CHAPTER

10

THE RADIATIVE TRANSFER EQUATION IN PARTICIPATING MEDIA (RTE)

10.1 INTRODUCTION

In previous chapters we have looked at radiative transfer between surfaces that were separated by vacuum or by a transparent ("radiatively nonparticipating") medium. However, in many engineering applications the interaction of thermal radiation with an absorbing, emitting, and scattering ("radiatively participating") medium must be accounted for. Examples in the heat transfer area are the burning of any fuel (be it gaseous, liquid,or solid; be it for power production, within fires, within explosions, etc.), rocket propulsion, hypersonic shock layers, ablation systems on reentry vehicles, nuclear explosions, plasmas in fusion reactors, and many more.

In the present chapter we shall develop the general relationships that govern the behavior of radiative heat transfer in the presence of an absorbing, emitting, and/or scattering medium. We shall begin by making a radiative energy balance, known as the *radiative transfer equation*, or RTE, which describes the radiative intensity field within the enclosure as a function of location (fixed by location vector $\mathbf{r}$), direction (fixed by unit direction vector $\hat{\mathbf{s}}$) and spectral variable (wavenumber η).[1] To obtain the net radiative heat flux crossing a surface element, we must sum the contributions of radiative energy irradiating the surface from all possible directions and for all possible wavenumbers. Therefore, integrating the radiative transfer equation over all directions and wavenumbers leads to a *conservation of radiative energy* statement applied to an infinitesimal volume. Finally, this will be combined with a balance for all types of energy (including conduction and convection), leading to the *Overall Conservation of Energy* equation.

In the following three chapters we shall deal with the radiation properties of participating media, i.e., with how a substance can absorb, emit, and scatter thermal radiation. In Chapter 11 we discuss how a molecular gas can absorb and emit photons by changing its energy states, how to predict the radiation properties, and how to measure them experimentally. Chapter 12 is concerned with how small particles interact with electromagnetic waves—how they absorb,

[1]In our discussion of surface radiative transport we have used wavelength λ as the spectral variable throughout, largely to conform with the majority of other publications. However, for gases, frequency ν or wavenumber η are considerably more convenient to use. Again, to conform with the majority of the literature, we shall use wavenumber throughout this part.

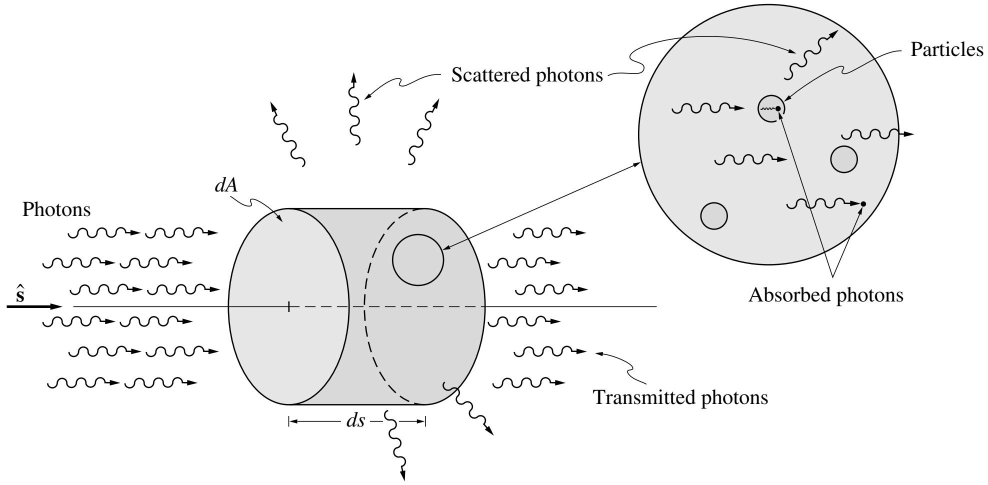

FIGURE 10-1
Attenuation of radiative intensity by absorption and scattering.

emit, and scatter radiative energy. Again, theoretical as well as experimental methods are covered. Finally, in Chapter 13 a very brief account is given of the radiation properties of solids and liquids that allow electromagnetic waves of certain wavelengths to penetrate into them for appreciable distances, known as semitransparent media.

10.2 ATTENUATION BY ABSORPTION AND SCATTERING

If the medium through which radiative energy travels is "participating," then any incident beam will be attenuated by absorption and scattering while it travels through the medium, as schematically shown in Fig. 10-1. In the following we shall develop expressions for this attenuation for a light beam which travels within a pencil of rays into the direction $\hat{\mathbf{s}}$. The present discussion will be limited to media with constant refractive index, i.e., media through which electromagnetic waves travel along straight lines [while a varying refractive index will bend the ray, as shown by Snell's law, equation (2.72), for an abrupt change]. It is further assumed that the medium is stationary (as compared to the speed of light), that it is nonpolarizing,and that it is (for most of the discussion) at local thermodynamic equilibrium (LTE).

Absorption

The absolute amount of absorption has been observed to be directly proportional to the magnitude of the incident energy as well as the distance the beam travels through the medium. Thus, we may write,

$$(dI_\eta)_{\rm abs} = -\kappa_\eta I_\eta \, ds, \tag{10.1}$$

where the proportionality constant κ_η is known as the *(linear) absorption coefficient*, and the negative sign has been introduced since the intensity decreases. As will be discussed in the following chapter, the absorption of radiation in molecular gases depends also on the number of receptive molecules per unit volume, so that some researchers use a *mass absorption coefficient* or a *pressure absorption coefficient*, defined by

$$(dI_\eta)_{\rm abs} = -\kappa_{\rho\eta} I_\eta \rho \, ds = -\kappa_{p\eta} I_\eta p \, ds. \tag{10.2}$$

The subscripts ρ and p are used here only to demonstrate the differences between the coefficients. The reader of scientific literature often must rely on the physical units to determine the coefficient used.

Integration of equation (10.1) over a geometric path s results in

$$I_\eta(s) = I_\eta(0)\exp\left(-\int_0^s \kappa_\eta\, ds\right) = I_\eta(0)\, e^{-\tau_\eta}, \tag{10.3}$$

where

$$\tau_\eta = \int_0^s \kappa_\eta\, ds \tag{10.4}$$

is the optical thickness (for absorption) through which the beam has traveled and $I_\eta(0)$ is the intensity entering the medium at $s = 0$. Note that the (linear) absorption coefficient is the inverse of the mean free path for a photon until it undergoes absorption. One may also define an *absorptivity* for the participating medium (for a given path within the medium) as

$$\alpha_\eta \equiv \frac{I_\eta(0) - I_\eta(s)}{I_\eta(0)} = 1 - e^{-\tau_\eta}. \tag{10.5}$$

Scattering

Attenuation by scattering, or "out-scattering" (away from the direction under consideration), is very similar to absorption, i.e., a part of the incoming intensity is removed from the direction of propagation, $\hat{\mathbf{s}}$. The only difference between the two phenomena is that absorbed energy is converted into internal energy, while scattered energy is simply redirected and appears as augmentation along another direction (discussed in the next section), also known as "in-scattering." Thus, we may write

$$(dI_\eta)_{\text{sca}} = -\sigma_{s\eta} I_\eta\, ds, \tag{10.6}$$

where the proportionality constant $\sigma_{s\eta}$ is the *(linear) scattering coefficient* for scattering from the pencil of rays under consideration into all other directions. Again, scattering coefficients based on density or pressure may be defined. It is also possible to define an optical thickness for scattering, where the scattering coefficient is the inverse of the mean free path for scattering.

Total Attenuation

The total attenuation of the intensity in a pencil of rays by both absorption and scattering is known as *extinction*. Thus, an *extinction coefficient* is defined[2] as

$$\beta_\eta = \kappa_\eta + \sigma_{s\eta}. \tag{10.7}$$

The optical distance based on extinction is defined as

$$\tau_\eta = \int_0^s \beta_\eta\, ds. \tag{10.8}$$

As for absorption and scattering, the extinction coefficient is sometimes based on density or pressure.

10.3 AUGMENTATION BY EMISSION AND SCATTERING

A light beam traveling through a participating medium in the direction of $\hat{\mathbf{s}}$ loses energy by absorption and by scattering away from the direction of travel. But at the same time it also gains energy by emission, as well as by scattering from other directions into the direction of travel $\hat{\mathbf{s}}$.

[2]Care must be taken to distinguish the dimensional extinction coefficient β_η from the absorptive index, i.e., the imaginary part of the index of refraction complex k (sometimes referred to in the literature as the "extinction coefficient").

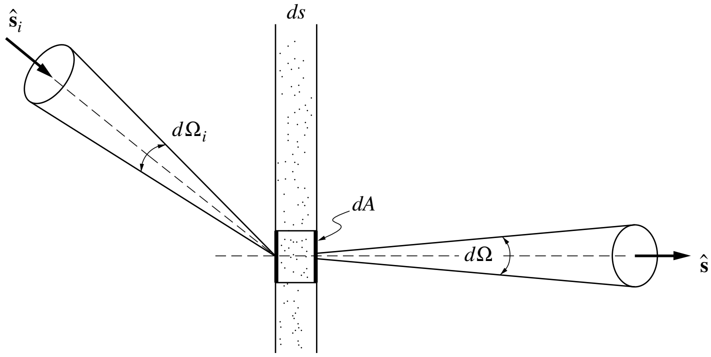

FIGURE 10-2
Redirection of radiative intensity by scattering.

Emission

The rate of emission from a volume element will be proportional to the magnitude of the volume. Therefore, the emitted intensity (which is the rate of emitted energy per unit area) along any path again must be proportional to the length of the path, and it must be proportional to the local energy content in the medium. Thus,

$$(dI_\eta)_{\rm em} = j_\eta\, ds, \tag{10.9}$$

where j_η is termed the *emission coefficient*. Since, at local thermodynamic equilibrium (LTE), the intensity everywhere must be equal to the blackbody intensity, it will be shown in Chapter 11, equation (11.22), that

$$j_\eta = \kappa_\eta I_{b\eta} \quad \text{and} \quad (dI_\eta)_{\rm em} = \kappa_\eta I_{b\eta}\, ds, \tag{10.10}$$

that is, at LTE the proportionality constant for emission is the same as for absorption. Similar to absorptivity, one may also define an *emissivity of an isothermal medium* as the amount of energy emitted over a certain path s that escapes into a given direction (without having been absorbed between point of emission and point of exit), as compared to the maximum possible. Combining equations (10.1) and (10.10) gives the complete radiative transfer equation for an absorbing–emitting (but not scattering) medium as

$$\frac{dI_\eta}{ds} = \kappa_\eta (I_{b\eta} - I_\eta), \tag{10.11}$$

where the first term of the right-hand side is augmentation due to emission and the second term is attenuation due to absorption. The solution to the radiative transfer equation for an isothermal gas layer of thickness s is

$$I_\eta(s) = I_\eta(0)\, e^{-\tau_\eta} + I_{b\eta}\left(1 - e^{-\tau_\eta}\right), \tag{10.12}$$

where the optical distance has been defined in equation (10.4). If only emission is considered, $I_\eta(0) = 0$, and the emissivity is defined as

$$\epsilon_\eta = I_\eta(s)/I_{b\eta} = 1 - e^{-\tau_\eta}, \tag{10.13}$$

which, as is the case with surface radiation, is identical to the expression for absorptivity.

