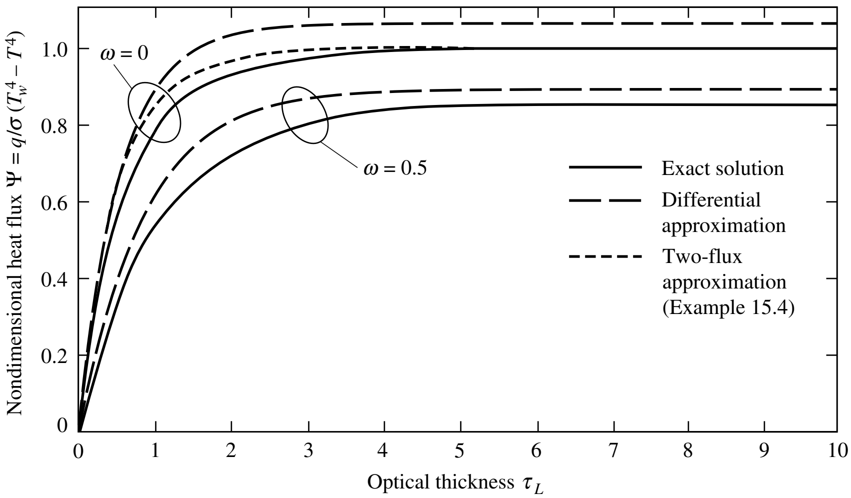

FIGURE 15-2
Radiative heat flux through a gray, isothermal, isotropically scattering medium bounded by black plates.

in which G has been eliminated using equation (15.34) and $J_1 = J_2 = \sigma T_w^4$. Substituting the expression for heat flux into the boundary condition gives

$$(2-2)C_1 + (2+2)C_2 = 4\sigma(T_w^4 - T^4),$$
$$-(2+2)C_1\, e^{2\tau_L} - (2-2)C_2\, e^{-2\tau_L} = 4\sigma(T_w^4 - T^4),$$

or

$$C_2 = -C_1\, e^{2\tau_L} = \sigma(T_w^4 - T^4)$$

and

$$\Psi = \frac{q}{\sigma(T^4 - T_w^4)} = e^{-2(\tau_L - \tau)} - e^{-2\tau}.$$

This nondimensional flux, evaluated at a wall, is plotted in Fig. 15-2, along with the exact solution and another approximate method.

The Schuster–Schwarzschild approximation always goes to the correct optically thin limit ($\tau_L \to 0$), since in that case the treatment is exact. Since it breaks up the intensity functions into two constant components for two directions, the method is also commonly referred to as the *two-flux approximation*. Obviously, the method is easily generalized to higher order (breaking up the 4π directions in more than two components and directions, or *discrete ordinates*), as well as to multidimensional geometries. For example, six-flux methods have been used by Chin and Churchill [5] and by Shih and Chen [6]; a review of the six-flux method has been given by Chan [7]. The general discrete ordinates method will be discussed in detail in Chapter 17.

15.4 THE MILNE–EDDINGTON APPROXIMATION (MOMENT METHOD)

Another simple method for the case of a one-dimensional, plane-parallel medium has been developed independently by Milne [8] and Eddington [9]. The method is also commonly referred to as the *differential approximation*, especially when generalized to more complicated geometries. Starting from equation (15.27) they took the *zeroth* and *first moments* of the equation, i.e., they integrated equation (15.27) over all directions after multiplication with $\mu^0 = 1$ (zeroth moment) and $\mu^1 = \mu$ (first moment). Defining intensity moments as

$$I_k = 2\pi \int_{-1}^{1} I\mu^k\, d\mu, \quad k = 0, 1, \ldots \tag{15.37}$$

leads to

$$\frac{dI_1}{d\tau} = (1-\omega)4\pi I_b - I_0 + \omega I_0 = (1-\omega)(4\pi I_b - I_0), \tag{15.38}$$

$$\frac{dI_2}{d\tau} = -I_1, \tag{15.39}$$

or two equations in three unknowns, I_0, I_1, and I_2. To make the system determinate, a *closing condition* must be found, i.e., a relationship between I_0, I_1, and I_2. Milne and Eddington, like Schuster and Schwarzschild, assumed the intensity to be isotropic over both the upper and lower hemisphere. Thus,

$$I_k = 2\pi\left(I^- \int_{-1}^{0} \mu^k\, d\mu + I^+ \int_0^1 \mu^k\, d\mu\right) = \frac{2\pi}{k+1}\left[(-1)^k I^- + I^+\right], \tag{15.40}$$

