CHAPTER 14

EXACT SOLUTIONS FOR ONE-DIMENSIONAL GRAY MEDIA

14.1 INTRODUCTION

The governing equation for radiative transfer of absorbing, emitting, and scattering media was developed in Chapter 10, resulting in an integro-differential equation for radiative intensity in five independent variables (three space coordinates and two direction coordinates). The problem becomes even more complicated if the medium is nongray (which introduces an additional variable, such as wavelength or frequency) and/or if other modes of heat transfer are present (which make it necessary to solve simultaneously for overall conservation of energy, to which intensity is related in a nonlinear way). Consequently, exact analytical solutions exist for only a few extremely simple situations. The simplest case arises when one considers thermal radiation in a one-dimensional plane-parallel gray medium that is either at radiative equilibrium (i.e., radiation is the only mode of heat transfer) or whose temperature field is known. Analytical solutions for such simple problems have been studied extensively, partly because of the great importance of one-dimensional plane-parallel media, partly because the simplicity of such solutions allows testing of more general solution methods, and partly because such a solution can give qualitative indications for more difficult situations.

In the present chapter we develop some analytical solutions for one-dimensional plane-parallel media and also include a few solutions for one-dimensional cylindrical and spherical media (without development). In general, we shall assume the medium to be gray, and all radiative intensity-related quantities are total, i.e., frequency-integrated quantities, for example, $I_b = \int_0^\infty I_{b\nu}\,d\nu = n^2\sigma T^4/\pi$. Most relations also hold, on a spectral basis, for nongray media, except for those that utilize the statement of radiative equilibrium, $\nabla \cdot \mathbf{q} = 0$ (since this relation does not hold on a spectral basis).

14.2 GENERAL FORMULATION FOR A PLANE-PARALLEL MEDIUM

The governing equation for the intensity field in an absorbing, emitting, and scattering medium is, from equation (10.24),

$$\hat{\mathbf{s}} \cdot \nabla I = \kappa I_b - \beta I + \frac{\sigma_s}{4\pi} \int_{4\pi} I(\hat{\mathbf{s}}_i)\, \Phi(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i, \tag{14.1}$$

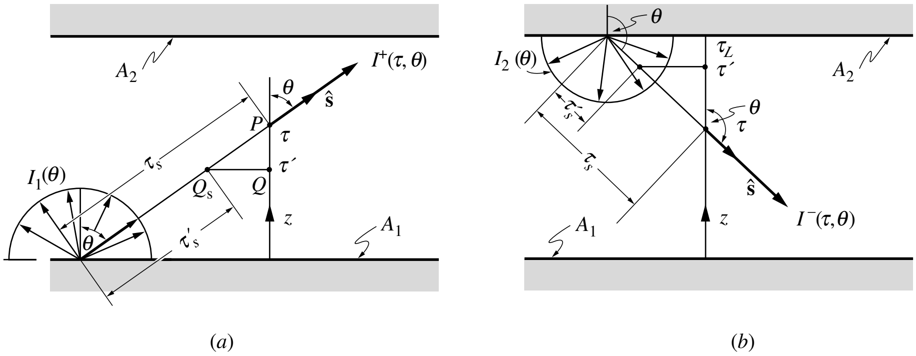

FIGURE 14-1
Coordinates for radiative intensities in a one-dimensional plane-parallel medium: (*a*) upward directions, (*b*) downward directions.

which describes the change of radiative intensity along a path in the direction of $\hat{\mathbf{s}}$. The formal solution to equation (14.1) is given by equation (10.28) as

$$I(\mathbf{r},\hat{\mathbf{s}}) = I_w(\hat{\mathbf{s}})\,e^{-\tau_s} + \int_0^{\tau_s} S(\tau'_s,\hat{\mathbf{s}})\,e^{-(\tau_s-\tau'_s)}\,d\tau'_s, \tag{14.2}$$

where S is the radiative source term, equation (10.25),

$$S(\tau'_s,\hat{\mathbf{s}}) = (1-\omega)I_b(\tau'_s) + \frac{\omega}{4\pi}\int_{4\pi} I(\tau'_s,\hat{\mathbf{s}}_i)\,\Phi(\hat{\mathbf{s}},\hat{\mathbf{s}}_i)\,d\Omega_i, \tag{14.3}$$

and $\tau_s = \int_0^s \beta(s)\,ds$ is *optical thickness* or *optical depth* based on extinction coefficient[1] measured from a point on the wall ($\tau'_s = 0$) toward the point under consideration ($\tau'_s = \tau_s$), in the direction of $\hat{\mathbf{s}}$. For a plane-parallel medium the change of intensity is illustrated in Fig. 14-1*a*, measuring polar angle θ from the direction perpendicular to the plates (z-direction), and azimuthal angle ψ in a plane parallel to the plates (x-y-plane): Radiative intensity of strength $I_w(\hat{\mathbf{s}}) = I_w(\theta,\psi)$ leaves the point on the bottom surface into the direction of θ,ψ, toward the point under consideration, P. This intensity is augmented by the radiative source (by emission and by in-scattering, i.e., scattering of intensity from other directions into the direction of P). The amount of energy $S(\tau'_s,\theta,\psi)\,d\tau'_s$ is released over the infinitesimal optical depth $d\tau'_s$ and travels toward P. Since this energy also undergoes absorption and out-scattering along its path from τ'_s to τ_s, only the fraction $e^{-(\tau_s-\tau'_s)}$ actually arrives at P. In general, the intensity leaving the bottom wall may vary across the bottom surface, and radiative source and medium properties may vary throughout the medium, i.e., in the directions parallel to the plates as well as normal to them.

We shall now assume that both plates are isothermal and isotropic, i.e., neither temperature nor radiative properties vary across each plate and properties may show a directional dependence on polar angle θ, but not on azimuthal angle ψ. Thus, the intensity leaving the bottom plate at a certain location is the same for all azimuthal angles and, indeed, for all positions on that plate; it is a function of polar angle θ alone. We also assume that the temperature field and radiative properties of the medium vary only in the direction perpendicular to the plates. This assumption implies that the radiative source at position Q, $S(\tau',\theta)$, is identical to the one at position Q_s, $S(\tau'_s,\theta)$, or any horizontal position with identical z-coordinate $\tau' = \int_0^{z'} \beta\,dz$ (based on extinction coefficient). Therefore, radiative source, $S(\tau,\theta)$, and radiative intensity, $I(\tau,\theta)$, both depend only on a single space coordinate plus a single direction coordinate. The radiative

[1]We use here the notation τ_s to describe optical depth along s so that we will be able to use the simpler τ for optical depth perpendicular to the plates, i.e., $\tau = \int_0^z \beta\,dz$.

source term may be simplified for the one-dimensional case to

$$S(\tau', \theta) = (1-\omega) I_b(\tau') + \frac{\omega}{4\pi} \int_{\psi_i=0}^{2\pi} \int_{\theta_i=0}^{\pi} I(\tau', \theta_i)\, \Phi\,(\theta, \psi, \theta_i, \psi_i) \sin\theta_i \, d\theta_i \, d\psi_i. \tag{14.4}$$

For *isotropic scattering*, $\Phi \equiv 1$, and we find immediately from the definition for incident radiation, G [equation (10.32)], that

$$S(\tau') = (1-\omega) I_b(\tau') + \frac{\omega}{4\pi} G(\tau'). \tag{14.5}$$

In other words, the source term does not depend on direction, that is, the radiative source due to isotropic emission and isotropic in-scattering is also isotropic.

If the scattering is *anisotropic*, we may write, from equation (12.99),[2]

$$\Phi(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i) = 1 + \sum_{m=1}^{M} A_m P_m(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i), \tag{14.6}$$

where it is assumed that the series may be truncated after M terms. Measuring the polar angle from the z-axis and the azimuthal angle from the x-axis (in the x-y-plane) for both $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}_i$, we get the direction vectors

$$\hat{\mathbf{s}} = \sin\theta(\cos\psi\hat{\mathbf{\imath}} + \sin\psi\hat{\mathbf{\jmath}}) + \cos\theta\hat{\mathbf{k}}, \tag{14.7}$$

$$\hat{\mathbf{s}}_i = \sin\theta_i(\cos\psi_i\hat{\mathbf{\imath}} + \sin\psi_i\hat{\mathbf{\jmath}}) + \cos\theta_i\hat{\mathbf{k}}, \tag{14.8}$$

and

$$\Phi(\theta, \psi, \theta_i, \psi_i) = 1 + \sum_{m=1}^{M} A_m P_m[\cos\theta\cos\theta_i + \sin\theta\sin\theta_i\cos(\psi-\psi_i)]. \tag{14.9}$$

Using a relationship between Legendre polynomials [1], one may separate the directional dependence in the last relationship by

$$\begin{aligned} P_m[\cos\theta\cos\theta_i + \sin\theta\sin\theta_i\cos(\psi-\psi_i)] &= P_m(\cos\theta)P_m(\cos\theta_i) \\ &+ 2\sum_{n=1}^{m} \frac{(m-n)!}{(m+n)!} P_n^m(\cos\theta)P_n^m(\cos\theta_i)\cos m(\psi-\psi_i), \end{aligned} \tag{14.10}$$

where the P_n^m are *associated Legendre polynomials*. Thus, the scattering phase function may be rewritten as

$$\begin{aligned} \Phi(\theta, \psi, \theta_i, \psi_i) = 1 &+ \sum_{m=1}^{M} A_m P_m(\cos\theta)P_m(\cos\theta_i) \\ &+ 2\sum_{m=1}^{M}\sum_{n=1}^{m} A_m \frac{(m-n)!}{(m+n)!} P_n^m(\cos\theta)P_n^m(\cos\theta_i)\cos m(\psi-\psi_i). \end{aligned} \tag{14.11}$$

For a one-dimensional plane-parallel geometry, the intensity does not depend on azimuthal angle, and we may carry out the ψ_i-integration in equation (14.4). This integration leads to a one-dimensional scattering phase function of

$$\Phi(\theta, \theta_i) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i)\, d\psi_i = 1 + \sum_{m=1}^{M} A_m P_m(\cos\theta)P_m(\cos\theta_i), \tag{14.12}$$

[2] In Chapter 12 we used Θ to denote the angle between the incoming and scattered ray and, therefore, $\cos\Theta = \hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i$.

since $\int_0^{2\pi} \cos m(\psi - \psi_i)\, d\psi_i = 0$. The radiative source then becomes

$$S(\tau', \theta) = (1-\omega) I_b(\tau') + \frac{\omega}{2} \int_0^{\pi} I(\tau', \theta_i)\, \Phi(\theta, \theta_i) \sin\theta_i \, d\theta_i. \tag{14.13}$$

For *linear-anisotropic scattering*, with

$$\Phi(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i) = 1 + A_1 P_1(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i) = 1 + A_1 \hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_i, \quad M = 1, \tag{14.14}$$

and, using the definitions for incident radiation and radiative heat flux, equations (10.32) and (10.52), respectively, equation (14.13) reduces to

$$S(\tau', \theta) = (1-\omega) I_b(\tau') + \frac{\omega}{4\pi} \left[G(\tau') + A_1 q(\tau') \cos\theta \right]. \tag{14.15}$$

We may now simplify the equation of radiative transfer, equation (14.1), using the geometric relations $\tau_s = \tau / \cos\theta$ and $\tau_s' = \tau' / \cos\theta$ (see Fig. 14-1*a*),

$$\frac{1}{\beta} \frac{dI}{ds} = \frac{dI}{d\tau_s} = \cos\theta \frac{dI}{d\tau} = (1-\omega) I_b - I + \frac{\omega}{2} \int_0^{\pi} I(\tau, \theta_i)\, \Phi(\theta, \theta_i) \sin\theta_i \, d\theta_i. \tag{14.16}$$

Similarly, the expression for intensity, equation (14.2), may be simplified to

$$I^+(\tau, \theta) = I_1(\theta)\, e^{-\tau/\cos\theta} + \int_0^{\tau} S(\tau', \theta)\, e^{-(\tau - \tau')/\cos\theta} \frac{d\tau'}{\cos\theta}, \quad 0 < \theta < \frac{\pi}{2}, \tag{14.17}$$

where the intensity is denoted by I^+ since equation (14.17) is limited to directions with wall intensities emanating from the lower wall, at $\tau = 0$ ("positive" directions). Here the radiative source $S(\tau', \theta)$ is given by equation (14.5) for *isotropic scattering* (or no scattering with $\omega = 0$), by equation (14.15) for *linear-anisotropic scattering*, and by equations (14.12) and (14.13) for *general anisotropic scattering*.