Scattering

Augmentation due to scattering, or “in-scattering,” has contributions from all directions and, therefore, must be calculated by integration over all solid angles. Consider the radiative heat flux impinging on a volume element $dV = dA\, ds$, from an infinitesimal pencil of rays in the direction $\hat{\mathbf{s}}_i$ as depicted in Fig. 10-2. Recalling the definition for radiative intensity as energy flux per unit area normal to the rays, per unit solid angle, and per unit wavenumber interval,

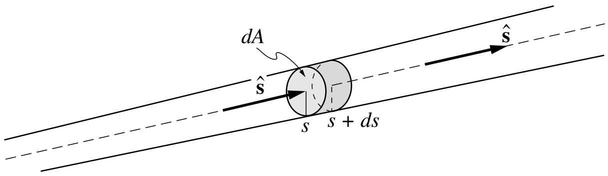

FIGURE 10-3
Pencil of rays for radiative energy balance.

one may calculate the spectral radiative heat flux impinging on dA from within the solid angle $d\Omega_i$ as

$$I_\eta(\hat{\mathbf{s}}_i)(dA\,\hat{\mathbf{s}}_i\cdot\hat{\mathbf{s}})\,d\Omega_i\,d\eta.$$

This flux travels through dV for a distance $ds/\hat{\mathbf{s}}_i\cdot\hat{\mathbf{s}}$. Therefore, the total amount of energy scattered away from $\hat{\mathbf{s}}_i$ is, according to equation (10.6),

$$\sigma_{s\eta}\Big(I_\eta(\hat{\mathbf{s}}_i)(dA\,\hat{\mathbf{s}}_i\cdot\hat{\mathbf{s}})\,d\Omega_i\,d\eta\Big)\left(\frac{ds}{\hat{\mathbf{s}}_i\cdot\hat{\mathbf{s}}}\right)=\sigma_{s\eta}I_\eta(\hat{\mathbf{s}}_i)\,dA\,d\Omega_i\,d\eta\,ds. \tag{10.14}$$

Of this amount, the fraction $\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})\,d\Omega/4\pi$ is scattered into the cone $d\Omega$ around the direction $\hat{\mathbf{s}}$. The function Φ_η is called the *scattering phase function* and describes the probability that a ray from one direction, $\hat{\mathbf{s}}_i$, will be scattered into a certain other direction, $\hat{\mathbf{s}}$. The constant 4π is arbitrary and is included for convenience [see equation (10.17) below].

The amount of energy flux from the cone $d\Omega_i$ scattered into the cone $d\Omega$ is then

$$\sigma_{s\eta}I_\eta(\hat{\mathbf{s}}_i)\,dA\,d\Omega_i\,d\eta\,ds\frac{\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})}{4\pi}\,d\Omega. \tag{10.15}$$

We can now calculate the energy flux scattered into the direction $\hat{\mathbf{s}}$ from *all* incoming directions $\hat{\mathbf{s}}_i$ by integrating:

$$\Big(dI_\eta\Big)_{\rm sca}(\hat{\mathbf{s}})\,dA\,d\Omega\,d\eta=\int_{4\pi}\sigma_{s\eta}I_\eta(\hat{\mathbf{s}}_i)\,dA\,d\Omega_i\,d\eta\,ds\,\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})\frac{d\Omega}{4\pi},$$

or

$$\Big(dI_\eta\Big)_{\rm sca}(\hat{\mathbf{s}})=ds\frac{\sigma_{s\eta}}{4\pi}\int_{4\pi}I_\eta(\hat{\mathbf{s}}_i)\,\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})\,d\Omega_i. \tag{10.16}$$

Returning to equation (10.15), we find that the amount of energy flux scattered from $d\Omega_i$ into all directions is

$$\sigma_{s\eta}I_\eta(\hat{\mathbf{s}}_i)\,dA\,d\Omega_i\,d\eta\,ds\frac{1}{4\pi}\int_{4\pi}\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})\,d\Omega,$$

which must be equal to the amount in equation (10.14). We conclude that

$$\frac{1}{4\pi}\int_{4\pi}\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})\,d\Omega\equiv 1. \tag{10.17}$$

Therefore, if Φ_η = const, i.e., if equal amounts of energy are scattered into all directions (called *isotropic scattering*), then $\Phi_\eta\equiv 1$. This is the reason for the inclusion of the factor 4π.

10.4 THE RADIATIVE TRANSFER EQUATION

We can now make an energy balance on the radiative energy traveling in the direction of $\hat{\mathbf{s}}$ within a small pencil of rays as shown in Fig. 10-3. The change in intensity is found by summing the contributions from emission, absorption, scattering away from the direction $\hat{\mathbf{s}}$, and scattering into the direction of $\hat{\mathbf{s}}$, from equations (10.1), (10.6), (10.9), and (10.16) as

$$\begin{aligned}I_\eta(s+ds,\hat{\mathbf{s}},t+dt)-I_\eta(s,\hat{\mathbf{s}},t)&=j_\eta(s,t)\,ds-\kappa_\eta I_\eta(s,\hat{\mathbf{s}},t)\,ds\\&\quad-\sigma_{s\eta}I_\eta(s,\hat{\mathbf{s}},t)\,ds+\frac{\sigma_{s\eta}}{4\pi}\int_{4\pi}I_\eta(\hat{\mathbf{s}}_i)\,\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})\,d\Omega_i\,ds.\end{aligned} \tag{10.18}$$

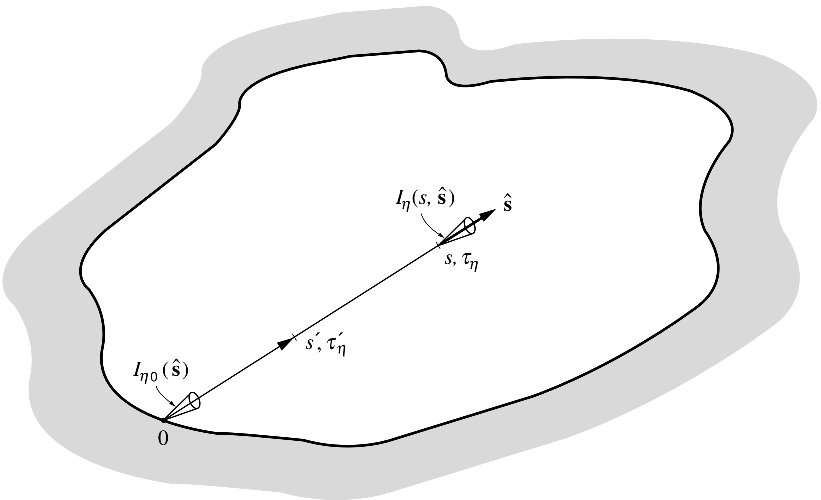

FIGURE 10-4
Enclosure for derivation of radiative transfer equation.

This equation is Lagrangian in nature, i.e., we are following a ray from s to $s+ds$; since the ray travels at the speed of light c, ds and dt are related through $ds = c\,dt$. The outgoing intensity may be developed into a truncated Taylor series, or

$$I_\eta(s+ds, \hat{\mathbf{s}}, t+dt) = I_\eta(s, \hat{\mathbf{s}}, t) + dt\frac{\partial I_\eta}{\partial t} + ds\frac{\partial I_\eta}{\partial s}, \tag{10.19}$$

so that equation (10.18) may be simplified to

$$\frac{1}{c}\frac{\partial I_\eta}{\partial t} + \frac{\partial I_\eta}{\partial s} = j_\eta - \kappa_\eta I_\eta - \sigma_{s\eta} I_\eta + \frac{\sigma_{s\eta}}{4\pi}\int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\,\Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\,d\Omega_i. \tag{10.20}$$

In this *radiative transfer equation* (commonly abbreviated as RTE), or *equation of transfer*, all quantities may vary with location in space, time, and wavenumber, while the intensity and the phase function also depend on direction $\hat{\mathbf{s}}$ (and $\hat{\mathbf{s}}_i$). Only the directional dependence, and only whenever necessary, has been explicitly indicated in this and the following equations, to simplify notation. As indicated earlier, the development of this equation is subject to a number of simplifying assumptions, *viz.*, the medium is homogeneous and at rest (as compared to the speed of light), the medium is nonpolarizing and the state of polarization is neglected, and the medium has a constant index of refraction. An elaborate discussion of these limitations has been given by Viskanta and Mengüç [1]. The RTE for a medium with varying refractive index has been given, e.g., by Pomraning [2], and some recent developments have been reported by Ben-Abdallah [3].

Equation (10.20) is valid anywhere inside an arbitrary enclosure. Its solution requires knowledge of the intensity for each direction at some location s, usually the intensity entering the medium through or from the enclosure boundary into the direction of $\hat{\mathbf{s}}$, as indicated in Fig. 10-4. We have not yet brought the radiative transfer equation into its most compact form so that the four different contributions to the change of intensity may be clearly identified. Equation (10.20) is the *transient* form of the radiative transfer equation, valid at local thermodynamic equilibrium as well as nonequilibrium.

Over the last few years, primarily due to the development of short-pulsed lasers, with pulse durations in the ps or fs range, transient radiation phenomena have been becoming of increasing importance [4]. However, for the vast majority of engineering applications, the speed of light is so large compared to local time and length scales that the first term in equation (10.20) may be neglected. There are also several important applications that take place at thermodynamic nonequilibrium, such as the strong nonequlibrium radiation hitting a hypersonic spacecraft entering Earth's atmosphere [5] (creating a high-temperature plasma ahead of it; cf. Fig. 11-7). Nevertheless, most engineering applications are at local thermodynamic equilibrium. We have presented here the full equation for completeness, but will omit the transient and nonequilibrium terms during the remainder of this book (with the exception of a very brief discussion of nonequlibrium properties in Chapter 11, and a somewhat more detailed consideration of transient radiation in Chapter 19).