or

$$I_2 = \tfrac{1}{3} I_0. \tag{15.41}$$

With $G = I_0$ and $q = I_1$ equation (15.41) transforms (15.38) and (15.39) to

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G), \tag{15.42}$$

$$\frac{dG}{d\tau} = -3q. \tag{15.43}$$

The boundary conditions are identical to those of the Schuster–Schwarzschild approximation, equation (15.31), again leading to

$$\tau = 0: \quad G + 2q = 4J_1, \tag{15.44a}$$

$$\tau = \tau_L: \quad G - 2q = 4J_2. \tag{15.44b}$$

In the case of radiative equilibrium we have $dq/d\tau = 0$ and, therefore, $G = 4\pi I_b$. For this case equation (15.43) reduces to

$$q = -\frac{4\pi}{3}\frac{dI_b}{d\tau}, \tag{15.45}$$

which is the same as for the diffusion approximation (although the boundary conditions are different and, for large τ_L, are carried to one additional order of accuracy in the diffusion approximation).

Example 15.5. Consider a gray medium with refractive index $n = 1$, confined between two isothermal, black, parallel plates at temperatures T_1 and T_2, respectively. As in Example 15.2 the medium is at radiative equilibrium and absorbs and emits, but does not scatter radiation. Determine the heat flux between the plates using the differential approximation.

Solution

For a gray, nonscattering medium at radiative equilibrium equations (15.42) and (15.43) reduce to

$$\frac{dq}{d\tau} = 4\sigma T^4 - G = 0, \quad \text{or} \quad q = \text{const},$$

$$\frac{dG}{d\tau} = -3q, \quad \text{or} \quad G = 4\sigma T^4 = C - 3q\tau.$$

Applying the boundary conditions we get

$$\tau = 0: \quad C + 2q = 4\sigma T_1^4,$$

$$\tau = \tau_L: \quad C - 3q\tau_L - 2q = 4\sigma T_2^4,$$

or

$$\Psi = \frac{q}{\sigma(T_1^4 - T_2^4)} = \frac{1}{1 + \frac{3}{4}\tau_L}$$

and

$$C = 4\sigma T_1^4 - 2q,$$

$$\Phi = \frac{T_1^4 - T^4}{T_1^4 - T_2^4} = \frac{2 + 3\tau}{4 + 3\tau_L}.$$

It is easy to show that this result is identical to the one obtained from the diffusion approximation with Deissler's jump boundary conditions (for example, by letting $\tau_2 = \tau_1 + \tau_L$ and $\tau_1 \to \infty$ in Example 15.3).

Example 15.6. Repeat Example 15.2 using the differential approximation.

Solution
For the one-dimensional case of a medium at radiative equilibrium between concentric cylinders the divergence of the radiative heat flux is, in cylindrical coordinates,

$$\frac{1}{\tau}\frac{d}{d\tau}(\tau q) = 4\sigma T^4 - G = 0,$$

or

$$q = \frac{C_1}{\tau}.$$

Substituting this expression into equation (15.45) gives

$$\sigma T^4 = -\tfrac{3}{4} C_1 \ln \tau + C_2,$$

which is, of course, the same as for the diffusion approximation (since we have radiative equilibrium). Applying the boundary conditions (with $G = 4\sigma T^4$) gives

$$\tau = \tau_1: \quad -\frac{3}{4}C_1 \ln \tau_1 + C_2 + \frac{C_1}{2\tau_1} = \sigma T_1^4,$$

$$\tau = \tau_2: \quad -\frac{3}{4}C_1 \ln \tau_2 + C_2 - \frac{C_1}{2\tau_2} = \sigma T_2^4,$$

or

$$C_1 = \frac{\sigma(T_1^4 - T_2^4)}{\frac{1}{2}\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) + \frac{3}{4}\ln\frac{\tau_2}{\tau_1}}$$

and

$$\Psi = \frac{q_1}{\sigma(T_1^4 - T_2^4)} = 1\bigg/\left[\frac{1}{2}\left(1 + \frac{\tau_1}{\tau_2}\right) + \frac{3}{4}\tau_1 \ln\frac{\tau_2}{\tau_1}\right].$$

Results from the differential approximation are also included in Fig. 15-1. Note that the diffusion approximation with jump conditions outperforms the differential approximation over a large range of optical depths (since it has a higher order boundary condition), but fails much more severely in optically thin cases.