A similar relationship is readily developed for intensity emanating from the top wall (traveling into "negative" directions). With $\tau_s' = -(\tau_L - \tau')/\cos\theta$ and $\tau_s = -(\tau_L - \tau)/\cos\theta$ (keeping in mind that $\cos\theta < 0$ for "negative" directions, $\theta > \pi/2$) we obtain (see Fig. 14-1*b*)

$$\begin{aligned} I^-(\tau, \theta) &= I_2(\theta)\, e^{(\tau_L - \tau)/\cos\theta} + \int_{\tau_L}^{\tau} S(\tau', \theta)\, e^{(\tau' - \tau)/\cos\theta} \frac{d\tau'}{\cos\theta} \\ &= I_2(\theta)\, e^{(\tau_L - \tau)/\cos\theta} - \int_{\tau}^{\tau_L} S(\tau', \theta)\, e^{(\tau' - \tau)/\cos\theta} \frac{d\tau'}{\cos\theta}, \quad \frac{\pi}{2} < \theta < \pi, \end{aligned} \tag{14.18}$$

where $I_2(\theta)$ is the intensity leaving the wall at $\tau = \tau_L$ (Wall 2). It is customary (and somewhat more compact) to rewrite equations (14.16) through (14.18) in terms of the direction cosine $\mu = \cos\theta$, or

$$\mu \frac{dI}{d\tau} + I = (1-\omega) I_b + \frac{\omega}{2} \int_{-1}^{1} I(\tau, \mu_i)\, \Phi(\mu, \mu_i)\, d\mu_i = S(\tau, \mu), \tag{14.19}$$

$$I^+(\tau, \mu) = I_1(\mu)\, e^{-\tau/\mu} + \int_0^{\tau} S(\tau', \mu)\, e^{-(\tau - \tau')/\mu} \frac{d\tau'}{\mu}, \qquad 0 < \mu < 1, \tag{14.20a}$$

$$I^-(\tau, \mu) = I_2(\mu)\, e^{(\tau_L - \tau)/\mu} - \int_{\tau}^{\tau_L} S(\tau', \mu)\, e^{(\tau' - \tau)/\mu} \frac{d\tau'}{\mu}, \qquad -1 < \mu < 0. \tag{14.20b}$$

For heat transfer purposes the incident radiation, G, and radiative heat flux, q, are of interest. From the definition of incident radiation, equation (10.32), it follows that

$$\begin{aligned}
G(\tau) &= \int_0^{2\pi}\int_0^{\pi} I(\tau,\theta)\sin\theta\,d\theta\,d\psi = 2\pi\int_{-1}^{+1} I(\tau,\mu)\,d\mu \\
&= 2\pi\left[\int_{-1}^{0} I^-(\tau,\mu)\,d\mu + \int_0^{+1} I^+(\tau,\mu)\,d\mu\right] \\
&= 2\pi\left[\int_0^1 I^-(\tau,-\mu)\,d\mu + \int_0^1 I^+(\tau,\mu)\,d\mu\right] \\
&= 2\pi\left\{\int_0^1 I_1(\mu)\,e^{-\tau/\mu}\,d\mu + \int_0^1 I_2(-\mu)\,e^{-(\tau_L-\tau)/\mu}\,d\mu\right. \\
&\quad\left. + \int_0^1\left[\int_0^{\tau} S(\tau',\mu)\,e^{-(\tau-\tau')/\mu}\,d\tau' + \int_{\tau}^{\tau_L} S(\tau',-\mu)\,e^{-(\tau'-\tau)/\mu}\,d\tau'\right]\frac{d\mu}{\mu}\right\}.
\end{aligned} \tag{14.21}$$

Similarly, for the radiative heat flux for a plane-parallel medium, equation (10.52),

$$\begin{aligned}
q(\tau) &= \int_0^{2\pi}\int_0^{\pi} I(\tau,\theta)\cos\theta\sin\theta\,d\theta\,d\psi = 2\pi\int_{-1}^{+1} I(\tau,\mu)\mu\,d\mu \\
&= 2\pi\left\{\int_0^1 I_1(\mu)\,e^{-\tau/\mu}\mu\,d\mu - \int_0^1 I_2(-\mu)\,e^{-(\tau_L-\tau)/\mu}\mu\,d\mu\right. \\
&\quad\left. + \int_0^1\left[\int_0^{\tau} S(\tau',\mu)\,e^{-(\tau-\tau')/\mu}\,d\tau' - \int_{\tau}^{\tau_L} S(\tau',-\mu)\,e^{-(\tau'-\tau)/\mu}\,d\tau'\right]d\mu\right\}.
\end{aligned} \tag{14.22}$$

During a large part of this chapter we shall study the solution to equations (14.21) and (14.22) for a number of different situations. We shall assume either that the temperature across the medium and, therefore, $I_b(\tau)$ is known or that radiative equilibrium prevails, $dq/d\tau = 0$. In either case we are interested in the direction-integrated form of the equation of transfer, equation (14.1), which has been given by equation (10.59) as

$$\nabla\cdot\mathbf{q} = \kappa(4\pi I_b - G), \tag{14.23}$$

or, for the present one-dimensional case after division by extinction coefficient β (and remembering that $\kappa/\beta = 1-\sigma_s/\beta$),

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G). \tag{14.24}$$

We note in passing that, up to this point, all relations, and in particular equations (14.21), (14.22), and (14.24), hold on a total basis for a gray medium and on a spectral basis for any medium. If radiative equilibrium prevails, then $dq/d\tau = 0$ or, in the presence of a heat source,[3]

$$\frac{dq}{d\tau} = \frac{\dot{Q}'''}{\beta}(\tau), \tag{14.25}$$

where $\dot{Q}'''$ is local heat generation per unit time and volume. Equation (14.25) is valid only for total radiative heat flux and may, therefore, in this form be applied only to gray media. For such a case we see that the incident radiation is closely related to the blackbody intensity (and, therefore, temperature) by

$$4\pi I_b(\tau) = G(\tau) + \frac{\dot{Q}'''}{\kappa}(\tau). \tag{14.26}$$

[3]Such heat sources are often used to couple the radiation problem with overall energy conservation.

14.3 PLANE LAYER OF A NONSCATTERING MEDIUM

Enclosure with Black Bounding Surfaces

Since this is the most basic of cases, we shall rederive the relationships for this simple problem. From equation (14.3), with $\omega = 0$, it follows that $S(\tau', \hat{\mathbf{s}}) = I_b(\tau')$; for black bounding surfaces, the intensity leaving the lower plate is $I_1(\theta) = I_{b1}$ and the intensity leaving the top plate is $I_2(\theta) = I_{b2}$. Thus, for this simple case, neither radiative source nor boundary intensities are direction-dependent. Equations (14.17) and (14.18) may then be rewritten as

$$I^+(\tau, \theta) = I_{b1}\, e^{-\tau/\cos\theta} + \int_0^{\tau} I_b(\tau')\, e^{-(\tau-\tau')/\cos\theta}\, \frac{d\tau'}{\cos\theta}, \qquad 0 < \theta < \frac{\pi}{2}, \tag{14.27a}$$

$$I^-(\tau, \theta) = I_{b2}\, e^{(\tau_L-\tau)/\cos\theta} - \int_{\tau}^{\tau_L} I_b(\tau')\, e^{(\tau'-\tau)/\cos\theta}\, \frac{d\tau'}{\cos\theta}, \qquad \frac{\pi}{2} < \theta < \pi. \tag{14.27b}$$

Making the substitution $\mu = \cos\theta$ transforms this to

$$I^+(\tau, \mu) = I_{b1}\, e^{-\tau/\mu} + \frac{1}{\mu}\int_0^{\tau} I_b(\tau')\, e^{-(\tau-\tau')/\mu}\, d\tau', \qquad 0 < \mu < 1, \tag{14.28a}$$

$$I^-(\tau, \mu) = I_{b2}\, e^{(\tau_L-\tau)/\mu} - \frac{1}{\mu}\int_{\tau}^{\tau_L} I_b(\tau')\, e^{(\tau'-\tau)/\mu}\, d\tau', \qquad -1 < \mu < 0. \tag{14.28b}$$

From the definition for incident radiation it follows, from equation (14.21), that

$$\begin{aligned} G(\tau) &= 2\pi \left[\int_{-1}^{0} I^-(\tau, \mu)\, d\mu + \int_0^1 I^+(\tau, \mu)\, d\mu \right] \\ &= 2\pi \Bigg[I_{b1} \int_0^1 e^{-\tau/\mu}\, d\mu + I_{b2} \int_0^1 e^{-(\tau_L-\tau)/\mu}\, d\mu \\ &\quad + \int_0^{\tau} I_b(\tau') \int_0^1 e^{-(\tau-\tau')/\mu}\, \frac{d\mu}{\mu}\, d\tau' + \int_{\tau}^{\tau_L} I_b(\tau') \int_0^1 e^{-(\tau'-\tau)/\mu} \frac{d\mu}{\mu}\, d\tau' \Bigg]. \end{aligned} \tag{14.29}$$

Taking advantage of the fact that wall intensities and radiative sources do not depend on direction, we have taken these terms out of the direction integrals and reversed the order of integration for the terms describing medium emission.