After introducing the extinction coefficient defined in equation (10.7), one may restate equation (10.20) in its equilibrium, *quasi-steady* form as

$$\frac{dI_\eta}{ds} = \hat{\mathbf{s}} \cdot \nabla I_\eta = \kappa_\eta I_{b\eta} - \beta_\eta I_\eta + \frac{\sigma_{s\eta}}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i, \tag{10.21}$$

where the intensity gradient has been converted into a total derivative since we assume the process to be quasi-steady. The radiative transfer equation is often rewritten in terms of nondimensional optical coordinates (see Fig. 10-4),

$$\tau_\eta = \int_0^s (\kappa_\eta + \sigma_{s\eta})\, ds = \int_0^s \beta_\eta\, ds, \tag{10.22}$$

and the *single scattering albedo*, first defined in equation (1.58) as

$$\omega_\eta \equiv \frac{\sigma_{s\eta}}{\kappa_\eta + \sigma_{s\eta}} = \frac{\sigma_{s\eta}}{\beta_\eta}, \tag{10.23}$$

leading to

$$\frac{dI_\eta}{d\tau_\eta} = -I_\eta + (1 - \omega_\eta) I_{b\eta} + \frac{\omega_\eta}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i. \tag{10.24}$$

The last two terms in equation (10.24) are often combined and are then known as the *source function* for radiative intensity,

$$S_\eta(\tau_\eta, \hat{\mathbf{s}}) = (1 - \omega_\eta) I_{b\eta} + \frac{\omega_\eta}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i. \tag{10.25}$$

Equation (10.24) then assumes the deceptively simple form of

$$\frac{dI_\eta}{d\tau_\eta} + I_\eta = S_\eta(\tau_\eta, \hat{\mathbf{s}}), \tag{10.26}$$

which is, of course, an integro-differential equation (in space, and in two directional coordinates with local origin). Furthermore, the Planck function $I_{b\eta}$ is generally not known and must be found by considering the overall energy equation (adding derivatives in the three space coordinates and integrations over two more directional coordinates and the wavenumber spectrum).

10.5 FORMAL SOLUTION TO THE RADIATIVE TRANSFER EQUATION

If the source function is known (or assumed known), equation (10.26) can be formally integrated by the use of an integrating factor. Thus, multiplying through by e^{τ_η} results in

$$\frac{d}{d\tau_\eta}\left(I_\eta\, e^{\tau_\eta}\right) = S_\eta(\tau_\eta, \hat{\mathbf{s}})\, e^{\tau_\eta}, \tag{10.27}$$

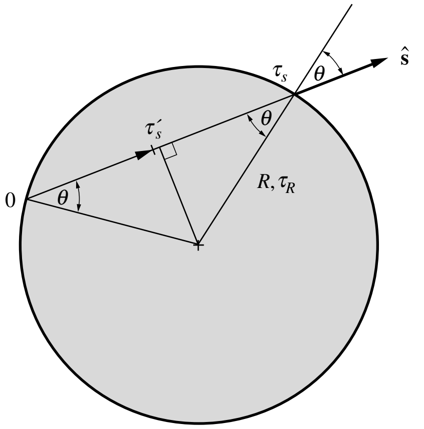

FIGURE 10-5
Isothermal sphere for Example 10.1.

which may be integrated from a point $s' = 0$ at the wall to a point $s' = s$ inside the medium (see Fig. 10-4), so that

$$I_\eta(\tau_\eta) = I_\eta(0)\, e^{-\tau_\eta} + \int_0^{\tau_\eta} S_\eta(\tau_\eta', \hat{\mathbf{s}})\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta', \tag{10.28}$$

where τ_η' is the optical coordinate at $s = s'$.

Physically, one can readily appreciate that the first term on the right-hand side of equation (10.28) is the contribution to the local intensity by the intensity entering the enclosure at $s = 0$, which decays exponentially due to extinction over the optical distance τ_η. The integrand of the second term, $S_\eta(\tau_\eta')\, d\tau_\eta'$, on the other hand, is the contribution from the local emission at τ_η', attenuated exponentially by self-extinction over the optical distance between the emission point and the point under consideration, $\tau_\eta - \tau_\eta'$. The integral, finally, sums all the contributions over the entire emission path.

Equation (10.28) is a third-order integral equation in intensity I_η. The integral over the source function must be carried out over the optical coordinate (for all directions), while the source function itself is also an integral over a set of direction coordinates (with varying local origin) containing the unknown intensity. Furthermore, usually the temperature and, therefore, the blackbody intensity are not known and must be found in conjunction with overall conservation of energy. There are, however, a few cases for which the radiative transfer equation becomes considerably simplified.

Nonscattering Medium

If the medium only absorbs and emits, the source function reduces to the local blackbody intensity, and

$$I_\eta(\tau_\eta) = I_\eta(0)\, e^{-\tau_\eta} + \int_0^{\tau_\eta} I_{b\eta}(\tau_\eta')\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta'. \tag{10.29}$$

This equation is an explicit expression for the radiation intensity if the temperature field is known. However, generally the temperature is not known and must be found in conjunction with overall conservation of energy.

Example 10.1. What is the spectral intensity emanating from an isothermal sphere bounded by vacuum or a cold black wall?

Solution
Because of the symmetry in this problem, the intensity emanating from the sphere surface is only a function of the exit angle. Examining Fig. 10-5, we see that equation (10.29) reduces to

$$I_\eta(\tau_R, \theta) = \int_0^{\tau_s} I_{b\eta}(\tau_s')\, e^{-(\tau_s - \tau_s')}\, d\tau_s'.$$

But for a sphere

$$\tau_s = 2\tau_R \cos\theta,$$

regardless of the azimuthal angle. Therefore, with $I_{b\eta}(\tau_s') = I_{b\eta} = \text{const}$, the desired intensity turns out to be

$$I_\eta(\tau_R, \theta) = I_{b\eta}\, e^{-(2\tau_R\cos\theta - \tau_s')}\Big|_0^{2\tau_R\cos\theta} = I_{b\eta}\left(1 - e^{-2\tau_R\cos\theta}\right).$$

Thus, for $\tau_R \gg 1$ the isothermal sphere emits equally into all directions, like a black surface at the same temperature.

The Cold Medium

If the temperature of the medium is so low that the blackbody intensity at that temperature is small as compared with incident intensity, then the radiative transfer equation is decoupled from other modes of heat transfer. However, the governing equation remains a third-order integral equation, namely,

$$I_\eta(\tau_\eta, \hat{\mathbf{s}}) = I_\eta(0)\, e^{-\tau_\eta} + \int_0^{\tau_\eta} \frac{\omega_\eta}{4\pi} \int_{4\pi} I_\eta(\tau_\eta', \hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta'. \tag{10.30}$$

If the scattering is isotropic, or $\Phi \equiv 1$, the directional integration in equation (10.30) may be carried out, so that

$$I_\eta(\tau_\eta, \hat{\mathbf{s}}) = I_\eta(0)\, e^{-\tau_\eta} + \frac{1}{4\pi} \int_0^{\tau_\eta} \omega_\eta G_\eta(\tau_\eta')\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta', \tag{10.31}$$

where

$$G_\eta(\tau) \equiv \int_{4\pi} I_\eta(\tau_\eta', \hat{\mathbf{s}}_i)\, d\Omega_i \tag{10.32}$$

is known as the *incident radiation function* (since it is the total intensity impinging on a point from all sides). The problem is then much simplified since it is only necessary to find a solution for G [by direction-integrating equation (10.31)] rather than determining the direction-dependent intensity.

Purely Scattering Medium

If the medium scatters radiation, but does not absorb or emit, then the radiative transfer is again decoupled from other heat transfer modes. In this case $\omega_\eta \equiv 1$, and the radiative transfer equation reduces to a form essentially identical to equation (10.30), i.e.,

$$I_\eta(\tau_\eta, \hat{\mathbf{s}}) = I_\eta(0)\, e^{-\tau_\eta} + \frac{1}{4\pi} \int_0^{\tau_\eta} \int_{4\pi} I_\eta(\tau_\eta', \hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta'. \tag{10.33}$$

Again, for isotropic scattering, this equation may be simplified by introducing the incident radiation, so that

$$I_\eta(\tau_\eta, \hat{\mathbf{s}}) = I_\eta(0)\, e^{-\tau_\eta} + \frac{1}{4\pi} \int_0^{\tau_\eta} G_\eta(\tau_\eta', \hat{\mathbf{s}})\, e^{-(\tau_\eta - \tau_\eta')}\, d\tau_\eta'. \tag{10.34}$$

Example 10.2. A large isothermal black plate is covered with a thin layer of isotropically scattering, nonabsorbing (and, therefore, nonemitting) material with unity index of refraction. Assuming that the layer is so thin that any ray emitted from the plate is scattered at most once before leaving the scattering layer, estimate the radiative intensity above the layer in the direction normal to the plate.

Solution

The exiting intensity in the normal direction (see Fig. 10-6) may be calculated from equation (10.34) by retaining only terms of order τ_η or higher (since $\tau_\eta \ll 1$). This process leads to $e^{-\tau_\eta} = 1 - \tau_\eta + \mathcal{O}(\tau_\eta^2)$,

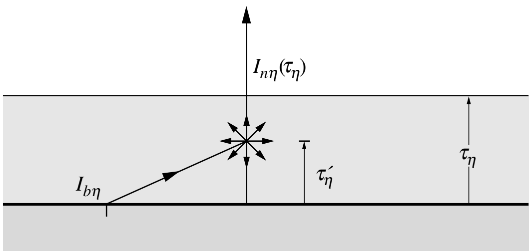

FIGURE 10-6
Geometry for Example 10.2.

$G(\tau'_\eta) = G(\tau_\eta) + \mathscr{O}(\tau_\eta)$ (radiation to be scattered arrives unattenuated at a point), and $e^{-(\tau_\eta - \tau'_\eta)} = 1 - \mathscr{O}(\tau_\eta)$ (scattered radiation will leave the medium without further attenuation), so that

$$I_{n\eta} = I_{b\eta}(1 - \tau_\eta) + \frac{1}{4\pi} G_\eta \tau_\eta + \mathscr{O}(\tau_\eta^2),$$

where the intensity emanating from the plate is known since the plate is black. The incident radiation at any point is due to unattenuated emission from the bottom plate arriving from the lower 2π solid angles, and nothing coming from the top 2π solid angles, i.e., $G_\eta \approx 2\pi I_{b\eta}$ and

$$I_{n\eta} = I_{b\eta}(1 - \tau_\eta) + \frac{1}{2} I_{b\eta}\tau_\eta + \mathscr{O}(\tau_\eta^2) = I_{b\eta}\left(1 - \frac{\tau_\eta}{2}\right) + \mathscr{O}(\tau_\eta^2).$$

Physically this result tells us that the emission into the normal direction is attenuated by the fraction τ_η (scattered away from the normal direction), and augmented by the fraction $\tau_\eta/2$ (scattered into the normal direction): Since scattering is isotropic, exactly half of the attenuation is scattered upward and half downward; the latter is then absorbed by the emitting plate. Thus, the scattering layer acts as a heat shield for the hot plate.