Like the Schuster–Schwarzschild approximation, the Milne–Eddington approximation may be generalized to higher order as well as to more general geometries. It is then known as the *moment method*, in which the radiative intensity is approximated by

$$\begin{aligned} I(\mathbf{r}, \hat{\mathbf{s}}) &= I_0(\mathbf{r}) + I_{1x}(\mathbf{r})s_x + I_{1y}(\mathbf{r})s_y + I_{1z}(\mathbf{r})s_z + I_{2xx}(\mathbf{r})s_x^2 + I_{2xy}(\mathbf{r})s_x s_y + \cdots \\ &= I_0(\mathbf{r}) + \mathbf{I}_1(\mathbf{r}) \cdot \hat{\mathbf{s}} + \mathbf{I}_2(\mathbf{r}) : \hat{\mathbf{s}}\hat{\mathbf{s}} + \cdots . \end{aligned} \tag{15.46}$$

Here the $s_x = \hat{\mathbf{s}} \cdot \hat{\mathbf{i}} = \sin\theta\cos\psi$, $s_y = \hat{\mathbf{s}} \cdot \hat{\mathbf{j}} = \sin\theta\sin\psi$, and $s_z = \hat{\mathbf{s}} \cdot \hat{\mathbf{k}} = \cos\theta$ are the direction cosines of the unit direction vector $\hat{\mathbf{s}}$. I_0 is a scalar to be determined (related to G), $\mathbf{I}_1$ is a vector (related to $\mathbf{q}$), $\mathbf{I}_2$ is a second-rank tensor (which may be related to radiation pressure), and so on. The unknowns are determined by taking moments of the equation of transfer, i.e., by integrating it over all directions after multiplication by $1, s_x, s_y, s_z, s_x^2, s_x s_y, \ldots$. The method has been shown by Krook [10] to be completely equivalent to the method of spherical harmonics (using spherical harmonics, which are functions of direction cosines, and exploiting their orthogonality properties). That method will be discussed in detail in Chapter 16.

15.5 THE EXPONENTIAL KERNEL APPROXIMATION

Another popular way to solve equations (15.1) and (15.2) in an approximate way is known as the *exponential kernel approximation*. In this method the *kernels* of the integrals [E_1 in equation (15.1), and E_2 in equation (15.2)], are approximated by functions of exponentials (e^x, $\cosh x$, $\sinh x$, $\cos x$, $\sin x$). Since these functions have repetitive derivatives (except for constant factors), this fact enables us to eliminate the integrals from equations (15.1) and (15.2) and transform them into differential equations. We shall demonstrate the method here by solving equation (15.2) with a very simple approximate kernel of the form

$$E_2(x) \simeq a\, e^{-bx}. \tag{15.47}$$

More elaborate approximations could consist of a sum of exponentials, for example (as long as derivatives with respect to x are repetitive). To determine the "best" values for a and b, one could choose either to satisfy equation (15.47) at two selected points, or to satisfy equation (15.47) in an integral sense. How exactly the values of a and b are determined is somewhat arbitrary and should be decided by studying the problem at hand (is the medium optically thin, thick, covering all ranges?). The most common method of finding a and b is to take the zeroth and first moments of equation (15.47):

$$\int_0^\infty E_2(x)\,dx = -E_3(x)\Big|_0^\infty = \frac{1}{2} = a\int_0^\infty e^{-bx}\,dx = -\frac{a}{b}\,e^{-bx}\Big|_0^\infty = \frac{a}{b},$$
$$\int_0^\infty xE_2(x)\,dx = \frac{1}{3} = a\int_0^\infty x\,e^{-bx}\,dx = \frac{a}{b^2},$$

or

$$a = \tfrac{3}{4}, \quad b = \tfrac{3}{2}, \quad E_2(x) \simeq \tfrac{3}{4}\,e^{-3x/2}.$$

While $E_1(x)$ and $E_3(x)$ could be found in a similar fashion, it is usually preferred (for consistency, and numerical simplicity) to use the recursion formulae for exponential integrals, as given in Appendix E. Thus,

$$E_1(x) = -\frac{dE_2}{dx} \simeq \tfrac{9}{8}\,e^{-3x/2},$$
$$E_3(x) = \int_x^\infty E_2(x)\,dx \simeq \tfrac{1}{2}\,e^{-3x/2}.$$