A similar relationship may be established for radiative heat flux, from equation (14.22), as

$$\begin{aligned} q(\tau) = 2\pi \Bigg[& I_{b1} \int_0^1 e^{-\tau/\mu} \mu\, d\mu - I_{b2} \int_0^1 e^{-(\tau_L-\tau)/\mu} \mu\, d\mu \\ & + \int_0^{\tau} I_b(\tau') \int_0^1 e^{-(\tau-\tau')/\mu}\, d\mu\, d\tau' - \int_{\tau}^{\tau_L} I_b(\tau') \int_0^1 e^{-(\tau'-\tau)/\mu}\, d\mu\, d\tau' \Bigg]. \end{aligned} \tag{14.30}$$

We see that none of the important parameters G, q, and I_b depends on direction, and that direction μ enters equations (14.29) and (14.30) only as a dummy integration variable. We may write these equations in more compact form by introducing the *exponential integral of order n*,

$$E_n(x) = \int_1^{\infty} e^{-xt} \frac{dt}{t^n} = \int_0^1 \mu^{n-2}\, e^{-x/\mu}\, d\mu. \tag{14.31}$$

Since exponential integrals are of great importance in radiative transfer, a sketch of them is shown in Fig. 14-2, and a somewhat more detailed discussion is given in Appendix E. For

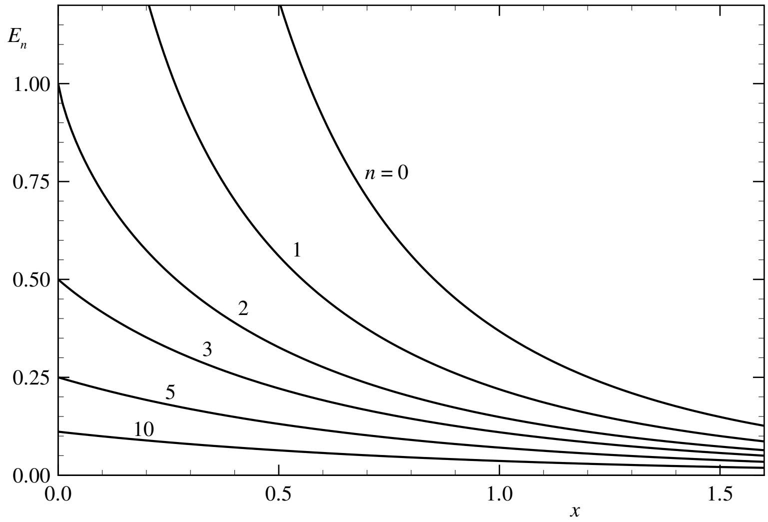

FIGURE 14-2
General behavior of exponential integrals $E_n(x)$.

our present purposes, we note that exponential integrals behave somewhat like "generalized negative exponentials" and that

$$E_n(0) = \int_1^\infty \frac{dt}{t^n} = \frac{1}{n-1}, \tag{14.32}$$

$$\frac{d}{dx}E_n(x) = -E_{n-1}(x); \quad \text{or} \quad E_n(x) = \int_x^\infty E_{n-1}(x)\,dx. \tag{14.33}$$

Substituting equation (14.31) into equations (14.29) and (14.30) then leads to

$$G(\tau) = 2\pi\left[I_{b1}E_2(\tau) + I_{b2}E_2(\tau_L - \tau) + \int_0^\tau I_b(\tau')E_1(\tau - \tau')\,d\tau' + \int_\tau^{\tau_L} I_b(\tau')E_1(\tau' - \tau)\,d\tau'\right], \tag{14.34}$$

$$q(\tau) = 2\pi\left[I_{b1}E_3(\tau) - I_{b2}E_3(\tau_L - \tau) + \int_0^\tau I_b(\tau')E_2(\tau - \tau')\,d\tau' - \int_\tau^{\tau_L} I_b(\tau')E_2(\tau' - \tau)\,d\tau'\right]. \tag{14.35}$$

Medium with Specified Temperature Field

If radiative heat transfer is not so dominant that conduction and/or convection can be neglected, the problem of finding the temperature distribution and heat fluxes is always nonlinear. For the simplest case of a gray medium with constant properties, the incident radiation, as calculated from equation (14.21), is proportional to temperature to the fourth power, while the conductive and/or convective terms are proportional to temperature itself. Therefore, the temperature field must always be determined through an iterative procedure. In general, this involves guessing a temperature field, which is then used to determine incident radiation G [from equation (14.34)] and divergence of radiative heat flux $\nabla \cdot \mathbf{q}$ [from equation (14.24)]. This radiative source term is then substituted into the equation for overall conservation of energy, equation (10.72), from which an improved temperature field is determined. This process is then repeated until the temperature field has converged to within specified criteria. The treatment of combined radiation together with conduction and/or convection is discussed in more detail in Chapter 22. There are also some important industrial applications where outright knowledge of the temperature field may be assumed, for example, swirling combustion chambers that are essentially isothermal as a result of very strong convection.

For a gray medium equations (14.34) and (14.35) give the total incident radiation G and radiative flux q. The radiative source term then follows from

$$\frac{dq}{d\tau} = (1-\omega)(4\pi I_b - G). \tag{14.36}$$

For a nongray medium equations (14.34) and (14.35) provide only spectral values, G_λ and q_λ, and the total radiative source must be found through integration of

$$\frac{dq}{dz} = \int_0^\infty \kappa_\lambda(4\pi I_{b\lambda} - G_\lambda)\,d\lambda. \tag{14.37}$$

Medium at Radiative Equilibrium

If other modes of heat transfer are negligible, or are not considered, the temperature distribution is unknown and must be determined from the statement of radiative equilibrium, $dq/dz = 0$. This is a total (spectrally integrated) heat flux and, to make the problem tractable, we will limit ourselves to a gray medium [i.e., equations (14.34) and (14.35) deal with total properties, or $I_b = n^2\sigma T^4/\pi$]. We find $q(\tau) = \text{const}$ and, from equation (14.26), $G = 4\pi I_b = 4n^2\sigma T^4$. Equation (14.34) now becomes an integral equation governing the temperature distribution within the medium, or

$$T^4(\tau) = \frac{1}{2}\left[T_1^4 E_2(\tau) + T_2^4 E_2(\tau_L-\tau) + \int_0^{\tau_L} T^4(\tau')E_1(|\tau'-\tau|)\,d\tau'\right]. \tag{14.38}$$

Since the heat flux,

$$\begin{aligned} q(\tau) &= 2n^2\sigma T_1^4 E_3(\tau) - 2n^2\sigma T_2^4 E_3(\tau_L-\tau) \\ &\quad + 2\int_0^{\tau} n^2\sigma T^4(\tau')E_2(\tau-\tau')\,d\tau' - 2\int_\tau^{\tau_L} n^2\sigma T^4(\tau')E_2(\tau'-\tau)\,d\tau', \end{aligned} \tag{14.39}$$

does not vary across the medium, it may be evaluated at any location, conveniently chosen as $\tau = 0$:

$$q = n^2\sigma T_1^4 - 2n^2\sigma T_2^4 E_3(\tau_L) - 2\int_0^{\tau_L} n^2\sigma T^4(\tau')E_2(\tau')\,d\tau'. \tag{14.40}$$

The difference between equations (14.40) and (14.39) is that equation (14.40) is only valid for radiative equilibrium, and equation (14.39) is valid for the more general case of any gray medium between black plates. For an overall solution the temperature field is found first by solving the integral equation (14.38), after which knowledge of the temperature field is used to determine radiative heat flux from equation (14.39). Unfortunately, no closed-form solution exists to integral equations such as (14.38); a solution has to be found by numerical and/or approximate means. Before proceeding to a solution it is advantageous to reduce the number of parameters in equations (14.38) and (14.39) to a minimum. We introduce a nondimensional emissive power or temperature

$$\Phi_b(\tau) = \frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4}, \tag{14.41}$$

and a nondimensional radiative heat flux

$$\Psi_b = \frac{q}{n^2\sigma(T_1^4 - T_2^4)}. \tag{14.42}$$

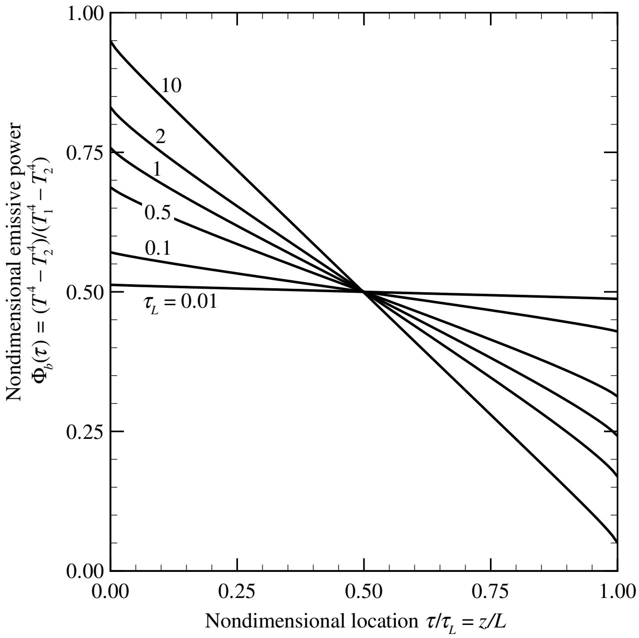

FIGURE 14-3
Nondimensional temperature distribution for a gray medium at radiative equilibrium between isothermal plates.

If we substitute these expressions into equations (14.38) and (14.39) and use equations (14.32) and (14.33), we find that

$$\Phi_b(\tau) = \frac{1}{2}\left[E_2(\tau) + \int_0^{\tau_L} \Phi_b(\tau')E_1(|\tau - \tau'|)\,d\tau'\right], \tag{14.43}$$

$$\Psi_b(\tau) = 2\left[E_3(\tau) + \int_0^{\tau} \Phi_b(\tau')E_2(\tau-\tau')\,d\tau' - \int_\tau^{\tau_L} \Phi_b(\tau')E_2(\tau'-\tau)\,d\tau'\right], \tag{14.44}$$

or

$$\Psi_b = 1 - 2\int_0^{\tau_L} \Phi_b\,(\tau')E_2(\tau')\,d\tau'. \tag{14.45}$$

Besides the independent variable τ, only one parameter, the medium's optical thickness τ_L, appears in the governing equations for Φ_b and Ψ_b: Once Φ_b has been determined for a given τ_L, the temperature field and radiative heat flux may be determined for any combination of surface temperatures. Equation (14.43) is a *Fredholm integral equation* and is readily solved by any of the methods described in Section 5.6. The numerical solution to equations (14.43) and (14.45) was first given by Heaslet and Warming [2]. Figure 14-3 shows the nondimensional temperature field for a range of optical thicknesses. Some representative nondimensional fluxes are given in Table 14.1.

Examination of Fig. 14-3 shows that, for radiative equilibrium, there may be a temperature discontinuity at the walls.[4] In the limiting case of a transparent medium, $\tau_L \to 0$, we have $\Phi_b = \frac{1}{2}$ or $T^4 \to (T_1^4 + T_2^4)/2 = \text{const}$, with corresponding temperature jumps at the boundaries (strictly speaking, a transparent or nonparticipating medium, $\tau_L = 0$, could have any temperature distribution since it would not enter the calculations). The temperature slip decreases as the optical thickness increases until it vanishes for $\tau_L \to \infty$. In that optically thick limit the nondimensional emissive power profile becomes linear. The situation is not unlike conduction in a rarefied gas: When the mean free path for collision (absorption) is very large, molecules (photons) travel between plates without interference with an average energy equal to the average of surface

[4]This discontinuity must, of course, vanish if heat is transferred by conduction and/or convection in addition to radiation.

TABLE 14.1
Nondimensional radiative heat flux for radiative equilibrium between parallel black plates, $\Psi_b = q/n^2\sigma(T_1^4 - T_2^4)$, from Heaslet and Warming [2].

Optical thickness, τ_L	Ψ_b	Optical thickness, τ_L	Ψ_b
0.0	1.0000	0.8	0.6046
0.1	0.9157	1.0	0.5532
0.2	0.8491	1.5	0.4572
0.3	0.7934	2.0	0.3900
0.4	0.7458	2.5	0.3401
0.5	0.7040	3.0	0.3016
0.6	0.6672	5.0	0.2077

For $\tau_L \gg 1$, $\quad \Psi_b = \dfrac{4/3}{1.42089 + \tau_L}$

temperatures (emissive powers). If the mean free path becomes very small compared with physical dimensions, the conductive flux obeys Fourier's law and the diffusion limit is reached.