10.6 BOUNDARY CONDITIONS FOR THE RADIATIVE TRANSFER EQUATION

The radiative transfer equation in its quasi-steady form, equation (10.21), is a first-order differential equation in intensity (for a fixed direction $\hat{\mathbf{s}}$). As such, the equation requires knowledge of the radiative intensity at a single point in space, into the direction of $\hat{\mathbf{s}}$. Generally, the point where the intensity can be specified independently lies on the surface of an enclosure surrounding the participating medium, as indicated by the formal solution in equation (10.28). This intensity, leaving a wall into a specified direction, may be determined by the methods given in Chapter 5 (diffusely emitting and reflecting surfaces), Chapter 6 (diffusely emitting and specularly reflecting surfaces) and Chapter 7 (surfaces with arbitrary characteristics).

Diffusely Emitting and Reflecting Opaque Surfaces

For a surface that emits and reflects diffusely, the exiting intensity is independent of direction. Therefore, at a point $\mathbf{r}_w$ on the surface, from equations (5.18) and (5.19),

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = I(\mathbf{r}_w) = J(\mathbf{r}_w)/\pi = \epsilon(\mathbf{r}_w)\, I_b(\mathbf{r}_w) + \rho(\mathbf{r}_w)\, H(\mathbf{r}_w)/\pi, \tag{10.35}$$

where $H(\mathbf{r}_w)$ is the hemispherical irradiation (i.e., incoming radiative heat flux) defined by equation (3.41), leading to

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = \epsilon(\mathbf{r}_w)\, I_b(\mathbf{r}_w) + \frac{\rho(\mathbf{r}_w)}{\pi} \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}'<0} I(\mathbf{r}_w, \hat{\mathbf{s}}')\, |\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}'|\, d\Omega', \tag{10.36}$$

where $\hat{\mathbf{n}}$ is the local outward surface normal and $\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}' = \cos\theta'$ is the cosine of the angle between any incoming direction $\hat{\mathbf{s}}'$ and the surface normal, as indicated in Fig. 10-7. Therefore, the

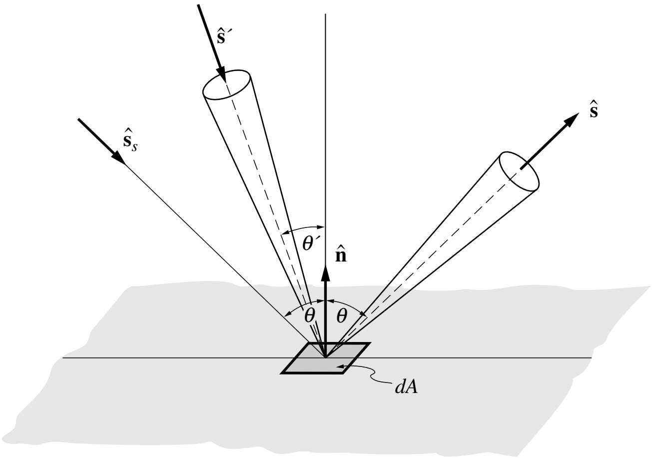

FIGURE 10-7
Radiative intensity reflected from a surface.

outgoing intensity is not generally known explicitly, but is related to the incoming intensity. An exception is the black surface, for which (with $\rho = 0$),

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = I_b(\mathbf{r}_w). \tag{10.37}$$

Diffusely Emitting, Specularly Reflecting, Opaque Surfaces

If the reflectance of the surface has a specular as well as a diffuse component, i.e., the reflectance obeys equation (6.1), then the outgoing intensity also consists of two components. One part of the outgoing intensity is due to diffuse emission as well as the diffuse fraction of reflected energy, as described by equation (10.36). In addition, the outgoing intensity has a specularly reflected component,[3] so that

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = \epsilon(\mathbf{r}_w)\, I_b(\mathbf{r}_w) + \frac{\rho^d(\mathbf{r}_w)}{\pi} \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}'<0} I(\mathbf{r}_w, \hat{\mathbf{s}}')\, |\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}'|\, d\Omega' + \rho^s(\mathbf{r}_w)\, I(\mathbf{r}_w, \hat{\mathbf{s}}_s), \tag{10.38}$$

where $\hat{\mathbf{s}}_s$ is the "specular direction," defined as the direction from which a light beam must hit the surface in order to travel into the direction of $\hat{\mathbf{s}}$ after a specular reflection. This direction is, from Fig. 10-7, $\hat{\mathbf{s}} + (-\hat{\mathbf{s}}_s) = 2\cos\theta\hat{\mathbf{n}}$, or

$$\hat{\mathbf{s}}_s = \hat{\mathbf{s}} - 2(\hat{\mathbf{s}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}. \tag{10.39}$$

Opaque Surfaces with Arbitrary Surface Properties

Reflection from a surface with nonideal radiative properties is governed by the bidirectional reflection function, as discussed in Chapter 7. From equation (7.10) it follows immediately that

$$I(\mathbf{r}_w, \hat{\mathbf{s}}) = \epsilon'(\mathbf{r}_w, \hat{\mathbf{s}})\, I_b(\mathbf{r}_w) + \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}<0} \rho''(\mathbf{r}_w, \hat{\mathbf{s}}', \hat{\mathbf{s}})\, I(\mathbf{r}_w, \hat{\mathbf{s}}')\, |\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}'|\, d\Omega'. \tag{10.40}$$

If the surface reflects diffusely, $\rho'' = \rho^d/\pi$ and equation (10.40) reduces to equation (10.36). For specular reflection the development of equation (7.15) shows that it reduces to equation (10.38).

[3]Note that the specularly reflected component cannot be "assigned" to the surface where it leaves in diffuse fashion, as was done for surface transport in Chapter 6. The reason is that the intensity changes while radiation travels from surface to surface within a participating medium.

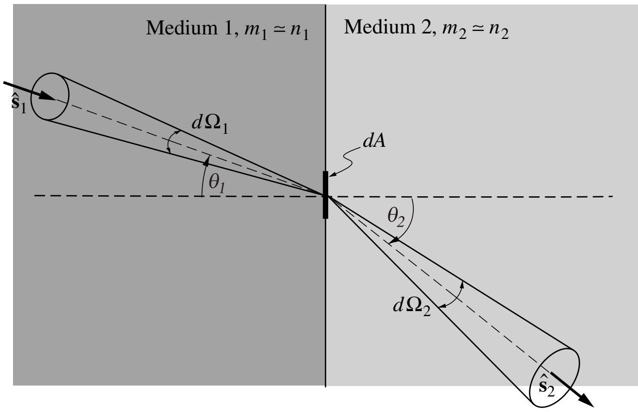

FIGURE 10-8
Radiation within a pencil of rays passing from medium to another with different refractive index (shown for $n_1 > n_2$).

Semitransparent Boundaries

If the boundary is a semitransparent wall, external radiation may penetrate into the enclosure and must be added to equations (10.36), (10.38), and (10.40) as $I_o(\mathbf{r}_w, \hat{\mathbf{s}})$. The emittance ϵ in these boundary conditions is then an effective value for the internal emission from the entire semitransparent wall thickness. If the bounding surface is totally transparent (or simply an opening), then there is no emission from the boundary and $\epsilon = 0$. This type of boundary condition was discussed in some detail in Section 6.6.

Interface Between Two Semitransparent Media

An interface between two semitransparent media is of interest only, if radiation can penetrate an appreciable distance through either medium—if not, the optically dense medium may be modeled as an "opaque surface." This implies that the absorptive indices of both media are very small [see equation (2.43)], and $m = n - ik \simeq n$. We will also assume that the interface is optically smooth, i.e., reflection can be modeled by Snell's law, as given by equations (2.72) and (3.59), together with Fresnel's relations, equations (2.96) and (3.60) (with $n = n_2/n_1$). If we perform an energy balance for a pencil of rays transmitted from Medium 1 into Medium 2 (as shown in Fig. 10-8 for the case of $n_1 > n_2$), we have from the definition of radiative intensity,

$$I_{\nu 1}(\theta_1)(1-\rho_{12})dt(dA\cos\theta_1)d\Omega_1 d\nu = I_{\nu 2}(\theta_2)dt(dA\cos\theta_2)d\Omega_2 d\nu, \tag{10.41}$$

where dA is an infinitesimal area element on the interface, and we have chosen frequency ν as the spectral variable, because only frequency remains unchanged as light passes through media with different refractive indices. Eliminating solid angle $d\Omega = \sin\theta d\theta d\psi$ (and azimuthal angle ψ, which is unaffected by passing from one medium to the next), this simplifies to

$$I_{\nu 1}(\theta_1)(1-\rho_{12})\sin\theta_1\cos\theta_1 d\theta_1 = I_{\nu 1}(\theta_2)\sin\theta_2\cos\theta_2 d\theta_2. \tag{10.42}$$

From Snell's law, equations (2.72) and (3.59), we have

$$n_1\sin\theta_1 = n_2\sin\theta_2,\ \text{and, after differentiation,}\ n_1\cos\theta_1 d\theta_1 = n_2\cos\theta_2 d\theta_2. \tag{10.43}$$

Finally, sticking these two relations into equation (10.42), we obtain

$$\frac{I_{\nu 1}(\theta_1)(1-\rho_{12})}{n_1^2} = \frac{I_{\nu 2}(\theta_2)}{n_2^2}. \tag{10.44}$$

Note that, since $n_1 > n_2$, refraction in Medium 2 is away from the surface normal, i.e., $\theta_2 > \theta_1$, and there is a critical angle $\theta_1 = \theta_c$, as given by equation (2.100), at which $\theta_2 = 90°$ and for

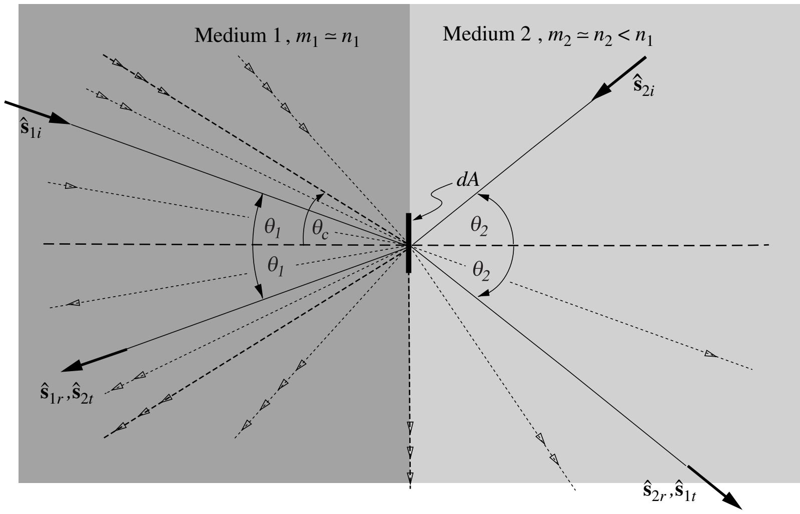

FIGURE 10-9
Intensities leaving an interface between two semitransparent media with different refractive indices (shown for $n_1 > n_2$).

larger θ_1 there will be total internal reflection, and nothing is transmitted into Medium 2:

$$\theta_1 > \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right): \quad \rho_{12} = 1; \; I_{\nu 2}(\theta_2) = 0. \tag{10.45}$$

This is indicated in Fig. 10-9 by showing several additional incident directions (with thin dashed lines and open arrows), together with their transmitted (for $\theta_1 < \theta_c$ only) and reflected directions.