A plot of these approximations is given in Fig. 15-3. With them equation (15.2) may be rewritten as

$$q(\tau) = J_1\,e^{-3\tau/2} - J_2\,e^{-3(\tau_L-\tau)/2} + \frac{3\pi}{2}\left[\int_0^\tau S(\tau')\,e^{-3(\tau-\tau')/2}\,d\tau' - \int_\tau^{\tau_L} S(\tau')\,e^{-3(\tau'-\tau)/2}\,d\tau'\right]. \tag{15.48}$$

Differentiating this equation twice with respect to τ results in

$$\frac{d^2q}{d\tau^2} = \frac{9}{4}J_1\,e^{-3\tau/2} - \frac{9}{4}J_2\,e^{-3(\tau_L-\tau)/2} + 3\pi\frac{dS}{d\tau} + \frac{27\pi}{8}\left[\int_0^\tau S(\tau')\,e^{-3(\tau-\tau')/2}\,d\tau' - \int_\tau^{\tau_L} S(\tau')\,e^{-3(\tau'-\tau)/2}\,d\tau'\right], \tag{15.49}$$

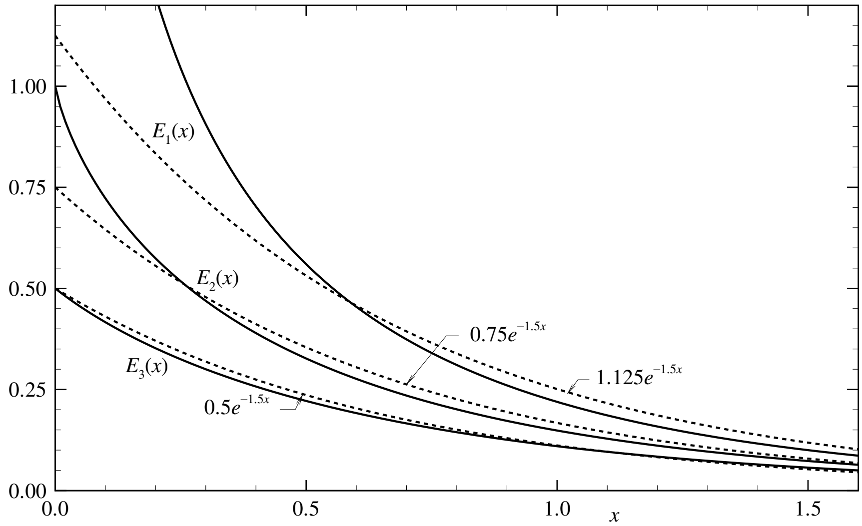

FIGURE 15-3
Approximations for exponential integrals.

or, using equation (15.48) to eliminate the integrals,

$$\frac{d^2q}{d\tau^2} - \frac{9}{4}q = 3\pi\frac{dS}{d\tau} = 3\pi\frac{d}{d\tau}\left[(1-\omega)I_b + \frac{\omega}{4\pi}G\right]. \tag{15.50}$$

The source function is either known or must be determined by performing a similar procedure on equation (15.1). Equation (15.50) is a second-order differential equation and, thus, requires two boundary conditions (while an integral equation does not require any boundary conditions). The problem is overcome by substituting the solution to equation (15.50) back into equation (15.48). Since two boundary conditions are required it is sufficient to do this at two selected locations, say $\tau = 0$ and $\tau = \tau_L$.

Example 15.7. Redo Example 15.5 using the exponential kernel approximation.

Solution
For $\omega = 0$ we have $S = I_b = \sigma T^4/\pi$, and for radiative equilibrium $dq/d\tau = 0$ (and, therefore, $d^2q/d\tau^2 = 0$). Thus, from equation (15.50), we get

$$q = -\frac{4\sigma}{3}\frac{dT^4}{d\tau} = \text{const},$$

which is the same as for the diffusion approximation as well as for the differential approximation. Integration gives

$$\sigma T^4 = C - \tfrac{3}{4}q\tau.$$

Substituting this expression into equation (15.48) leads to

$$\begin{aligned} q &= \sigma T_1^4\, e^{-3\tau/2} - \sigma T_2^4\, e^{-3(\tau_L-\tau)/2} + \frac{3}{2}\left[\int_0^\tau \left(C - \frac{3}{4}q\tau'\right)e^{-3(\tau-\tau')/2}\,d\tau' - \int_\tau^{\tau_L}\left(C - \frac{3}{4}q\tau'\right)e^{-3(\tau'-\tau)/2}\,d\tau'\right] \\ &= \left(\sigma T_1^4 - C - \frac{q}{2}\right)e^{-3\tau/2} - \left(\sigma T_2^4 - C + \frac{q}{2} + \frac{3}{4}q\tau_L\right)e^{-3(\tau_L-\tau)/2} + q. \end{aligned}$$