Gray, Diffuse Boundaries

If the walls are not black, but are gray, diffuse emitters and reflectors, the entire development of this section still holds, except that the fluxes leaving the bottom and top plates are no longer πI_{b1} and πI_{b2}, but must be replaced by the radiosities J_1 and J_2, respectively:

$$G(\tau) = 2J_1E_2(\tau) + 2J_2E_2(\tau_L - \tau) + 2\pi \int_0^\tau I_b(\tau')E_1(\tau - \tau')\,d\tau' + 2\pi \int_\tau^{\tau_L} I_b(\tau')E_1(\tau' - \tau)\,d\tau', \tag{14.46}$$

$$q(\tau) = 2J_1E_3(\tau) - 2J_2E_3(\tau_L - \tau) + 2\pi \int_0^\tau I_b(\tau')E_2(\tau - \tau')\,d\tau' - 2\pi \int_\tau^{\tau_L} I_b(\tau')E_2(\tau' - \tau)\,d\tau'. \tag{14.47}$$

The radiosities, accounting for emission as well as diffuse reflection, may be related to the Planck function through equation (5.26) as

$$\mathbf{q}_w \cdot \hat{\mathbf{n}} = \frac{\epsilon_w}{1 - \epsilon_w}\left(\pi I_{bw} - J_w\right), \tag{14.48}$$

or

$$\tau = 0: \qquad q_1 = \frac{\epsilon_1}{1 - \epsilon_1}\left(n^2\sigma T_1^4 - J_1\right), \tag{14.49a}$$

$$\tau = \tau_L: \qquad q_2 = -\frac{\epsilon_2}{1 - \epsilon_2}\left(n^2\sigma T_2^4 - J_2\right). \tag{14.49b}$$

Medium with Specified Temperature Field

For nonblack surfaces the equations for incident radiation and radiative flux are coupled through the radiosity. First, sticking equations (14.49) into equation (14.47) one can determine the unknown radiosities, after which incident radiation is calculated. The radiative source term is then evaluated as for black surfaces.

Example 14.1. A gray, nonscattering medium with refractive index $n = 1$ is contained between two parallel, gray plates. The medium is isothermal at temperature T_m, with constant absorption coefficient κ. The two plates are both isothermal at temperature T_w, have the same gray-diffuse emittance ϵ, and are spaced a distance L apart. Determine the radiative heat flux between the plates as well as its divergence.

Solution
The radiative heat flux is determined from equation (14.47) with $\tau = \kappa z$,

$$\begin{aligned} q(\tau) &= 2J_w E_3(\tau) - 2J_w E_3(\tau_L - \tau) + 2\sigma T_m^4 \int_0^\tau E_2(\tau - \tau')\,d\tau' - 2\sigma T_m^4 \int_\tau^{\tau_L} E_2(\tau' - \tau)\,d\tau' \\ &= 2J_w E_3(\tau) - 2J_w E_3(\tau_L - \tau) + 2\sigma T_m^4 \left[E_3(\tau - \tau')\Big|_0^\tau + E_3(\tau' - \tau)\Big|_\tau^{\tau_L} \right] \\ &= (J_w - \sigma T_m^4)\, 2\,[E_3(\tau) - E_3(\tau_L - \tau)], \end{aligned}$$

where we have made use of the symmetry of the problem, i.e., $J_1 = J_2 = J_w$, and $\pi I_{bm} = \sigma T_m^4 =$ const. The necessary relationship between surface flux, temperature, and radiosity has been given by equation (14.48), or for the plane slab at $\tau = 0$,

$$q(0) = \frac{\epsilon}{1-\epsilon}(\sigma T_w^4 - J_w) = (J_w - \sigma T_m^4)\,[1 - 2E_3(\tau_L)].$$

Solving for J_w, we find

$$J_w = \frac{\sigma T_w^4 + (1/\epsilon - 1)\,[1 - 2E_3(\tau_L)]\,\sigma T_m^4}{1 + (1/\epsilon - 1)[1 - 2E_3(\tau_L)]},$$

and

$$\frac{q(\tau)}{\sigma(T_w^4 - T_m^4)} = \frac{2\,[E_3(\tau) - E_3(\tau_L - \tau)]}{1 + (1/\epsilon - 1)[1 - 2E_3(\tau_L)]}.$$

The divergence of the flux may be evaluated by first calculating the incident radiation from equation (14.53) and then using equation (14.36). While this method is preferable for numerical and/or multidimensional calculations, it is more convenient here simply to differentiate the above expression for the heat flux. Thus,

$$\frac{dq}{d\tau}(\tau)\Big/\sigma(T_w^4 - T_m^4) = -\frac{2[E_2(\tau) + E_2(\tau_L - \tau)]}{1 + (1/\epsilon - 1)[1 - 2E_3(\tau_L)]}.$$

If $T_w > T_m$, then $dq/d\tau$ is always negative: The flux is positive at $\tau = 0$ (going into the medium), zero at the midplane, and turning more and more negative as the $\tau = \tau_L$ plate is approached.

Medium at Radiative Equilibrium

We will again limit our discussion to a gray medium. Replacing $\pi I_{bi} = n^2 \sigma T_i^4$ by J_i in the nondimensionalized equations for radiative equilibrium, equations (14.41) and (14.42), and setting $q_1 = q_2 = q =$ const transforms equation (14.45) into

$$\frac{q}{J_1 - J_2} = \Psi_b = 1 - 2\int_0^{\tau_L} \Phi_b(\tau') E_2(\tau')\,d\tau',$$

and from equation (14.49)

$$J_1 - J_2 = n^2\sigma(T_1^4 - T_2^4) - \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 2\right) q.$$

Thus,

$$q = \Psi_b(J_1 - J_2) = \Psi_b \left[n^2\sigma(T_1^4 - T_2^4) - \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 2\right) q \right],$$

or

$$\Psi = \frac{q}{n^2\sigma(T_1^4 - T_2^4)} = \frac{\Psi_b}{1 + \Psi_b\left(\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 2\right)}. \tag{14.50}$$

Similarly, for the nondimensional temperature distribution one obtains

$$\Phi(\tau) = \frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4} = \frac{\Phi_b(\tau) + \left(\dfrac{1}{\epsilon_2} - 1\right)\Psi_b}{1 + \Psi_b\left(\dfrac{1}{\epsilon_1} + \dfrac{1}{\epsilon_2} - 2\right)}. \tag{14.51}$$

Example 14.2. A gray, nonscattering medium with refractive index $n = 1$ and an absorption coefficient $\kappa = 0.1\,\text{cm}^{-1}$ is contained between two isothermal cylinders. The inner cylinder is hot ($T_1 = 2000\,\text{K}$) and highly reflective ($\epsilon_1 = 0.1$); the outer cylinder is a strong absorber ($\alpha_2 = \epsilon_2 = 0.9$), and it must be kept relatively cool ($T_2 \leq 400\,\text{K}$). The gap between the two cylinders is 25 cm. Assuming that conductive and convective heat transfer can be neglected as compared to radiation, and assuming that the cylinders have large diameters ($D_1 \gg 25\,\text{cm}$), determine the necessary cooling rate for the outer cylinder to avoid overheating.

Solution
Since the thickness of the medium is small as compared with the diameters of the cylinders, we may model the gap as a one-dimensional plane-parallel slab of optical thickness $\tau_L = 0.1\,\text{cm}^{-1} \times 25\,\text{cm} = 2.5$. Thus, from Table 14.1 $\Psi_b = 0.3401$ and from equation (14.50)

$$\Psi = \frac{q}{\sigma(T_1^4 - T_2^4)} = \frac{0.3401}{1 + 0.3401\left(\frac{1}{0.1} + \frac{1}{0.9} - 2\right)} = 0.0830,$$

and

$$\begin{aligned} q_{\text{min}} &= \Psi\sigma\left(T_1^4 - T_{2,\text{max}}^4\right) \\ &= 0.0830 \times 5.670 \times 10^{-12}(2000^4 - 400^4)\,\text{W/cm}^2 = 7.52\,\text{W/cm}^2. \end{aligned}$$

14.4 PLANE LAYER OF A SCATTERING MEDIUM

Isotropic Scattering

For isotropic scattering the source function is found from equation (14.5) as

$$S(\tau) = (1 - \omega)I_b(\tau) + \frac{\omega}{4\pi}G(\tau), \tag{14.52}$$

and the $I_b(\tau')$ in equations (14.46) and (14.47) must be replaced by $S(\tau')$:

$$G(\tau) = 2J_1E_2(\tau) + 2J_2E_2(\tau_L - \tau) + 2\pi\int_0^\tau S(\tau')E_1(\tau - \tau')\,d\tau' + 2\pi\int_\tau^{\tau_L} S(\tau')E_1(\tau' - \tau)\,d\tau', \tag{14.53}$$

$$q(\tau) = 2J_1E_3(\tau) - 2J_2E_3(\tau_L - \tau) + 2\pi\int_0^\tau S(\tau')E_2(\tau - \tau')\,d\tau' - 2\pi\int_\tau^{\tau_L} S(\tau')E_2(\tau' - \tau)\,d\tau'. \tag{14.54}$$

Medium with Specified Temperature Field

In the presence of isotropic scattering equation (14.53) becomes an integral equation for the unknown $G(\tau)$. Defining a general nondimensional function similar to equation (14.41),

$$\Phi(\tau) = \frac{\pi S(\tau) - J_2}{J_1 - J_2}, \tag{14.55}$$

equation (14.53) may be simplified to

$$\Phi(\tau) = (1-\omega)\frac{\pi I_b(\tau) - J_2}{J_1 - J_2} + \frac{\omega}{2}\left[E_2(\tau) + \int_0^{\tau_L} \Phi(\tau) E_1(|\tau' - \tau|)\, d\tau'\right]. \tag{14.56}$$

Equation (14.56) reduces to equation (14.43) for the case of $\omega \to 1$ (purely scattering medium). For such a medium thermal radiation is decoupled from the temperature field (since there is no emission), and radiative equilibrium prevails regardless of the temperature distribution. This behavior is also seen from equation (14.24), which states that $dq/d\tau = 0$ if $\omega = 1$, regardless of $I_b(\tau)$.

Similarly, equation (14.54) may be nondimensionalized as

$$\Psi(\tau) = \frac{q(\tau)}{J_1 - J_2} = 2\left[E_3(\tau) + \int_0^{\tau} \Phi(\tau') E_2(\tau - \tau')\, d\tau' - \int_{\tau}^{\tau_L} \Phi(\tau') E_2(\tau' - \tau)\, d\tau'\right]. \tag{14.57}$$

Once a solution for $\Phi(\tau)$ has been obtained, the radiative flux is determined from equation (14.57), and incident radiation and radiative source term are found through equation (14.55).

Medium at Radiative Equilibrium

In the case of radiative equilibrium in a gray medium (and assuming no internal heat generation takes place) equation (14.26) further simplifies the source function to

$$S(\tau) = (1-\omega) I_b(\tau) + \frac{\omega}{4\pi} G(\tau) = I_b(\tau). \tag{14.58}$$

Thus all relations developed in the previous section on radiative equilibrium are equally valid for isotropically scattering media with optical thickness $\tau = \int_0^z \beta\, dz$ based on the extinction coefficient rather than the absorption coefficient. For a gray medium at radiative equilibrium there is no distinction between absorption and isotropic scattering: Any energy absorbed at τ must be reemitted isotropically at the same location, although at different wavelengths; any isotropically scattered energy is simply redirected isotropically (without change of wavelength). Since a gray medium is "colorblind" it cannot distinguish between emission and isotropic scattering. However, for the purely scattering case, $\omega \to 1$, there is no emission and, therefore, I_b no longer enters the calculations. For this extreme case the $T^4(\tau)$ in equations (14.41) and (14.51) should be replaced by $G(\tau)/4n^2\sigma$.