Employing equations (10.44) and (10.45), we can now make a full energy balance for the interface, comprising intensity coming in from inside Medium 1, $I_{\nu 1i}(\theta_1)$, the fraction of it that is reflected, $I_{\nu 1r}(\theta_1)$ (with specular reflection angle $\theta_r = \theta_1$), and the fraction transmitted into Medium 2, $I_{\nu 1t}(\theta_2)$, along with similar contributions from intensity striking the interface from inside Medium 2, as depicted in Fig. 10-9:

$$I_{\nu 2}(\theta_2) = \rho_{21} I_{\nu 2i}(\theta_2) + I_{\nu 1t}(\theta_2) = \rho_{21} I_{\nu 2i}(\theta_2) + (1-\rho_{12})\left(\frac{n_2}{n_1}\right)^2 I_{\nu 1i}(\theta_1), \tag{10.46a}$$

$$I_{\nu 1}(\theta_1) = \rho_{12} I_{\nu 1i}(\theta_1) + I_{\nu 2t}(\theta_1) = \rho_{12} I_{\nu 1i}(\theta_1) + (1-\rho_{21})\left(\frac{n_1}{n_2}\right)^2 I_{\nu 2i}(\theta_2), \tag{10.46b}$$

where, from equation (2.96),

$$\rho_{12} = \rho_{21} = \begin{cases} \dfrac{1}{2}\left[\left(\dfrac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}\right)^2 + \left(\dfrac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos_1 + n_2\cos\theta_2}\right)^2\right], & \theta_1 < \theta_c, \\ 1 & \theta_1 \geq \theta_c. \end{cases} \tag{10.47}$$

The intensity entering the optically less dense Medium 2 from the interface, $I_{\nu 2}(\theta_2)$, will have a transmitted contribution from Medium 1 for all values of θ_2 (but coming from within a cone with opening angle θ_c). Intensity entering Medium 1, $I_{\nu 1}(\theta_1)$, on the other hand, will have a transmitted component from Medium 2 only if $\theta_1 < \theta_c$.

10.7 RADIATION ENERGY DENSITY

A volume element inside an enclosure is irradiated from all directions and, at any instant in time t, contains a certain amount of radiative energy in the form of photons. Consider, for example, an element $dV = dA\, ds$ irradiated perpendicularly to dA with intensity $I_\eta(\hat{\mathbf{s}})$ as shown in Fig. 10-3. Therefore, per unit time radiative energy in the amount of $I_\eta(\hat{\mathbf{s}})\, d\Omega\, dA$ enters dV. From the development in Chapter 1, equation (1.48), we see that this energy remains inside dV for a duration of $dt = ds/c$, before exiting at the other side. Thus, due to irradiation from a single direction, the volume contains the amount of radiative energy $I_\eta(\hat{\mathbf{s}})\, d\Omega\, dA\, ds/c = I_\eta(\hat{\mathbf{s}})\, d\Omega\, dV/c$ at any instant in time. Adding the contributions from all possible directions, we find the total radiative energy stored within dV is $u_\eta\, dV$, where u_η is the *spectral radiation energy density*

$$u_\eta \equiv \frac{1}{c}\int_{4\pi} I_\eta(\hat{\mathbf{s}})\, d\Omega. \tag{10.48}$$

Integration over the spectrum gives the *total radiation energy density*,

$$u = \int_0^\infty u_\eta\, d\eta = \frac{1}{c}\int_{4\pi}\int_0^\infty I_\eta(\hat{\mathbf{s}})\, d\eta\, d\Omega = \frac{1}{c}\int_{4\pi} I(\hat{\mathbf{s}})\, d\Omega. \tag{10.49}$$

Although the radiation energy density is a very basic quantity akin to internal energy for energy stored within matter, it is not widely used by heat transfer engineers. Instead, it is common practice to employ the *incident radiation* G_η, which is related to the energy density through

$$G_\eta \equiv \int_{4\pi} I_\eta(\hat{\mathbf{s}})\, d\Omega = c u_\eta; \quad G = cu. \tag{10.50}$$

10.8 RADIATIVE HEAT FLUX

The spectral radiative heat flux onto a surface element has been expressed in terms of incident and outgoing intensity in equation (1.39) as

$$\mathbf{q}_\eta \cdot \hat{\mathbf{n}} = \int_{4\pi} I_\eta\, \hat{\mathbf{n}} \cdot \hat{\mathbf{s}}\, d\Omega. \tag{10.51}$$

This relationship also holds, of course, for a hypothetical (i.e., totally transmissive) surface element placed arbitrarily inside an enclosure. Removing the surface normal from equation (1.39), we obtain the definition for the *spectral, radiative heat flux vector* inside a participating medium. To obtain the *total radiative heat flux*, equation (10.51) needs to be integrated over the spectrum, and

$$\mathbf{q} = \int_0^\infty \mathbf{q}_\eta\, d\eta = \int_0^\infty \int_{4\pi} I_\eta(\hat{\mathbf{s}})\, \hat{\mathbf{s}}\, d\Omega\, d\eta. \tag{10.52}$$

Depending on the coordinate system used, or the surface being described, the radiative heat flux vector may be separated into its coordinate components, for example q_x, q_y, and q_z (for a Cartesian coordinate system), or into components normal and tangential to a surface, and so on.

Example 10.3. Evaluate the total heat loss from an isothermal spherical medium bounded by vacuum, assuming that κ_η = const (i.e., does not vary with location, temperature, or wavenumber).

Solution
Here we are dealing with a spherical coordinate system, and we are interested in the radial component of the radiative heat flux (the other two being equal to zero by symmetry). We saw in Example 10.1 that the intensity emanating from the sphere is

$$I_\eta(\tau_R, \theta) = I_{b\eta}\left(1 - e^{-2\tau_R \cos\theta}\right), \quad 0 \le \theta \le \frac{\pi}{2},$$

where θ is measured from the surface normal pointing away from the sphere (Fig. 10-5). Since the sphere is bounded by vacuum, there is no incoming radiation and

$$I_\eta(\tau_R, \theta) = 0, \quad \frac{\pi}{2} \le \theta \le \pi.$$

Therefore, from equation (10.52),

$$\begin{aligned} q(\tau_R) &= \int_0^\infty \int_0^{2\pi} \int_0^\pi I_\eta(\tau_R, \theta) \cos\theta \sin\theta \, d\theta \, d\psi \, d\eta \\ &= 2\pi \int_0^\infty \int_0^{\pi/2} I_{b\eta} \left(1 - e^{-2\tau_R \cos\theta}\right) \cos\theta \sin\theta \, d\theta \, d\eta \\ &= \pi I_b \left\{1 - \frac{1}{2\tau_R^2}\left[1 - (1 + 2\tau_R)\, e^{-2\tau_R}\right]\right\} = n^2 \sigma T^4 \left\{1 - \frac{1}{2\tau_R^2}\left[1 - (1 + 2\tau_R)\, e^{-2\tau_R}\right]\right\}, \end{aligned}$$

where n is the refractive index of the medium (usually $n \approx 1$ for gases, but $n > 1$ for semitransparent liquids and solids). As discussed in the previous example, if $\tau_R \to \infty$ the heat flux approaches the same value as the one from a black surface.

If the sphere in the last example is optically thin $\tau_R \ll 1$ (i.e., the medium *emits* radiative energy, but does not *absorb* any of the emitted energy), then the total heat loss (total emission) from the sphere is

$$Q = 4\pi R^2 q = 4\pi R^2 \times \tfrac{4}{3} \tau_R n^2 \sigma T^4 = 4\kappa n^2 \sigma T^4 V. \tag{10.53}$$

This result may be generalized to govern emission from any isothermal volume V *without self-absorption*, or

$$Q_{\text{emission}} = 4\kappa n^2 \sigma T^4 V. \tag{10.54}$$

10.9 DIVERGENCE OF THE RADIATIVE HEAT FLUX

While the heat transfer engineer is interested in the radiative heat flux, this interest usually holds true only for fluxes at physical boundaries. Inside the medium, on the other hand, we need to know how much net radiative energy is deposited into (or withdrawn from) each volume element. Thus, making a radiative energy balance on an infinitesimal volume $dV = dx\,dy\,dz$ as shown in Fig. 10-10, we have

$$\begin{pmatrix}\text{radiative energy}\\ \text{stored in } dV\\ \text{per unit time}\end{pmatrix} - \begin{pmatrix}\text{rad. energy generated}\\ \text{(emitted) by } dV\\ \text{per unit time}\end{pmatrix} + \begin{pmatrix}\text{rad. energy destroyed}\\ \text{(absorbed) by } dV\\ \text{per unit time}\end{pmatrix}$$

$$= \begin{pmatrix}\text{flux in at } x & - \text{ flux out at } x + dx\\ + \text{ flux in at } y & - \text{ flux out at } y + dy\\ + \text{ flux in at } z & - \text{ flux out at } z + dz\end{pmatrix}.$$

The right-hand side may be written in mathematical form as

$$\left.\begin{aligned} q(x)\,dy\,dz - q(x+dx)\,dy\,dz\\ +\,q(y)\,dx\,dz - q(y+dy)\,dx\,dz\\ +\,q(z)\,dx\,dy - q(z+dz)\,dx\,dy \end{aligned}\right\} = -\left(\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial q}{\partial z}\right) dx\,dy\,dz = -\boldsymbol{\nabla}\cdot\mathbf{q}\,dV.$$

Thus, within the overall energy equation, it is the divergence of the radiative heat flux that is of interest inside the participating medium.[4]

We have already established an energy balance for thermal radiation, the radiative transfer equation [for example, equation (10.21)],

$$\frac{dI_\eta}{ds} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} I_\eta = \kappa_\eta I_{b\eta} - \beta_\eta I_\eta(\hat{\mathbf{s}}) + \frac{\sigma_{s\eta}}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i, \tag{10.55}$$

[4]For simplicity, this equation was derived for a Cartesian coordinate system but the result holds, of course, for any arbitrary coordinate system.