Since this equation must hold for all values of τ, both expressions within the parentheses must vanish,

$$\begin{aligned} \sigma T_1^4 &= C + \frac{q}{2}, \\ \sigma T_2^4 &= C - \frac{q}{2} - \frac{3}{4}\tau_L q, \end{aligned}$$

or

$$\Psi = \frac{q}{\sigma(T_1^4 - T_2^4)} = \frac{1}{1 + \frac{3}{4}\tau_L}.$$

This result is identical to the ones from the diffusion approximation with jump conditions, and from the differential approximation, Example 15.5.

References

1. Rosseland, S.: *Theoretical Astrophysics; Atomic Theory and the Analysis of Stellar Atmospheres and Envelopes*, Clarendon Press, Oxford, 1936.
2. Deissler, R. G.: "Diffusion approximation for thermal radiation in gases with jump boundary conditions," *ASME Journal of Heat Transfer*, vol. 86, pp. 240–246, 1964.
3. Schuster, A.: "Radiation through a foggy atmosphere," *Astrophysical Journal*, vol. 21, pp. 1–22, 1905.
4. Schwarzschild, K.: "Über das Gleichgewicht der Sonnenatmosphären (Equilibrium of the sun's atmosphere)," *Akad. Wiss. Göttingen, Math.-Phys. Kl. Nachr.*, vol. 195, pp. 41–53, 1906.
5. Chin, J. H., and S. W. Churchill: "Anisotropic, multiply scattered radiation from an arbitrary, cylindrical source in an infinite slab," *ASME Journal of Heat Transfer*, vol. 87, pp. 167–172, 1965.
6. Shih, T. M., and Y. N. Chen: "A discretized-intensity method proposed for two-dimensional systems enclosing radiative and conductive media," *Numerical Heat Transfer*, vol. 6, pp. 117–134, 1983.
7. Chan, S. H.: "Numerical methods for multidimensional radiative transfer analysis in participating media," in *Annual Review of Numerical Fluid Mechanics and Heat Transfer*, vol. 1, Hemisphere, New York, pp. 305–350, 1987.
8. Milne, F. A.: "Thermodynamics of the stars," in *Handbuch der Astrophysik*, Springer-Verlag, OHG, Berlin, pp. 65–255, 1930.
9. Eddington, A. S.: *The Internal Constitution of the Stars*, Dover Publications, New York, 1959.
10. Krook, M.: "On the solution of the equation of transfer, I," *Astrophysical Journal*, vol. 122, pp. 488–497, 1955.

Problems

15.1 Derive the jump boundary condition for the diffusion approximation, equation (15.26), for the case of concentric cylinders. Assume the heat transfer to be one-dimensional (only radial, no azimuthal or axial dependence).
Hint: Introduce a local Cartesian coordinate system at a point at the boundary, express any other r-location within the medium in terms of x, y, z, and transform the derivatives in equation (15.26) to r-derivatives; finally let x, y, z go to zero (since the derivatives at the boundary are needed).

15.2 The gap between two parallel black plates at T_1 and T_2, respectively, is filled with a particle-laden gas. Radiative equilibrium prevails, and the particle loading is a fixed volume fraction, with particles manufactured from two different materials (one a specular reflector, the other a diffuse reflector, both having the same ϵ). Sketch the nondimensional heat flux $\Psi = q/\sigma(T_1^4 - T_2^4)$ *vs.* particle size (but keeping volume fraction constant).

15.3 Consider radiative equilibrium of a gray, absorbing, emitting, and isotropically scattering medium contained between two isothermal, gray-diffuse, parallel plates spaced a distance L apart. Determine the nondimensional temperature variation within the medium, $\Phi = (\sigma T^4 - J_2)/(J_1 - J_2)$ for the optically thin case ($\tau_L \ll 1$).

15.4 Consider a gray, absorbing–emitting, linear-anisotropically scattering medium at radiative equilibrium. The medium is confined between two parallel, isothermal, black plates (at temperatures T_1 and T_2). Determine an expression for the radiative heat flux between the two plates using the diffusion approximation with jump boundary conditions.