Anisotropic Scattering

For demonstrative purposes we will only consider the case of a gray slab at *radiative equilibrium* with *linear-anisotropic scattering*. The source function is then given by equation (14.15), which reduces to

$$S(\tau, \mu) = I_b(\tau) + \frac{A_1 \omega}{4\pi} q \mu. \tag{14.59}$$

Therefore, equations (14.21) and (14.22) become

$$\begin{aligned}\frac{G(\tau)}{4\pi} &= I_b(\tau) = \frac{J_1}{2\pi} E_2(\tau) + \frac{J_2}{2\pi} E_2(\tau_L - \tau) + \frac{1}{2}\int_0^{\tau} I_b(\tau') E_1(\tau - \tau')\, d\tau' \\ &\quad + \frac{1}{2}\int_{\tau}^{\tau_L} I_b(\tau') E_1(\tau' - \tau)\, d\tau' + \frac{A_1\omega}{8\pi} q \left[E_3(\tau_L - \tau) - E_3(\tau)\right],\end{aligned} \tag{14.60}$$

$$\begin{aligned}q(\tau) &= q = 2J_1 E_3(\tau) - 2J_2 E_3(\tau_L - \tau) + 2\pi\left[\int_0^{\tau} I_b(\tau') E_2(\tau - \tau')\, d\tau'\right. \\ &\quad \left. - \int_{\tau}^{\tau_L} I_b(\tau') E_2(\tau' - \tau)\, d\tau'\right] + \frac{A_1\omega}{2} q \left[\frac{2}{3} - E_4(\tau) - E_4(\tau_L - \tau)\right].\end{aligned} \tag{14.61}$$

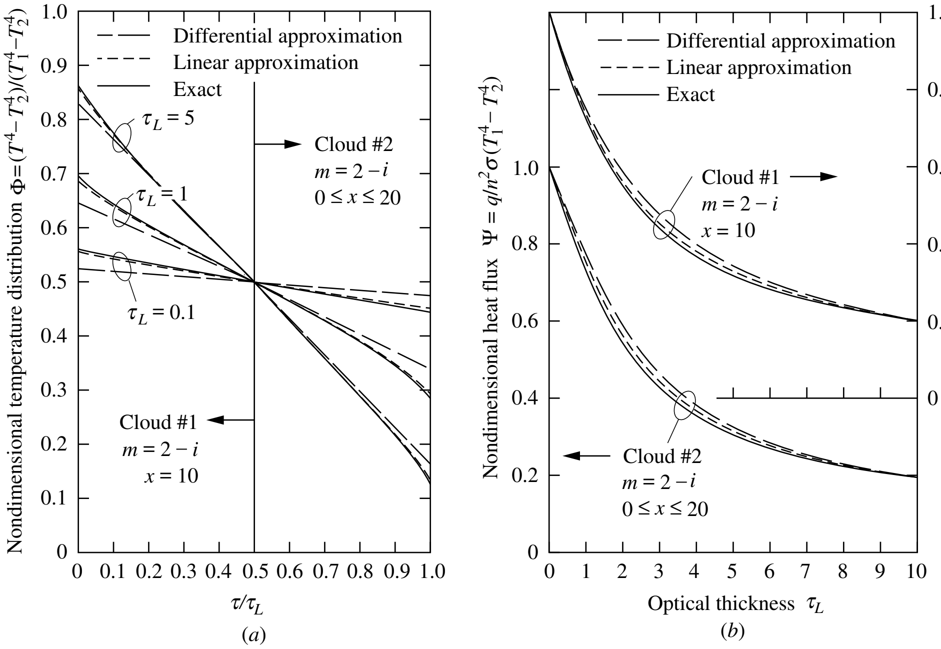

FIGURE 14-4
(*a*) Nondimensional temperature profiles, and (*b*) nondimensional heat flux rates; for a slab at radiative equilibrium.

In nondimensional form these relations reduce to

$$\Phi_b(\tau) = \frac{\pi I_b(\tau) - J_2}{J_1 - J_2} = \frac{1}{2}\left\{E_2(\tau) + \int_0^{\tau_L} \Phi_b(\tau')E_1(|\tau - \tau'|)\,d\tau' + \frac{A_1\omega}{4}\Psi_b\left[E_3(\tau_L - \tau) - E_3(\tau)\right]\right\}, \quad (14.62)$$

$$\Psi_b = \frac{q}{J_1 - J_2} = 2\left\{E_3(\tau) + \int_0^{\tau} \Phi_b(\tau')E_2(\tau - \tau')\,d\tau' - \int_{\tau}^{\tau_L} \Phi_b(\tau')E_2(\tau' - \tau)\,d\tau' + \frac{A_1\omega}{4}\Psi_b\left[\frac{2}{3} - E_4(\tau) - E_4(\tau_L - \tau)\right]\right\}. \quad (14.63)$$

The problem of radiative equilibrium in a one-dimensional, plane-parallel, anisotropically scattering medium has been solved by Modest and Azad [3]. They considered full Mie-anisotropic scattering for a number of particulate clouds, whose relevant parameters have been given in Chapter 12, Table 12.1. Figure 14-4 shows representative results for radiative heat fluxes and temperature distributions in Clouds 1 and 2. Also included are results for linear-anisotropic scattering (approximating the phase functions as indicated in Fig. 12-6), using the exact relations, equations (14.62) and (14.63), as well as the differential approximation (to be discussed in the following two chapters). It was observed that approximating a complicated phase function by a linear-anisotropic one (after removing forward- and backward-scattering peaks) always leads to accurate results for heat transfer applications.

14.5 RADIATIVE TRANSFER IN SPHERICAL MEDIA

In a plane-parallel medium, if the temperature field (as well as any radiative property) varies only in the direction normal to the plates, the problem is one-dimensional. If the polar angle

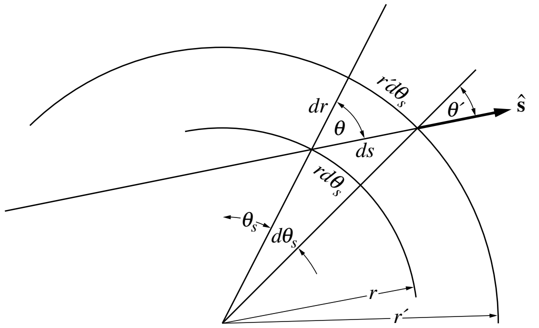

FIGURE 14-5
Coordinates for a one-dimensional spherical medium.

θ is measured from the direction normal to the plates, the radiative intensity depends only on the spatial coordinate z and polar angle θ (but not on azimuthal angle ψ, because of symmetry). A similar situation exists in a one-dimensional spherical medium. Let the temperature field vary only in the radial direction r, but not with polar angle θ_s or azimuthal angle ψ_s (where the subscripts s have been added to emphasize that these angles specify *position* in a spherical coordinate system, and are independent of angles θ and ψ, which are employed to describe *direction*). If the polar direction angle θ is measured from the radial position vector as shown in Fig. 14-5, then the radiative intensity depends only on polar angle θ and—owing to symmetry—not on azimuthal angle ψ. However, unlike in the plane layer of Fig. 14-1, in spherical symmetry the polar angle changes as a beam travels in the direction of $\hat{\mathbf{s}}$ through the medium (with θ steadily decreasing with increasing path length s). Therefore, with intensity depending on radial location r and direction angle θ, the left-hand side of equation (14.1) must be expressed[5] as

$$\hat{\mathbf{s}}\cdot\nabla I = \frac{dI}{ds} = \frac{\partial I}{\partial r}\frac{dr}{ds} + \frac{\partial I}{\partial \theta}\frac{d\theta}{ds}. \tag{14.64}$$

From inspection of Fig. 14-5, we find that $\cos\theta = dr/ds$. Also, from the law of sines,

$$\frac{\sin\theta}{r'\,d\theta_s} = \frac{\sin\theta'}{r\,d\theta_s} \quad \text{or} \quad r\sin\theta = r'\sin\theta' = \text{const} \tag{14.65}$$

along s. Differentiating this relation gives

$$dr\sin\theta + r\cos\theta\,d\theta = 0, \quad \text{or} \quad \frac{d\theta}{dr} = -\frac{\tan\theta}{r}. \tag{14.66}$$

Substituting both relations into equation (14.64) leads to

$$\frac{dI}{ds} = \cos\theta\frac{\partial I}{\partial r}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial I}{\partial\theta}(r,\theta), \tag{14.67}$$

or, if the shorthand $\mu = \cos\theta$ is preferred,

$$\frac{dI}{ds} = \mu\frac{\partial I}{\partial r}(r,\mu) + \frac{1-\mu^2}{r}\frac{\partial I}{\partial\mu}(r,\mu), \tag{14.68}$$

where we have used $d\mu = -\sin\theta\,d\theta$. Substituting equation (14.68) into equations (14.1) and (14.3), we obtain, with $\tau = \int_0^r \beta\,dr$,

$$\mu\frac{\partial I}{\partial\tau}(\tau,\mu) + \frac{1-\mu^2}{\tau}\frac{\partial I}{\partial\mu}(\tau,\mu) = S(\tau,\mu) - I(\tau,\mu), \tag{14.69}$$

[5]While this expression is also valid for a one-dimensional plane layer (with r replaced by z), the polar angle does not change along a path through the slab, or $d\theta/ds = 0$.

where the radiative source for linear-anisotropic scattering has been given by equation (14.15).

The problem of one-dimensional heat transfer through a spherical medium was first considered by Sparrow and coworkers [4], who investigated radiative equilibrium in a gray nonscattering medium contained between concentric black spheres. They assumed that there was uniform heat generation within the medium and that both surfaces had identical and constant temperatures. Ryhming [5] considered the same problem, but without heat generation and with the two surfaces at different temperatures T_1 and T_2. The condition of black walls was relaxed by Viskanta and Crosbie [6], who considered a nonscattering, gray, heat-generating medium between two gray, isothermal spheres of radius R_1 and R_2, respectively (and at temperatures T_1 and T_2, and with gray diffuse emittances ϵ_1 and ϵ_2). They found that the temperature field, in terms of emissive power E_b, may be calculated from

$$E_b(\tau) = J_1 + (J_2 - J_1)\,\Phi(\tau) + \frac{\dot{Q}'''}{\kappa}\,\Phi_s(\tau). \tag{14.70}$$

Here J_1 and J_2 are the radiosities of the two spherical surfaces, and $\dot{Q}'''$ is the (uniform) heat generation within the medium. $\Phi(\tau)$ is the nondimensional emissive power for a medium without heat generation, determined from

$$\Phi(\tau) = \frac{E_b(\tau)-J_1}{J_2-J_1} = \frac{1}{2\tau}\left[g(\tau) + \int_{\tau_1}^{\tau_2} K(\tau,t)\,\Phi(t)\,dt\right], \tag{14.71}$$

$$\begin{aligned} g(\tau) = {} & \tau_2 E_2(\tau_2-\tau) - \sqrt{\tau_2^2-\tau_1^2}\,E_2\left(\sqrt{\tau_2^2-\tau_1^2} + \sqrt{\tau^2-\tau_1^2}\right) \\ & + E_3(\tau_2-\tau) - E_3\left(\sqrt{\tau_2^2-\tau_1^2} + \sqrt{\tau^2-\tau_1^2}\right), \end{aligned} \tag{14.72}$$

$$K(\tau,t) = \left[E_1(|\tau-t|) - E_1\left(\sqrt{\tau_2^2-\tau_1^2} + \sqrt{\tau^2-\tau_1^2}\right)\right] t. \tag{14.73}$$

$\Phi_s(\tau)$ is the nondimensional emissive power for a medium with uniform heat generation, but with both surfaces having the same radiosity, J_1, obtained from

$$\Phi_s(\tau) = \frac{E_b-J_1}{\dot{Q}'''/\kappa} = \frac{1}{4} + \frac{1}{2\tau}\int_{\tau_1}^{\tau_2} K(\tau,t)\,\Phi_s(t)\,dt. \tag{14.74}$$