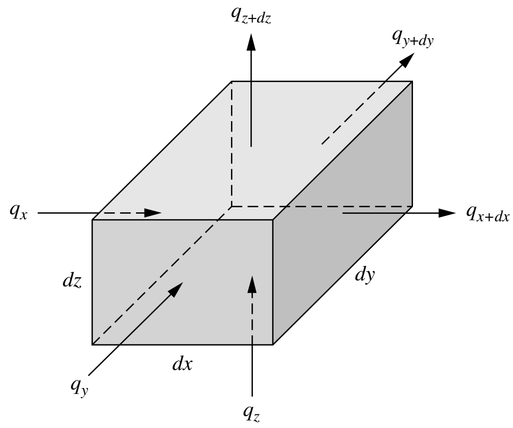

FIGURE 10-10
Control volume for derivation of divergence of radiative heat flux.

which is a radiation balance for an infinitesimal pencil of rays. Thus, in order to get a volume balance, we integrate this equation over all solid angles, or

$$\int_{4\pi} \hat{\mathbf{s}} \cdot \nabla I_\eta \, d\Omega = \int_{4\pi} \kappa_\eta I_{b\eta} \, d\Omega - \int_{4\pi} \beta_\eta I_\eta(\hat{\mathbf{s}}) \, d\Omega + \int_{4\pi} \frac{\sigma_{s\eta}}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i) \, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}}) \, d\Omega_i \, d\Omega, \tag{10.56}$$

and

$$\nabla \cdot \int_{4\pi} I_\eta \hat{\mathbf{s}} \, d\Omega = 4\pi \kappa_\eta I_{b\eta} - \int_{4\pi} \beta_\eta I_\eta(\hat{\mathbf{s}}) \, d\Omega + \frac{\sigma_{s\eta}}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i) \left(\int_{4\pi} \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}}) \, d\Omega \right) d\Omega_i. \tag{10.57}$$

On the left side of equation (10.57) the integral and the direction vector were taken into the gradient since direction and space coordinates are all independent from one another.[5] The expression inside the operator is now, of course, the spectral radiative heat flux. On the right side of equation (10.57) the order of integration has been changed, applying the Ω-integration to the only part depending on it, the scattering phase function Φ_η. This last integration can be carried out using equation (10.17), leading to

$$\nabla \cdot \mathbf{q}_\eta = 4\pi \kappa_\eta I_{b\eta} - \beta_\eta \int_{4\pi} I_\eta(\hat{\mathbf{s}}) \, d\Omega + \sigma_{s\eta} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i) \, d\Omega_i. \tag{10.58}$$

Since Ω and Ω_i are dummy arguments for integration over all solid angles, the last two terms can be pulled together, using $\kappa_\eta = \beta_\eta - \sigma_{s\eta}$:

$$\nabla \cdot \mathbf{q}_\eta = \kappa_\eta \left(4\pi I_{b\eta} - \int_{4\pi} I_\eta \, d\Omega \right) = \kappa_\eta \left(4\pi I_{b\eta} - G_\eta \right). \tag{10.59}$$

Equation (10.59) states that physically the net loss of radiative energy from a control volume is equal to emitted energy minus absorbed irradiation. This direction-integrated form of the radiative transfer equation no longer contains the scattering coefficient. This fact is not surprising since scattering only redirects the stream of photons; it does not affect the energy content of any given unit volume.

Equation (10.59) is a spectral relationship, i.e., it gives the heat flux per unit wavenumber at a certain spectral position. If the divergence of the total heat flux is desired, the integration over the spectrum is carried out to give

$$\nabla \cdot \mathbf{q} = \nabla \cdot \int_0^\infty \mathbf{q}_\eta \, d\eta = \int_0^\infty \kappa_\eta \left(4\pi I_{b\eta} - \int_{4\pi} I_\eta \, d\Omega \right) d\eta = \int_0^\infty \kappa_\eta \left(4\pi I_{b\eta} - G_\eta \right) d\eta. \tag{10.60}$$

[5]While this statement is always true, care must be taken in non-Cartesian coordinate systems: Although the direction *vector* is independent from space coordinates, the *three components* may be tied to locally defined unit vectors. For example, in a cylindrical coordinate system the direction vector is usually defined in terms of $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$, which vary with r and θ.

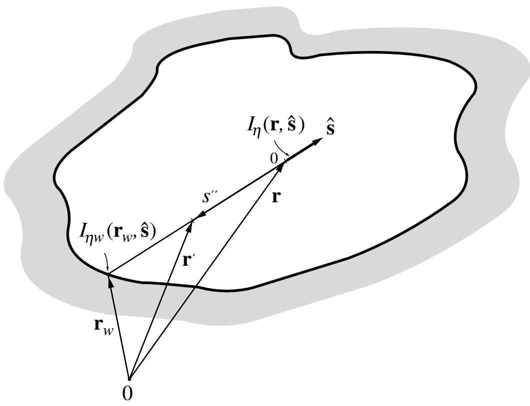

FIGURE 10-11
Enclosure for the derivation of the integral form of the radiative transfer equation.

Equation (10.60) is a statement of the *conservation of radiative energy.* For the special case of a gray medium ($\kappa_\eta = \kappa =$ constant) this may be simplified to

$$\nabla \cdot \mathbf{q} = \kappa \left(4\sigma T^4 - \int_{4\pi} I \, d\Omega \right) = \kappa \left(4\sigma T^4 - G \right). \tag{10.61}$$

Example 10.4. Calculate the divergence of the total radiative heat flux at the center and at the surface of the gray, isothermal spherical medium in the previous example.

Solution
We already know the intensity at the surface of the sphere and, therefore,

$$G_\eta(\tau_R) = 2\pi \int_0^\pi \sin\theta I_\eta \, d\theta = 2\pi I_{b\eta} \int_0^{\pi/2} \left(1 - e^{-2\tau_R \cos\theta}\right) \sin\theta \, d\theta$$
$$= 2\pi I_{b\eta} \left(1 - \frac{e^{-2\tau_R \cos\theta}}{2\tau_R} \Bigg|_0^{\pi/2} \right) = \frac{\pi I_{b\eta}}{\tau_R} \left(2\tau_R - 1 + e^{-2\tau_R}\right),$$

and

$$\nabla \cdot \mathbf{q}(\tau_R) = \kappa \left(4\pi I_b - G\right) = \frac{\sigma T^4}{R} \left(2\tau_R + 1 - e^{-2\tau_R}\right). \tag{10.62}$$

At the center of the sphere the intensity is easily evaluated as

$$I_\eta(0) = I_{b\eta} \left(1 - e^{-\tau_R}\right),$$

and

$$G_\eta(0) = 4\pi I_{b\eta} \left(1 - e^{-\tau_R}\right),$$

so that

$$\nabla \cdot \mathbf{q}(0) = \kappa 4\sigma T^4 e^{-\tau_R}. \tag{10.63}$$

The right-hand sides of equations (10.62) and (10.63) are radiative heat losses per unit time and volume, which must be made up for by a volumetric heat source if the sphere is to stay isothermal.

10.10 INTEGRAL FORMULATION OF THE RADIATIVE TRANSFER EQUATION

In order to obtain incident radiation, radiative heat flux, or its divergence, it is sometimes desirable to use an integral formulation of the radiative transfer equation. We start with the formal solution, equation (10.28), but rewritten in terms of the vectors shown in Fig. 10-11,

$$I_\eta(\mathbf{r}, \hat{\mathbf{s}}) = I_{w\eta}(\mathbf{r}_w, \hat{\mathbf{s}}) \exp\left[-\int_0^s \beta_\eta \, ds''\right] + \int_0^s S_\eta(\mathbf{r}', \hat{\mathbf{s}}) \exp\left[-\int_0^{s''} \beta_\eta \, ds''\right] \beta_\eta \, ds'', \tag{10.64}$$

where $s = |\mathbf{r} - \mathbf{r}_w|$ and the direction of integration has been switched to go along s'' (from point $\mathbf{r}$ toward the wall). From the definition of the incident radiation, equation (10.32), we have

$$G_\eta(\mathbf{r}) = \int_{4\pi} I_{w\eta}(\mathbf{r}_w, \hat{\mathbf{s}}) \exp\left[-\int_0^s \beta_\eta\, ds''\right] d\Omega + \int_{4\pi}\int_0^s S_\eta(\mathbf{r}', \hat{\mathbf{s}}) \exp\left[-\int_0^{s''} \beta_\eta\, ds''\right] \beta_\eta\, ds''\, d\Omega, \quad (10.65)$$

with, from equation (1.26),

$$d\Omega = \begin{cases} \dfrac{dA''}{|\mathbf{r} - \mathbf{r}'|^2}, & \text{inside volume,} \\[2ex] \dfrac{\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}\, dA_w}{|\mathbf{r} - \mathbf{r}_w|^2}, & \text{at the wall,} \end{cases} \quad (10.66)$$

where dA'' is an infinitesimal area perpendicular to the integration path (and ds''), such that $dV = ds''dA''$ is an infinitesimal volume. Therefore, equation (10.65) may be rewritten as

$$G_\eta(\mathbf{r}) = \int_{A_w} I_{w\eta}(\mathbf{r}_w, \hat{\mathbf{s}}) \exp\left[-\int_0^s \beta_\eta\, ds''\right] \frac{\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}\, dA_w}{|\mathbf{r} - \mathbf{r}_w|^2} + \int_V S_\eta(\mathbf{r}', \hat{\mathbf{s}}) \exp\left[-\int_0^{s''} \beta_\eta\, ds''\right] \frac{\beta_\eta\, dV}{|\mathbf{r} - \mathbf{r}'|^2}, \quad (10.67)$$

with the local unit direction vector found from

$$\hat{\mathbf{s}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \quad (10.68)$$

The radiative flux (and any higher moment) can be determined similarly, after first multiplying equation (10.64) by $\hat{\mathbf{s}}$, as

$$\mathbf{q}_\eta(\mathbf{r}) = \int_{A_w} I_{w\eta}(\mathbf{r}_w, \hat{\mathbf{s}}) \exp\left[-\int_0^s \beta_\eta\, ds''\right] \frac{(\hat{\mathbf{n}} \cdot \hat{\mathbf{s}})\hat{\mathbf{s}}\, dA_w}{|\mathbf{r} - \mathbf{r}_w|^2} + \int_V S_\eta(\mathbf{r}', \hat{\mathbf{s}}) \exp\left[-\int_0^{s''} \beta_\eta\, ds''\right] \frac{\beta_\eta \hat{\mathbf{s}}\, dV}{|\mathbf{r} - \mathbf{r}'|^2}. \quad (10.69)$$

For a nonscattering medium $S_\eta = I_{b\eta}$, and equation (10.67) is the explicit solution for incident radiation G_η, provided the temperature field is known, and if the walls are black. For isotropic scattering the source function depends only on $I_{b\eta}$ (or temperature) and incident radiation. For such a case (and if the walls are black) equation (10.67) is a single, independent integral equation for the incident radiation; once G_η has been determined $\mathbf{q}_\eta$ is found from equation (10.69). For reflecting walls and anisotropic scattering, equations (10.67) and (10.69) (and, perhaps, higher-order moments) must be solved simultaneously. Also, for a nonparticipating medium ($\beta_\eta = 0$) with diffusely reflecting surfaces ($I_w = J/\pi$), equation (5.25) is readily recovered from equation (10.69); this is left as an exercise (Problem 10.15).