15.5 Do Problem 15.4 using the Schuster–Schwarzschild (2-flux) approximation.

15.6 Do Problem 15.4 using the Milne–Eddington (differential) approximation.

15.7 Do Problem 15.4 using the exponential kernel approximation method.

15.8 Do Problem 14.7 using the Milne–Eddington (differential) approximation.

15.9 Do Problem 14.11 using the Milne–Eddington (differential) approximation.

15.10 Consider a space enclosed by infinite, diffuse-gray, parallel plates 1 m apart filled with a gray, nonscattering medium ($\kappa = 5\,\mathrm{m}^{-1}$). The surfaces are isothermal (both at $T_w = 500\,\mathrm{K}$ with emittance $\epsilon_w = 0.6$), and there is uniform and constant heat generation within the medium per unit volume, $\dot{Q}''' = 10^6\,\mathrm{W/m}^3$. Conduction and convection are negligible such that $\nabla \cdot \mathbf{q} = \dot{Q}'''$. Determine the radiative heat flux to the walls as well as the maximum temperature within the medium, using the diffusion approximation with jump boundary conditions.

15.11 Do Problem 15.10 using the Schuster–Schwarzschild approximation.

15.12 Do Problem 15.10 using the Milne–Eddington approximation.

15.13 Do Problem 15.10 using the exponential kernel approximation.
Note: The necessary exact integral relations have been given in Problem 14.3.

15.14 Consider (*a*) two parallel plates and (*b*) two concentric spheres. The bottom/inner surface needs to dissipate a heat flux of $30\,\mathrm{W/cm}^2$ and has a gray-diffuse emittance $\epsilon_1 = 0.5$. The top/outer surface is at $T_2 = 1000\,\mathrm{K}$ with $\epsilon_2 = 0.8$. The medium in between the surfaces is gray and nonscattering ($\kappa = 0.1\,\mathrm{cm}^{-1}$), has a thickness of $L = 5\,\mathrm{cm}$, and is at radiative equilibrium. Determine the temperature at the bottom/inner surface necessary to dissipate the supplied heat for the two different cases (the radius of the inner sphere is $R_1 = 5\,\mathrm{cm}$) using the Milne–Eddington approximation. Compare with the results of Problem 14.12.

15.15 A material produces an amount of heat that is constant per unit volume, i.e., $\dot{Q}''' = \mathrm{const}$. This heat production needs to be removed by thermal radiation. It is proposed to grind up the (fixed volume of) material into small particles, which are to be suspended evenly between two cold plates of (identical) emittance ϵ. Since it is important to keep the overall temperature level in the particles as low as possible, should the particles be ground as fine as possible, as large as possible, or does some optimum radius exist? What is the optimum particle size, and what is the maximum temperature if this size is employed? You may assume one-dimensional parallel plates with a constant volume fraction of particles, black particles with relatively large size parameters, and you may use the Schuster–Schwarzschild approximation.

15.16 Do Problem 15.15 using the Milne–Eddington (differential) approximation.

15.17 Do Problem 15.15 using the exponential kernel approximation.

15.18 Consider parallel, black plates, spaced 1 m apart, at constant temperatures T_1 and T_2. Due to pressure variations, the (gray) absorption coefficient is equal to

$$\kappa = \kappa_0 + \kappa_1 z; \quad \kappa_0 = 0.01\,\mathrm{cm}^{-1}; \quad \kappa_1 = 0.0002\,\mathrm{cm}^{-2},$$

where z is measured from Plate 1. The medium does not scatter radiation. Determine, for radiative equilibrium, the nondimensional heat flux $\Psi = q/\sigma(T_1^4 - T_2^4)$ by (*a*) the exact method, (*b*) the regular diffusion approximation, (*c*) the diffusion approximation with jump boundary conditions, (*d*) the two-flux method, (*e*) the differential approximation, and (*f*) the kernel approximation.

15.19 An infinite, black, isothermal plate at 1000 K bounds a semi-infinite space filled with black spheres of uniform radius $a = 100\,\mu\mathrm{m}$. The particle number density is maximum adjacent to the surface, and decays exponentially away from the surface according to

$$N_T = N_0\, e^{-Cz}; \qquad N_0 = 10^8\,\mathrm{m}^{-3}, \qquad C = \pi\,\mathrm{m}^{-1}.$$

(*a*) Determine the absorption and extinction coefficients as functions of z.

(*b*) Determine the optical coordinate as a function of z. What is the total optical thickness of the semi-infinite space?

(*c*) Assuming that radiative equilibrium prevails and using the Milne–Eddington approximation, set up the boundary conditions and solve for heat flux and temperature distribution (as a function of z).