Once the functions $\Phi(\tau)$ and/or $\Phi_s(\tau)$ have been determined, the radiative heat fluxes can be calculated from

$$\tau^2 q(\tau) = (J_1 - J_2)\tau_1^2\,\Psi(\tau) + \frac{\dot{Q}'''}{\kappa}\left[\frac{\tau^3}{3} - \tau_1^2\,\Psi_s(\tau)\right], \tag{14.75}$$

where

$$\Psi(\tau) = \frac{2}{\tau_1^2}\left[h(\tau) + \int_{\tau_1}^{\tau_2} H(\tau,t)\,\Phi(t)\,dt\right], \tag{14.76}$$

$$\Psi_s(\tau) = \frac{1}{\tau_1^2}\left[\frac{\tau^3}{3} - 2\int_{\tau_1}^{\tau_2} H(\tau,t)\,\Phi_s(t)\,dt\right], \tag{14.77}$$

$$\begin{aligned} h(\tau) = {} & -\tau\tau_2 E_3(\tau_2-\tau) - \sqrt{(\tau_2^2-\tau_1^2)(\tau^2-\tau_1^2)}\,E_3\left(\sqrt{\tau_2^2-\tau_1^2} + \sqrt{\tau^2-\tau_1^2}\right) \\ & + (\tau_2-\tau)E_4(\tau_2-\tau) - \left(\sqrt{\tau_2^2-\tau_1^2} + \sqrt{\tau^2-\tau_1^2}\right)E_4\left(\sqrt{\tau_2^2-\tau_1^2} + \sqrt{\tau^2-\tau_1^2}\right) \\ & + E_5(\tau_2-\tau) - E_5\left(\sqrt{\tau_2^2-\tau_1^2} + \sqrt{\tau^2-\tau_1^2}\right), \end{aligned} \tag{14.78}$$

$$\begin{aligned} H(\tau,t) = {} & \Big[\tau\,\mathrm{sgn}(\tau-t)E_2(|\tau-t|) - \sqrt{\tau^2-\tau_1^2}\,E_2\left(\sqrt{\tau^2-\tau_1^2} + \sqrt{t^2-\tau_1^2}\right) \\ & + E_3(|\tau-t|) - E_3\left(\sqrt{\tau^2-\tau_1^2} + \sqrt{t^2-\tau_1^2}\right)\Big]\,t, \end{aligned} \tag{14.79}$$

TABLE 14.2
Values of nondimensional flux functions for radiative equilibrium between concentric spheres, from Viskanta and Crosbie [6].

	Ψ			Ψ_s
τ_2	$R_1/R_2 = 0.1$	$R_1/R_2 = 0.5$	$R_1/R_2 = 0.9$	$R_1/R_2 = 0.5$
0	1.0000	1.0000	1.0000	0.0000
0.1	0.9970	0.9900	0.9946	0.0321
0.5	0.9844	0.9488	0.9728	0.1678
1.0	0.9680	0.8976	0.9459	0.3525
2.0		0.8006	0.8944	0.7619
5.0	0.8316	0.5797	0.7625	2.1552
10.0	0.6839	0.3834	0.6077	
20.0		0.2250	0.4312	

where $\text{sgn}(t) = t/|t| = \pm 1$, depending on the sign of t. Solutions to Φ, Ψ, Φ_s, and Ψ_s have been tabulated by Viskanta and Crosbie [6] for a number of radius ratios R_1/R_2 and optical thicknesses τ_2. Their results for the nondimensional flux functions Ψ and Ψ_s are given in Table 14.2. As for the plane slab, the statement of radiative equilibrium,

$$\frac{1}{\tau^2}\frac{d}{d\tau}(\tau^2 q) = \frac{\dot{Q}'''}{\kappa}, \tag{14.80}$$

implies that Ψ and Ψ_s are constants, and equations (14.76) and (14.77) may be evaluated for any arbitrary value of τ.

It remains to eliminate the radiosities J_1 and J_2 from equation (14.75) for the case of nonblack boundaries. Similar to the development for parallel plates, equations (14.48) through (14.50), we have

$$\tau = \tau_1: \qquad q_1 = \frac{\epsilon_1}{1-\epsilon_1}\left(n^2\sigma T_1^4 - J_1\right), \tag{14.81a}$$

$$\tau = \tau_2: \qquad -q_2 = \frac{\epsilon_2}{1-\epsilon_2}\left(n^2\sigma T_2^4 - J_2\right). \tag{14.81b}$$

Performing an energy balance (i.e., stating that energy coming in at Sphere 1, plus energy generated in the volume between spheres, equals energy going out at Sphere 2), we obtain

$$4\pi R_1^2 q_1 + \dot{Q}'''\frac{4}{3}\pi(R_2^3 - R_1^3) = 4\pi R_2^2 q_2,$$

or

$$q_2 = \left(\frac{\tau_1}{\tau_2}\right)^2 q_1 + \frac{1}{3}\frac{\dot{Q}'''}{\kappa}\frac{\tau_2^3 - \tau_1^3}{\tau_2^2}. \tag{14.82}$$

Substituting equation (14.82) into (14.81) leads to

$$\left(\frac{1}{\epsilon_1} - 1\right)q_1 + \left(\frac{1}{\epsilon_2} - 1\right)\left[\left(\frac{\tau_1}{\tau_2}\right)^2 q_1 + \frac{1}{3}\frac{\dot{Q}'''}{\kappa}\frac{\tau_2^3 - \tau_1^3}{\tau_2^2}\right] = n^2\sigma(T_1^4 - T_2^4) - (J_1 - J_2),$$

or

$$(J_1 - J_2)\tau_1^2 = n^2\sigma(T_1^4 - T_2^4)\tau_1^2 - \left[\frac{1}{\epsilon_1} - 1 + \left(\frac{\tau_1}{\tau_2}\right)^2\left(\frac{1}{\epsilon_2} - 1\right)\right]\tau^2 q - \frac{1}{3}\frac{\dot{Q}'''}{\kappa}\left(\frac{1}{\epsilon_2} - 1\right)\left(\frac{\tau_1}{\tau_2}\right)^2(\tau_2^3 - \tau_1^3), \tag{14.83}$$

which may be employed to eliminate the radiosities from equation (14.75).

More recently, a few investigators have considered somewhat more involved situations. The governing integral equations for an isotropically scattering spherical medium were first stated by Pomraning and Siewert [7]. These equations were used by Thynell and Özişik [8] to investigate the gray isotropically scattering solid sphere with gray, diffusely reflecting boundary. Their analysis applied to any given temperature fields or variable internal heat generation. Finally, the problem of nondiffuse reflectance at the outer face, obeying Fresnel's laws, was investigated by Wu and Wang [9] for an isothermal, isotropically scattering, solid sphere.

Example 14.3. A gray, nonscattering medium with refractive index $n = 1$ and an absorption coefficient $\kappa = 0.1\,\text{cm}^{-1}$ is contained between two concentric isothermal spheres with radii $R_1 = 25\,\text{cm}$ and $R_2 = 50\,\text{cm}$. The inner sphere is hot ($T_1 = 2000\,\text{K}$) and highly reflective ($\epsilon_1 = 0.1$); the outer sphere is a strong absorber ($\alpha_2 = \epsilon_2 = 0.9$), which must be kept relatively cool ($T_2 = 400\,\text{K}$). Assuming that conductive and convective heat transfer can be neglected as compared with radiation, determine the necessary cooling rate.

Solution

We have the situation of one-dimensional radiative equilibrium between concentric spheres, and equation (14.75) applies with $\dot{Q}''' = 0$. With $R_1/R_2 = 25/50 = 0.5$ and $\tau_2 = \kappa R_2 = 0.1 \times 50 = 5$ we obtain, from Table 14.2, $\Psi = 0.5797$. The radiosities are eliminated with equation (14.83) so that

$$\tau^2 q = \sigma(T_1^4 - T_2^4)\tau_1^2\Psi - \left[\frac{1}{\epsilon_1} - 1 + \left(\frac{\tau_1}{\tau_2}\right)^2\left(\frac{1}{\epsilon_2} - 1\right)\right]\tau^2 q\,\Psi,$$

or

$$\begin{aligned}\frac{\tau^2 q}{\tau_1^2\sigma(T_1^4 - T_2^4)} &= \frac{q_1}{\sigma(T_1^4 - T_2^4)} = \frac{\Psi}{1 + \left[\dfrac{1}{\epsilon_1} - 1 + \left(\dfrac{\tau_1}{\tau_2}\right)^2\left(\dfrac{1}{\epsilon_2} - 1\right)\right]\Psi}\\ &= \frac{0.5797}{1 + \left[\frac{1}{0.1} - 1 + 0.5^2\left(\frac{1}{0.9} - 1\right)\right]0.5797} = 0.0930.\end{aligned}$$

This result should be compared with the value of 0.0830 found in Example 14.2 for the identical situation between parallel plates. The flux density at the inner sphere then turns out to be

$$\begin{aligned}q_1 &= 0.0930\sigma(T_1^4 - T_2^4)\\ &= 0.0930 \times 5.670 \times 10^{-12}\,(2000^4 - 400^4)\,\text{W/cm}^2 = 8.42\,\text{W/cm}^2.\end{aligned}$$

14.6 RADIATIVE TRANSFER IN CYLINDRICAL MEDIA

We shall now briefly consider the case of a one-dimensional cylindrical medium, with temperature and radiative properties varying only in the radial direction r, but not with axial position z or azimuthal angle ψ_c (where we have again added the subscript c to emphasize that this angle specifies *position* in the cylindrical coordinate system, and is independent of azimuthal *direction* angle ψ). For this geometry it is advantageous to place the direction coordinate system such that polar angle θ is measured from the positive z-axis, while the azimuthal angle ψ is measured in the r-ψ_c-plane perpendicular to it, as shown in Fig. 14-6. Measuring the azimuthal angle from the radial coordinate as indicated in the figure, we recognize that radiative intensity may vary with radial position r and *both* direction angles θ and ψ. Therefore, similar to equation (14.64), we have

$$\hat{\mathbf{s}} \cdot \nabla I = \frac{dI}{ds} = \frac{\partial I}{\partial r}\frac{dr}{ds} + \frac{\partial I}{\partial \theta}\frac{d\theta}{ds} + \frac{\partial I}{\partial \psi}\frac{d\psi}{ds}. \tag{14.84}$$

From symmetry, it follows that $I(r,\theta,\psi) = I(r,\theta,-\psi) = I(r,\pi-\theta,\psi)$. Inspecting Fig. 14-6, we find

$$\cos\psi = \frac{dr}{ds\sin\theta}, \quad \text{or} \quad \frac{dr}{ds} = \sin\theta\cos\psi, \tag{14.85}$$

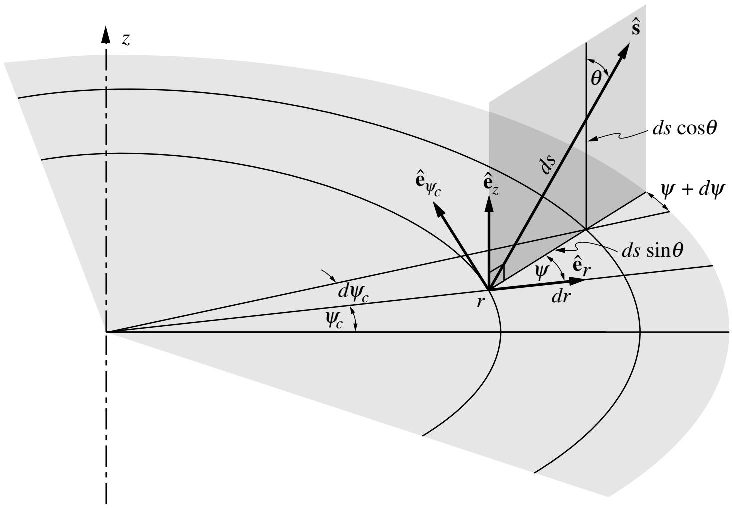

FIGURE 14-6
Coordinates for a one-dimensional cylindrical medium.