Example 10.5. Repeat Example 10.3 using the integral formulation of the RTE.

Solution

In this simple problem with a cold, black (i.e., nonreflecting) wall with $I_{w\eta} = 0$, and in the absence of scattering with $S_\eta = I_{b\eta} = \text{const}$ we can determine q_η directly from equation (10.69) as

$$q_\eta(R) = -\mathbf{q}_\eta(\mathbf{r}_w) \cdot \hat{\mathbf{n}} = -I_{b\eta}\kappa_\eta \int_V e^{-\kappa_\eta s''} \frac{\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}\, dV}{(s'')^2},$$

where s'' is the distance between any point inside the medium (at $\mathbf{r}'$) and the chosen point on the wall, $\mathbf{r} = \mathbf{r}_w$. It is tempting at this point to introduce a spherical coordinate system at the center of the sphere to evaluate the volume integral for q_η; however, this would lead to a very difficult integral. Instead, we introduce a spherical coordinate system at the chosen point at the wall, i.e., $\mathbf{r}_w = \mathbf{0}$ (point τ_s in Fig. 10-5). An arbitrary location inside the sphere can then be specified as

$$\mathbf{r}' = -\hat{\mathbf{s}}s'' = s''(\cos\psi \sin\theta\, \hat{\mathbf{i}} + \sin\psi \sin\theta\, \hat{\mathbf{j}} + \cos\theta\, \hat{\mathbf{k}}),$$

where $\hat{\mathbf{k}} = \hat{\mathbf{n}}$ is pointing toward the center of the sphere and $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ are arbitrary (as long as they form a right-handed coordinate system). Then, with a maximum value for $s''_{\max} = 2R\cos\theta$, as given in Example 10.1,

$$\begin{aligned} q_\eta(R) &= -I_{b\eta}\kappa_\eta \int_{\psi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{s''=0}^{2R\cos\theta} e^{-\kappa_\eta s''}\,\frac{(-\cos\theta)\sin\theta\,d\theta\,d\psi(s'')^2\,ds''}{(s'')^2} \\ &= 2\pi I_{b\eta}\int_{\theta=0}^{\pi/2}(1-e^{-2\tau_R\cos\theta})\cos\theta\sin\theta\,d\theta, \end{aligned}$$

exactly as in Example 10.3.

10.11 OVERALL ENERGY CONSERVATION

Thermal radiation is only one mode of transferring heat which, in general, must compete with conductive and convective heat transfer. Therefore, the temperature field must be determined through an energy conservation equation that incorporates all three modes of heat transfer. The radiation intensity, through emission and temperature-dependent properties, depends on the temperature field and, therefore, cannot be decoupled from the overall energy equation.

The general form of the energy conservation equation for a moving compressible fluid may be stated as

$$\rho\frac{Du}{Dt} = \rho\left(\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u\right) = -\nabla\cdot\mathbf{q} - p\nabla\cdot\mathbf{v} + \mu\Phi + \dot{Q}''', \tag{10.70}$$

where u is internal energy, $\mathbf{v}$ is the velocity vector, $\mathbf{q}$ is the total heat flux vector, Φ is the dissipation function, and $\dot{Q}'''$ is heat generated within the medium (such as energy release due to chemical reactions). For a detailed derivation of equation (10.70), the reader is referred to standard textbooks such as [6,7]. If the medium is radiatively participating through emission, absorption, and scattering, then the conservation equations for momentum and energy are altered by three effects [8]:

1. The heat flux term in equation (10.70), which without radiation is in most applications due only to molecular diffusion (heat conduction), now has a second component, the radiative heat flux, due to radiative energy interacting with the medium within the control volume.

2. The internal energy now contains a radiative contribution [the incident radiation G, due to the first term in equation (10.20)].

3. The radiation pressure tensor must be added to the traditional fluid dynamics pressure tensor.

We have already seen that the second effect is almost always negligible, and the same is true for the augmentation of the pressure tensor. Under these conditions the energy conservation equation can be simplified. If we assume that $du = c_v\,dT$, and that Fourier's law for heat conduction holds,

$$\mathbf{q} = \mathbf{q}_C + \mathbf{q}_R = -k\nabla T + \mathbf{q}_R, \tag{10.71}$$

equation (10.70) becomes

$$\begin{aligned} \rho c_v\frac{DT}{Dt} &= \rho c_v\left(\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right) \\ &= \nabla\cdot(k\nabla T) - p\nabla\cdot\mathbf{v} + \mu\Phi + \dot{Q}''' - \nabla\cdot\mathbf{q}_R. \end{aligned} \tag{10.72}$$

This is an integro-differential equation for the calculation of the temperature field, since the evaluation of the divergence of the radiative heat flux must come from (10.59), which is an integral equation in temperature. Obviously, a complete solution of this equation, even with the recent advent of supercomputers, is a truly formidable task.

Example 10.6. State the radiative transfer equation and its boundary conditions for the case of combined steady-state conduction and radiation within a one-dimensional, planar, gray, and nonscattering medium, bounded by isothermal black walls.

Solution
Since the problem is steady state and there is no movement in the medium, the left side of equation (10.72) vanishes, and only the first (conduction) and last (radiation) terms on the right side remain. For a one-dimensional planar medium this reduces to[6]

$$\frac{d}{dz}\left(k\frac{dT}{dz} - q_R\right) = 0, \tag{10.73}$$

and the divergence of radiative heat flux is related to temperature and incident radiation through equation (10.59),

$$\frac{dq_R}{dz} = \kappa(4\sigma T^4 - G),$$

where the spectral integration for the gray medium has been carried out by simply dropping the subscript η. Finally, the incident radiation is found from direction-integrating equation (10.29) (not a trivial task). The necessary boundary conditions are $T = T_i$, $i = 1, 2$ at the two walls (for conduction) and $I(0, \hat{\mathbf{s}}) = \sigma T_i^4/\pi$ (for radiation) needed in equation (10.29). Solution of this seemingly simple problem is by no means trivial, and can only be achieved through relatively involved numerical analysis.

Radiative Equilibrium

Much attention in the following chapters will be given to the situation in which radiation is the dominant mode of heat transfer, meaning that when conduction and convection are negligible. This situation is referred to as *radiative equilibrium*, meaning that thermodynamic equilibrium within the medium is achieved by virtue of thermal radiation alone. As is commonly done in the discussion of "pure" conduction or convection, we allow volumetric heat sources throughout the medium. Thus, we may write

$$\rho c_v \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}_R = \dot{Q}''', \tag{10.74}$$

which is identical in form to the basic transient heat conduction equation (before substitution of Fourier's law). In the vast majority of cases radiative transfer occurs so fast that radiative equilibrium is achieved before a noticeable change in temperature occurs [i.e., when the unsteady term in equation (10.20) can be dropped]. Then the statement of radiative equilibrium reduces to its steady-state form

$$\nabla \cdot \mathbf{q}_R = \dot{Q}'''. \tag{10.75}$$

Radiative equilibrium is often a good assumption in applications with extremely high temperatures, such as plasmas, nuclear explosions, and such. The inclusion of a volumetric heat source allows the treatment of conduction and convection "through the back door:" A guess is made for the temperature field and the nonradiation terms in equation (10.72) are calculated to give $\dot{Q}'''$ for the radiation calculations. This process is then repeated until a convergence criterion is met.

[6]While in the science of conduction the variable x is usually employed for one-dimensional planar problems, for thermal radiation problems the variable z is more convenient. The reason for this is that, by convention, the polar angle for the direction vector is measured from the z-axis.

10.12 SOLUTION METHODS FOR THE RADIATIVE TRANSFER EQUATION

Exact analytical solutions to the radiative transfer equation [equation (10.21)] are exceedingly difficult, and explicit solutions are impossible for all but the very simplest situations. Therefore, research on radiative heat transfer in participating media has generally proceeded in two directions: (*i*) exact (analytical and numerical) solutions of highly idealized situations, and (*ii*) approximate solution methods for more involved scenarios. Phenomena that make a radiative heat transfer problem difficult may be placed into four different categories:

Geometry: The problem may be one-dimensional, two-dimensional, or three-dimensional. Most investigations to date have dealt with one-dimensional geometries, and the vast majority of these dealt with the simplest case of a one-dimensional plane-parallel slab.

Temperature Field: The least difficult situation arises if the temperature profile within the medium is known, making equation (10.21) a relatively "simple" integral equation. Consequently, the most basic case of an isothermal medium has been studied extensively. Alternatively, if radiative equilibrium prevails, the temperature field is unknown but uncoupled from conduction and convection, and must be found from directional and spectral integration of the radiative transfer equation. In the most complicated scenario, radiative heat transfer is combined with conduction and/or convection, resulting in a highly nonlinear integro-differential equation.

Scattering: The solution to a radiation problem is greatly simplified if the medium does not scatter. In that case the radiative transfer equation reduces to a simple first-order differential equation if the temperature field is known, and a relatively simpler integral equation if radiative equilibrium prevails. If scattering must be considered, isotropic scattering is often assumed. Relatively few investigations have dealt with the case of anisotropic scattering, and most of those are limited to the case of linear-anisotropic scattering (see Section 12.9).

Properties: Although most participating media display strong nongray character, as discussed in the following three chapters, the vast majority of investigations to date have centered on the study of gray media. In addition, while radiative properties also generally depend strongly on temperature, concentration, etc., most calculations are limited to situations with constant properties.

Most "exact" solutions are limited to gray media with constant properties in one-dimensional, mainly plane-parallel geometries. The media are isothermal or at radiative equilibrium, and if they scatter, the scattering is usually isotropic. Since the usefulness of such one-dimensional solutions in heat transfer applications is limited, they are only briefly discussed in Chapter 14.

Several chapters are devoted to the various approximate methods that have been devised for the solution of the radiative transfer equation. Still, these seven chapters by no means cover all the different methods that have been and still are used by investigators in the field. A number of approximate methods for one-dimensional problems are discussed in Chapter 15. The *optically thin* and *diffusion* (or optically thick) approximations have historically been developed for a one-dimensional plane-parallel medium, but can readily be applied to more complicated geometries. Similarly, the *Schuster–Schwarzschild* or *two-flux approximation* [9, 10] is a forerunner to the multidimensional *discrete ordinates method*. In this method the intensity is assumed to be constant over discrete parts of the total solid angle of 4π. Several other flux methods exist, but they are usually tailored toward special geometries, and cannot easily be applied to other scenarios, for example, the *six-flux methods* of Chu and Churchill [11] and Shih and coworkers [12, 13]. Another early one-dimensional model was the *moment method* or *Eddington approximation* [14]. In this model the directional dependence is expressed by a truncated series representation (rather than discretized). In general geometries this expansion is usually achieved through the use of spherical harmonics, leading to the *spherical harmonics method*. Several variations to the moment method that are tailored toward specific geometries have been proposed [15, 16], but these are of limited general utility. Finally, the exponential kernel approximation, already discussed in Chapter 5 for surface radiation problems, may be used as a tool for many one-dimensional problems. However, its extension to multidimensional geometries is problematic.

A survey of the literature over the past forty years demonstrates that some solution methods have been used frequently, while others that appeared promising at one time are no longer employed on a regular basis. Apparently, some methods have been found to be more readily adapted to more difficult situations than others (such as multidimensionality, variable properties, anisotropic scattering, and/or nongray effects). The majority of radiative heat transfer analyses today appear to use one of four methods: (*i*) the *spherical harmonics method* or a variation of it, (*ii*) the *discrete ordinates method* or its more modern form, the *finite volume method*, (*iii*) the *zonal method*, and (*iv*) the *Monte Carlo method*. The first two of these have already been discussed briefly above with the one-dimensional approximations. The zonal method was developed by Hottel [17] in his pioneering work on furnace heat transfer. Unlike the spherical harmonics and discrete ordinates methods, the zonal method approximates spatial, rather than directional, behavior by breaking up an enclosure into finite, isothermal subvolumes. On the other hand, the Monte Carlo method [18] is a statistical method, in which the history of bundles of photons is traced as they travel through the enclosure. While the statistical nature of the Monte Carlo method makes it difficult to match it with other calculations, it is the only method that can satisfactorily deal with effects of irregular radiative properties (nonideal directional and/or nongray behavior).

Because of their importance, an entire chapter is devoted to each of these four solution methods. Several other methods that can be found in the literature are not covered in this book (except for brief descriptions in appropriate places). For example, the *discrete transfer method*, proposed by Shah [19] and Lockwood and Shah [20], combines features of the discrete ordinates, zonal, and Monte Carlo methods. Another hybrid proposed by Edwards [21] combines elements of the Monte Carlo and zonal methods.

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Problems

10.1 A semi-infinite medium $0 \leq z < \infty$ consists of a gray, absorbing–emitting gas that does not scatter, bounded by vacuum at the interface $z = 0$. The gas is isothermal at 1000 K, and the absorption coefficient is $\kappa = 1\,\mathrm{m}^{-1}$. The interface is nonreflecting; conduction and convection may be neglected.

(*a*) What is the local heat generation that is necessary to keep the gas at 1000 K?
(*b*) What is the intensity distribution at the interface, that is, $I(z = 0,\ \theta,\ \psi)$, for all θ and ψ?
(*c*) What is the total heat flux leaving the semi-infinite medium?

10.2 Reconsider the semi-infinite medium of Problem 10.1 for a temperature distribution of $T = T_0\, e^{-z/L}$, $T_0 = 1000\,\mathrm{K}$, $L = 1\,\mathrm{m}$. What are the exiting intensity and heat flux for this case? Discuss how the answer would change if κ varied between 0 and ∞.

10.3 Repeat Problem 10.1 for a medium of thickness $L = 1\,\mathrm{m}$. Discuss how the answer would change if κ varied between 0 and ∞.

10.4 A semi-infinite, gray, nonscattering medium ($n = 2,\ \kappa = 1\,\mathrm{m}^{-1}$) is irradiated by the sun normal to its surface at a rate of $q_{\mathrm{sun}} = 1000\,\mathrm{W/m^2}$. Neglecting emission from the relatively cold medium, determine the local heat generation rate due to absorption of solar energy.
Hint: The solar radiation may be thought of as being due to a radiative intensity which has a large value I_o over a very small cone of solid angles $\delta\Omega$, and is zero elsewhere, i.e.,

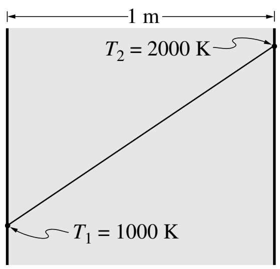
$$I(\hat{\mathbf{s}}) = \begin{cases} I_o & \text{over } \delta\Omega \text{ along } \hat{\mathbf{n}}, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$q_{\mathrm{sun}} = \int_{4\pi} I(\hat{\mathbf{s}})\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}\, d\Omega = I_o\,\delta\Omega.$$

10.5 A 1 m thick slab of an absorbing–emitting gas has an approximately linear temperature distribution as shown in the sketch. On both sides the medium is bounded by vacuum with nonreflecting boundaries.

(*a*) If the medium has a constant and gray absorption coefficient of $\kappa = 1\,\mathrm{m}^{-1}$, what is the intensity (as a function of direction) leaving the hot side of the slab?
(*b*) Give an expression for the radiative heat flux leaving the hot side.



10.6 A semitransparent sphere of radius $R = 10\,\mathrm{cm}$ has a parabolic temperature profile $T = T_c(1 - r^2/R^2)$, $T_c = 2000\,\mathrm{K}$. The sphere is gray with $\kappa = 0.1\,\mathrm{cm}^{-1}$, $n = 1.0$, does not scatter, and has nonreflective boundaries. Outline how to calculate the total heat loss from the sphere (i.e., there is no need actually to carry out cumbersome integrations).

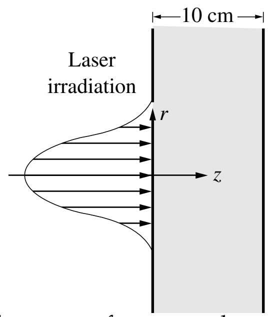
10.7 Repeat Problem 10.6, but assume that the temperature is uniform at 2000 K. What must the local production of heat be if the sphere is to remain at 2000 K everywhere? Note: The answer may be left in integral form (which must be solved numerically). Carry out the integration for $r = 0$ and $r = R$.

10.8 Repeat Problem 10.6, but assume that the temperature is uniform at 2000 K. Also, there is no heat production, meaning that the sphere cools down. How long will it take for the sphere to cool down to 500 K (the heat capacity of the medium is $\rho c = 1000\,\text{kJ/m}^3\,\text{K}$ and the conductivity is very large, i.e., the sphere is isothermal at all times)?

10.9 A relatively cold sphere with a radius of $R_o = 1\,\text{m}$ consists of a nonscattering gray medium that absorbs with an absorption coefficient of $\kappa = 0.1\,\text{cm}^{-1}$ and has a refractive index $n = 2$. At the center of the sphere is a small black sphere with radius $R_i = 1\,\text{cm}$ at a temperature of 1000 K. On the outside, the sphere is bounded by vacuum. What is the total heat flux leaving the sphere? Explain what happens as κ is increased from zero to a large value.

10.10 A laser beam is directed onto the atmosphere of a (hypothetical) planet. The planet's atmosphere contains 0.01% by volume of an absorbing gas. The absorbing gas has a molecular weight of 20 and, at the laser wavelength, an absorption coefficient $\kappa_\eta = 10^{-4}\,\text{cm}^{-1}/(\text{g/m}^3)$. It is known that the pressure and temperature distributions of the atmosphere can be approximated by $p = p_0\,e^{-2z/L}$ and $T = T_0\,e^{-z/L}$, where $p_0 = 0.75\,\text{atm}$, $T_0 = 400\,\text{K}$ are values at the planet surface $z = 0$, and $L = 2\,\text{km}$ is a characteristic length. What fraction of the laser energy arrives at the planet's surface?

10.11 A CO_2 laser with a total power output of $Q = 10\,\text{W}$ is directed (at right angle) onto a 10 cm thick, isothermal, absorbing/emitting (but not scattering) medium at 1000 K. It is known that the laser beam is essentially monochromatic at a wavelength of 10.6 μm with a Gaussian power distribution. Thus, the intensity falling onto the medium is



$$I(0) \propto e^{-(r/R)^2}/(\delta\Omega\,\delta\eta), \quad 0 \le r \le \infty;$$
$$Q = \int_A I(0)\,dA\,\delta\Omega\,\delta\eta,$$

where r is distance from beam center, $R = 100\,\mu\text{m}$ is the "effective radius" of the laser beam, $\delta\Omega = 5 \times 10^{-3}$ sr is the range of solid angles over which the laser beam outputs intensity (assumed uniform over $\delta\Omega$), and $\delta\eta$ is the range of wavenumbers over which the intensity is distributed (also assumed uniform). At 10.6 μm the medium is known to have an absorption coefficient $\kappa_\eta = 0.15\,\text{cm}^{-1}$. Assuming that the medium has nonreflecting boundaries, determine the exiting total intensity in the normal direction (transmitted laser radiation plus emission, assuming the medium to be gray). Is the emission contribution important? How thick would the medium have to be to make transmission and emission equally important?

10.12 Repeat Problem 10.11 for a medium with refractive index $n = 2$, bounded by vacuum (i.e., a slab with reflecting surfaces). Hint: (1) Part of the laser beam will be reflected when first hitting the slab, part will penetrate into the slab. Part of this energy will be absorbed by the layer, part will hit the rear face, where a fraction will be reflected back into the slab, and the rest will emerge from the slab, etc. Similar multiple internal reflections will take place with the emitted energy before emerging from the slab. (2) To calculate the slab–surroundings reflectance, show that the value of the absorptive index is negligible.

10.13 A thin column of gas of cross-section δA and length L contains a uniform suspension of small particles that absorb and scatter radiation. The scattering is according to the phase function (*a*) $\Phi = 1$ (isotropic scattering), (*b*) $\Phi = 1 + A_1 \cos\Theta$ (linear anisotropic scattering, A_1 is a constant), and (*c*) $\Phi = \frac{3}{4}(1+\cos^2\Theta)$ (Rayleigh scattering), where Θ is the angle between incoming and scattered directions. A laser beam hits the column normal to δA. What is the transmitted fraction of the laser power? What fraction of the laser flux goes through an infinite plane at L normal to the gas column? What fraction goes back through a plane at 0? What happens to the rest?

10.14 Repeat Example 10.2 for (*a*) $\Phi = 1 + A_1 \cos\Theta$ (linear anisotropic scattering, $A_1 = \text{const}$), and (*b*) $\Phi = \frac{3}{4}(1+\cos^2\Theta)$ (Rayleigh scattering), and Θ is the angle between incoming and scattered directions.

10.15 Show that, by setting $\beta_\eta = 0$ and $I_w = J/\pi$, the radiosity integral equation (5.25) can be recovered from equation (10.69) for a nonparticipating medium surrounded by diffusely reflecting walls.
Hint: Break up the heat flux in equation (10.69) into two parts, incoming radiation H and exiting radiation J. For the latter assume $\mathbf{r}$ to be an infinitesimal distance above the surface and evaluate the integral in equation (10.69).