where $ds \sin\theta$ is the projection of ds into the r-ψ_c-plane. Traveling along a beam in the direction of $\hat{\mathbf{s}}$ we see that, similar to the spherical case, the azimuthal angle ψ steadily decreases (instead of θ in the spherical case, see Fig. 14-5). Therefore, replacing θ by ψ in equation (14.66), we find

$$\frac{d\psi}{dr} = -\frac{\tan\psi}{r}, \quad \text{or} \quad \frac{d\psi}{ds} = \frac{d\psi}{dr}\frac{dr}{ds} = -\frac{\sin\theta\sin\psi}{r}. \tag{14.86}$$

On the other hand, traveling along $\hat{\mathbf{s}}$, we see that the angle with the z-axis remains unchanged, or $d\theta/ds = 0$. Sticking these relations into equation (14.84) yields

$$\frac{dI}{ds} = \sin\theta\left[\cos\psi\frac{\partial I}{\partial r}(r,\theta,\psi) - \frac{\sin\psi}{r}\frac{\partial I}{\partial\psi}(r,\theta,\psi)\right]. \tag{14.87}$$

The equation of transfer appropriate for the one-dimensional cylindrical medium follows then from equations (14.1) and (14.3) as

$$\sin\theta\left[\cos\psi\frac{\partial I}{\partial\tau}(\tau,\theta,\psi) - \frac{\sin\psi}{\tau}\frac{\partial I}{\partial\psi}(\tau,\theta,\psi)\right] = S(\tau,\theta,\psi) - I(\tau,\theta,\psi), \tag{14.88}$$

where again $\tau = \int_0^r \beta\, dr$. This relationship can also be found from the left-hand side of equation (14.84) by recognizing (cf. Fig. 14-6) that

$$\hat{\mathbf{s}} = \sin\theta\cos\psi\,\hat{\mathbf{e}}_r + \sin\theta\sin\psi\,\hat{\mathbf{e}}_{\psi_c} + \cos\theta\,\hat{\mathbf{e}}_z, \tag{14.89a}$$

$$\psi + \psi_c = \text{const along } \hat{\mathbf{s}}, \tag{14.89b}$$

and using, for cylindrical coordinates,

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial}{\partial\psi_c}\hat{\mathbf{e}}_{\psi_c} + \frac{\partial}{\partial z}\hat{\mathbf{e}}_z,$$

with $d\psi_c = -d\psi$ and $\partial I/\partial z = 0$. The general form for the radiative source function is given by equation (14.3); for linear-anisotropic scattering, equation (14.15) remains valid with $\hat{\mathbf{e}}_r\cdot\hat{\mathbf{s}} = \cos\theta$ (valid for slab and sphere, cf. Figs. 14-1 and 14-5) replaced by $\hat{\mathbf{e}}_r\cdot\hat{\mathbf{s}} = \sin\theta\cos\psi$ (valid for the cylinder, cf. Fig. 14-6), or

$$S(\tau,\theta,\psi) = (1-\omega)\,I_b(\tau) + \frac{\omega}{4\pi}\left[G(\tau) + A_1 q(\tau)\sin\theta\cos\psi\right]. \tag{14.90}$$

TABLE 14.3
Nondimensional heat loss from a gray, nonscattering, isothermal cylinder, $\Psi = q(\tau_R)/(n^2\sigma T^4 - J_w)$.

τ_R	Ψ	τ_R	Ψ
0.1	0.1770	1.0	0.8143
0.2	0.3172	1.5	0.9047
0.3	0.4299	2.0	0.9458
0.4	0.5213	2.5	0.9662
0.5	0.5960	3.0	0.9772
0.6	0.6573	3.5	0.9836
0.7	0.7080	4.0	0.9877
0.8	0.7500	4.5	0.9904
0.9	0.7850	5.0	0.9923

The problem of one-dimensional heat transfer through a cylindrical medium was first considered by Heaslet and Warming [10]. They investigated the case of an isotropically scattering medium contained within an isothermal black cylindrical container. Two cases were treated: (*i*) radiative equilibrium with uniform heat generation within the medium, and (*ii*) an isothermal medium. While they displayed a few graphical results, no tabulated results were given.

The solution for a one-dimensional, gray, linear-anisotropically scattering cylinder with arbitrary, but specified, temperature distribution has been given by Azad and Modest [11], using a different approach. We list here their solution for the simple case of no scattering ($\omega = 0$), for which

$$q(\tau) = \left[n^2\sigma T^4(\tau_R) - J_w\right] F(\tau,\tau_R) - \int_0^{\tau} \frac{d}{d\tau'}\left(n^2\sigma T^4\right)\frac{\tau'}{\tau}F(\tau',\tau)\,d\tau' - \int_{\tau}^{\tau_R} \frac{d}{d\tau'}\left(n^2\sigma T^4\right)F(\tau,\tau')\,d\tau, \tag{14.91}$$

with

$$F(\tau,\tau') = -\frac{4}{\pi}\int_0^{\pi}\int_0^{\pi/2} \exp\left[-\frac{\tau}{\sin\theta}\left(\cos\psi + \sqrt{\left(\frac{\tau'}{\tau}\right)^2 - \sin^2\psi}\right)\right]\sin^2\theta\cos\psi\,d\theta\,d\psi. \tag{14.92}$$

Equation (14.91) is very similar to the equivalent expression for the one-dimensional slab, equation (14.39) (after integration by parts): Instead of the relatively simple (and widely tabulated) exponential integral $E_3(\tau' - \tau)$ we have another, somewhat more complicated geometric function, $F(\tau,\tau')$. For an isothermal cylinder equation (14.91) may be evaluated in explicit form at $\tau = \tau_R$, with $F(\tau_R,\tau_R)$ expressed in terms of modified Bessel functions. Some representative results of equation (14.91) for $\tau = \tau_R$ have been tabulated in Table 14.3.

Thermal radiation between concentric cylinders was first treated by Kesten [12], who considered a nonscattering, gray gas with known temperature distribution. The case of a gray, isotropically scattering medium at radiative equilibrium between concentric cylinders has been studied by Pandey and Cogley [13] and Loyalka [14]. The governing equations become rather involved and will not be reproduced here. Because of this complexity, both solutions are not quite exact: Pandey and Cogley used some approximate geometric functions, while Loyalka used a simple variational approach to solve the governing integral equation. Comparison with the "exact" Monte Carlo solution[6] by Perlmutter and Howell [15] shows that Loyalka's results may essentially be taken as exact. Some representative results for the nondimensional radiative

[6]For a discussion of the Monte Carlo method, see Chapter 21.

TABLE 14.4
Nondimensional radiative heat transfer between concentric cylinders at radiative equilibrium, $\Psi = q(\tau_1)/(J_1 - J_2)$.

Optical Thickness $\tau_2 - \tau_1$	Ψ Radius Ratio R_1/R_2 0.1	0.5	0.9
0.1	0.9893	0.9677	0.9462
0.5	0.9464	0.8476	0.7688
1.0	0.8937	0.7225	0.6167
2.0	0.7956	0.5446	0.4371
3.0	0.7105	0.4313	0.3367
4.0	0.6377	0.3549	0.2727
5.0	0.5763	0.3010	0.2291
6.0	0.5250	0.2615	0.1976
7.0	0.4810	0.2308	0.1738
8.0	0.4429	0.2060	0.1549
9.0	0.4102	0.1864	0.1398
10.0	0.3821	0.1703	0.1278

heat flux in terms of surface radiosities,

$$\Psi = \frac{q(\tau_1)}{J_1 - J_2}, \tag{14.93}$$

are given in Table 14.4.

For nonblack walls the radiosities may be eliminated from equation (14.93) in precisely the same fashion as was done for concentric spheres. From equation (14.83), with $A_1/A_2 = R_1/R_2 = \tau_1/\tau_2$ and $\dot{Q}''' = 0$,

$$J_1 - J_2 = n^2\sigma(T_1^4 - T_2^4) - \left[\frac{1}{\epsilon_1} - 1 + \frac{\tau_1}{\tau_2}\left(\frac{1}{\epsilon_2} - 1\right)\right] q_1,$$

and

$$\frac{q_1}{n^2\sigma(T_1^4 - T_2^4)} = \frac{\Psi}{1 + \left[\frac{1}{\epsilon_1} - 1 + \frac{\tau_1}{\tau_2}\left(\frac{1}{\epsilon_2} - 1\right)\right]\Psi}. \tag{14.94}$$

Example 14.4. Repeat Example 14.3 for the case of concentric cylinders.

Solution
With $\tau_2 = 5$ and $\tau_1 = 2.5$ we have $\tau_2 - \tau_1 = 2.5$ and, after interpolating (somewhat nonlinearly) between values from Table 14.4 for $\tau_2 - \tau_1 = 2.0$ and $\tau_2 - \tau_1 = 3.0$, $\Psi \simeq 0.48$. Thus,

$$\frac{q_1}{\sigma(T_1^4 - T_2^4)} = \frac{0.48}{1 + \left[\frac{1}{0.1} - 1 + 0.5\left(\frac{1}{0.9} - 1\right)\right] 0.48} = 0.0898,$$

and

$$q_1 = 0.0898 \times 5.670 \times 10^{-12}\,(2000^4 - 400^4) = 8.13\,\mathrm{W/cm^2}.$$

We observe that, for identical conditions, the heat loss is greatest between concentric spheres, followed by concentric cylinders, and finally by parallel plates. Also, from Tables 14.2 and 14.4 we see that heat loss increases with decreasing radius ratio R_1/R_2. This observation may be explained by the fact that, per unit area, the surface of an (inner) sphere exchanges heat with a larger area on the (outer) sphere ($A_2/A_1 = R_2^2/R_1^2$) than is the case for concentric cylinders ($A_2/A_1 = R_2/R_1$) or parallel plates ($A_2/A_1 = 1$). The same argument applies to decreasing radius ratios.

14.7 NUMERICAL SOLUTION OF THE GOVERNING INTEGRAL EQUATIONS

The governing integral equations may be solved with several analytical and/or numerical techniques, which will not be discussed in this text in any detail.

An example of analytical techniques is the use of *Chandrasekhar's X- and Y-functions*, based on the *principle of invariance*, which is described in some detail in Chandrasekhar's book [16]. For example, Heaslet and Warming [2] expressed the nondimensional temperature Φ_b in equation (14.43) in terms of moments of these X- and Y-functions. The moments are determined by numerical quadrature, using tabulated values for Chandrasekhar's X- and Y-functions. *Case's normal-mode expansion technique* [17, 18] is to linear integral equations what separation-of-variables is to partial differential equations. The technique was originally developed for neutron transport theory, and has been applied to radiative heat transfer by Ferziger and Simmons [19, 20] and Siewert and Özişik [21–27]. A detailed account may be found in the book by Özişik [28]. Siewert and coworkers [29, 30] have further developed Case's normal-mode approach, by finding solutions in terms of power series. This approach, known as the F_N-method, was also originally applied to neutron transport, and only later to thermal radiation [31].

Numerical solutions to the governing integral equations may be found by a variety of methods. The simplest such method is the standard numerical quadrature as discussed in Section 5.6. Since integrands often contain singularities [cf. equation (14.43)], these must be removed before quadrature can be applied [2]. The problem of singularities may also be overcome by approximating the unknown variable in functional form, which allows the analytical evaluation of such integrals. For example, Özişik and coworkers [32–35] solved the governing integral equation for several plane-parallel problems with the *Galerkin method* [36]. In this method, the unknown dependent variable [say, $\Phi(\tau)$ in equation (14.43)] is approximated by a series of independent functions $\varphi_i(\tau)$, that is,

$$\Phi(\tau) = C_1\,\varphi_1(\tau) + C_2\,\varphi_2(\tau) + \cdots = \sum_{i=1}^{N} C_i\,\varphi_i(\tau), \tag{14.95}$$

where the C_i are unknown constants. These are determined by multiplying the governing equation by each of the functions $\varphi_i(\tau)$, followed by integration over the entire domain, resulting in N simultaneous algebraic equations for the unknown constants. Most often the independent functions in equation (14.95) are chosen to be powers in the independent variable, for example, $\varphi_i(\tau) = \tau^{i-1}$ $(i = 1, 2, \ldots)$, although Legendre polynomials have also been used [37], in order to exploit the orthogonality properties of such polynomials. The Galerkin method offers results of great accuracy even for series truncated after very few terms, albeit at the price of very tedious analytical or numerical integrations. For more general geometries the use of the related *finite element method* becomes more practical, as applied by Reddy and Murty [38].

A somewhat simpler method is the *point collocation method* together with approximating the unknown variable by piecewise-continuous splines. In this method the unknown dependent variable, say $\Phi(\tau)$, is approximated by a spline function involving $N+1$ nodal values $\Phi_i = \Phi(\tau_i)$, $i = 0, 1, 2, \ldots, N$. For example, if standard cubic splines are employed [39],

$$\Phi = \Phi_i \;+\; B_i\,(\tau-\tau_i) + C_i\,(\tau-\tau_i)^2 + D_i\,(\tau-\tau_i)^3, \qquad \tau_i \le \tau \le \tau_{i+1}, \quad i = 0, 1, \ldots, N-1, \tag{14.96}$$

where the constants B_i, C_i, and D_i depend on all values of Φ_i and are readily found through standard software packages resident on most computers. More sophisticated splines, such as B-splines [40–42] or Chebyshev polynomials [43], result in more complicated expressions. Equation (14.96) is now substituted into the governing integral equation, and the piecewise integrals are evaluated analytically. Applying the governing equation to the $N+1$ nodal points (point collocation) results in $N+1$ simultaneous, linear algebraic equations for the unknown Φ_i.

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Problems

14.1 The gap between two parallel black plates at T_1 and T_2, respectively, is filled with a particle-laden gas. Radiative equilibrium prevails, and the particle loading is a fixed volume fraction, with particles manufactured from two different materials (one a specular reflector, the other a diffuse reflector, both having the same ϵ). Sketch the nondimensional heat flux $\Psi = q/\sigma(T_1^4 - T_2^4)$ *vs.* particle size (but keeping volume fraction constant).

14.2 Consider radiative equilibrium in a one-dimensional, gray, nonscattering, plane-parallel medium bounded by isothermal black plates at temperatures T_1 and T_2. To make a simple closed-form solution possible for the determination of the heat flux between the plates, it has been proposed to replace the radiative-equilibrium slab by a constant temperature slab, with its temperature evaluated at $T_{\rm av}^4 = \frac{1}{2}(T_1^4 + T_2^4)$. Under what optical conditions is this a good idea, if ever?

To determine this compare the different "exact" solutions for various optical thicknesses, say, $\tau_L = 0$, 1, 5.

14.3 Consider a space enclosed by infinite, diffuse-gray parallel plates filled with a gray nonscattering medium. The surfaces are isothermal (both at T_w), and there is uniform and constant heat generation within the medium per unit volume, $\dot{Q}'''$. Conduction and convection are negligible so that $\nabla\cdot\mathbf{q} = \dot{Q}'''$. Set up the integral equations describing temperature and heat flux distribution in the enclosure, i.e., show that

$$\begin{aligned}\Phi(\tau) &= \frac{\sigma T^4 - J_w}{\dot{Q}'''/4\kappa} = 1 + \frac{1}{2}\int_0^{\tau_L} \Phi(\tau')E_1\left(|\tau - \tau'|\right)d\tau', \\ \Psi(\tau) &= \frac{q}{\dot{Q}'''/\kappa} = \tau - \frac{\tau_L}{2}.\end{aligned}$$

14.4 A semi-infinite, gray, isotropically scattering medium, originally at a temperature of 0 K, is subjected to collimated irradiation with a constant heat flux q_0 normal to its nonreflecting surface. Set up the integral relationships governing steady-state temperature and radiative heat flux within the medium, assuming radiative equilibrium. Hint: Collimated irradiation with heat flux q_0 has the radiative intensity

$$I_0(\theta, \psi) = \begin{cases} \dfrac{q_0}{2\pi \sin\theta\, \delta\theta}, & 0 \leq \theta < \delta\theta,\ 0 \leq \psi \leq 2\pi, \\ 0, & \text{elsewhere.} \end{cases}$$

14.5 Two large, isothermal gray plates at $T_1 = 2000\,\text{K}$, $T_2 = 1000\,\text{K}$ with $\epsilon = \epsilon_1 = \epsilon_2 = 0.5$ are separated by a gap of width $L = 1\,\text{m}$ filled with purely (isotropically) scattering particles. If the heat flux between the plates has been measured as $223.3\,\text{kW/m}^2$, what is the medium's scattering coefficient?

14.6 Consider two parallel, black, isothermal plates spaced 1 m apart with $T_1 = 2000\,\text{K}$ and $T_2 = 1000\,\text{K}$. The medium between the plates is gray and at radiative equilibrium with a nonconstant absorption coefficient of

$$\kappa = \kappa_0 + \kappa_1' z; \quad \kappa_0 = 10^{-2}\,\text{cm}^{-1}, \quad \kappa_1' = 2 \times 10^{-4}\,\text{cm}^{-2}.$$

The medium does not scatter.

(*a*) What is the heat flux between the plates?
(*b*) What is the temperature at the medium's center ($z = \frac{1}{2}L$)?

14.7 An infinite, black, isothermal plate bounds a semi-infinite space filled with black spheres. At any given distance, z, away from the plate the particle number density is identical, namely $N_T = 6.3662\times10^8\,\text{m}^{-3}$. However, the radius of the suspended spheres diminishes monotonically away from the surface as

$$a = a_0\, e^{-z/L}; \quad a_0 = 10^{-4}\,\text{m}, \quad L = 1\,\text{m}.$$

(*a*) Determine the absorption coefficient as a function of z (you may make the large-particle assumption).
(*b*) Determine the optical coordinate as a function of z. What is the total optical thickness of the semi-infinite space?
(*c*) Assuming that radiative equilibrium prevails, determine the heat loss from the plate.

14.8 The radiative heat transfer between two isothermal, black plates at temperatures T_1 and T_2 and separated by a nonparticipating gas is to be minimized. Enough of a black material is available to place a 1 mm thick radiation shield between the plates. Alternatively, the same amount of material could be used in the form of small spheres of 0.1 mm radius to be suspended between the plates. Which possibility results in lower heat flux, assuming conduction and convection to be negligible?

14.9 Two infinite, isothermal plates at temperatures T_1 and T_2 are separated by a cold, gray medium of optical thickness $\tau_L = \kappa L$ (no scattering).

(*a*) Calculate the radiative heat flux at the bottom plate and the top plate, and the net radiative energy going into the gray medium, assuming that both plates are black.
(*b*) Repeat (*a*), but assume that both plates have the same temperature T, and that both plates are gray with equal emittance ϵ (diffuse emission and reflection).

14.10 A semi-infinite, absorbing–emitting, nonscattering medium at uniform temperature is in contact with a gray-diffuse wall at T_w and with emittance ϵ_w.

(*a*) The medium is gray, and has a constant absorption coefficient. Determine the net radiative heat flux at the wall.
(*b*) Let the medium be nongray with nonconstant absorption coefficient κ_λ, and the wall be nongray and nondiffuse with spectral, directional emittance ϵ_λ'. How would this affect the wall heat flux?

14.11 A 1 m thick, isothermal slab bounded by two cold black plates has a temperature of 3000 K, and a nongray absorption coefficient that can be approximated by

$$\kappa_\lambda = \begin{cases} 0, & \lambda < 2\,\mu\text{m}, \\ 0.20\,\text{cm}^{-1}, & \lambda > 2\,\mu\text{m}. \end{cases}$$

Calculate the total heat loss by radiation from the slab (in W/cm^2).

14.12 Consider (*a*) two parallel plates, (*b*) two concentric spheres, and (*c*) two concentric cylinders. The bottom/inner surface needs to dissipate a heat flux of $30\,\mathrm{W/cm^2}$ and has a gray-diffuse emittance $\epsilon_1 = 0.5$. The top/outer surface is at $T_2 = 1000\,\mathrm{K}$ with $\epsilon_2 = 0.8$. The medium in between the surfaces is gray and nonscattering ($\kappa = 0.1\,\mathrm{cm}^{-1}$), has a thickness of $L = 5\,\mathrm{cm}$, and is at radiative equilibrium. Determine the temperature at the bottom/inner surface necessary to dissipate the supplied heat for the three different cases (the radii of the inner cylinder and sphere are $R_1 = 5\,\mathrm{cm}$). Discuss the results.

14.13 Consider a very hot sphere of a nongray gas of radius $R = 1\,\mathrm{m}$ in $0\,\mathrm{K}$ surroundings that have been evacuated. The gas has a single absorption–emission band in the infrared, with an absorption coefficient

$$\kappa_\eta = \begin{cases} 0, & \eta < 3000\,\mathrm{cm}^{-1} = \eta_0, \\ \kappa_0\, e^{-(\eta-\eta_0)/\omega}, & \eta > 3000\,\mathrm{cm}^{-1}, \end{cases}$$

where $\kappa_0 = 1\,\mathrm{cm}^{-1}$, $\omega = 200\,\mathrm{cm}^{-1}$. During cool-down the sphere is always isothermal, and remains of constant size (i.e., constant density $\rho = 1000\,\mathrm{g/m^3}$). The heat capacity of the gas is $c_p = 1\,\mathrm{kJ/kg\,K}$. Determine the time required to cool the gas from $T_i = 6000\,\mathrm{K}$ to $T_e = 1000\,\mathrm{K}$. Sketch qualitatively the behavior of $\Psi = q/\sigma T^4$ *vs.* T.
Hint: To make an analytical solution possible, you may make the following assumptions:

(*a*) $\mathrm{Ein}(x) = \int_0^1 (1-e^{-x\xi})\,d\xi/\xi = E_1(x) + \ln x + \gamma_E \simeq \ln x + \gamma_E$ (for sufficiently large x; see also Appendix E).

(*b*) Wien's distribution may be used.

14.14 It is proposed to construct a high-temperature heating element by guiding hot combustion gases through a silicon carbide tube. The outside of the SiC tube then radiates heat toward the load. Such devices are known as "radiant tubes." For the design of such a radiant tube you may make the following assumptions:

(*a*) The combustion gas inside the radiant tube is essentially gray and isothermal with $\kappa = 0.2\,\mathrm{cm}^{-1}$ and $T_{\mathrm{gas}} = 2000\,\mathrm{K}$.

(*b*) The silicon carbide tube wall is essentially isothermal and of negligible thickness, with a gray-diffuse emittance of 0.8 on both sides.

(*c*) The long tube is contained in a large furnace with a background temperature of 1000 K.

Determine the necessary tube diameter to achieve a radiant heating rate of 100 kW per m length of tube.