CHAPTER
1

FUNDAMENTALS OF THERMAL RADIATION

1.1 INTRODUCTION

The terms *radiative heat transfer* and *thermal radiation* are commonly used to describe the science of the heat transfer caused by electromagnetic waves. Obvious everyday examples of thermal radiation include the heating effect of sunshine on a clear day, the fact that—when one is standing in front of a fire—the side of the body facing the fire feels much hotter than the back, and so on. More subtle examples of thermal radiation are that the clear sky is blue, that sunsets are red, and that, during a clear winter night, we feel more comfortable in a room whose curtains are drawn than in a room (heated to the same temperature) with open curtains.

All materials continuously emit and absorb electromagnetic waves, or photons, by lowering or raising their molecular energy levels. The strength and wavelengths of emission depend on the temperature of the emitting material. As we shall see, for heat transfer applications wavelengths between 10^{-7} m and 10^{-3} m (ultraviolet, visible, and infrared) are of greatest importance and are, therefore, the only ones considered here.

Before embarking on the analysis of *thermal radiation* we want briefly to compare the nature of this mode of heat transfer with the other two possible mechanisms of transferring energy, conduction and convection. In the case of conduction in a solid, energy is carried through the atomic lattice by free electrons or by phonon–phonon interactions (i.e., excitation of vibrational energy levels for interatomic bonds). In gases and liquids, energy is transferred from molecule to molecule through collisions (i.e., the faster molecule loses some of its kinetic energy to the slower one). Heat transfer by convection is similar, but many of the molecules with raised kinetic energy are carried away by the flow and are replaced by colder fluid (low-kinetic-energy molecules), resulting in increased energy transfer rates. Thus, both conduction and convection require the presence of a medium for the transfer of energy. Thermal radiation, on the other hand, is transferred by electromagnetic waves, or photons, which may travel over a long distance without interacting with a medium. The fact that thermal radiation does not require a medium for its transfer makes it of great importance in vacuum and space applications. This so-called "action at a distance" also manifests itself in a number of everyday thermodynamic applications. For example, on a cold winter day in a heated room we feel more comfortable when the curtains are closed: our bodies exchange heat by convection with the warm air surrounding us, but also by radiation with walls (including cold window panes if they are without curtains); we feel the heat from a fire a distance away from us, and so on.

Another distinguishing feature between conduction and convection on the one hand and thermal radiation on the other is the difference in their temperature dependencies. For the vast majority of conduction applications heat transfer rates are well described by *Fourier's law* as

$$q_x = -k\frac{\partial T}{\partial x}, \tag{1.1}$$

where q_x is conducted heat flux[1] in the x-direction, T is temperature, and k is the thermal conductivity of the medium. Similarly, convective heat flux may usually be calculated from a correlation such as

$$q = h(T - T_\infty), \tag{1.2}$$

where h is known as the convective heat transfer coefficient, and T_∞ is a reference temperature. While k and h may depend on temperature, this dependence is usually not very strong. Thus, for most applications, conductive and convective heat transfer rates are *linearly proportional* to temperature differences. As we shall see, radiative heat transfer rates are generally proportional to differences in temperature to the fourth (or higher) power, i.e.,

$$q \propto T^4 - T_\infty^4. \tag{1.3}$$

Therefore, radiative heat transfer becomes more important with rising temperature levels and may be totally dominant over conduction and convection at very high temperatures. Thus, thermal radiation is important in combustion applications (fires, furnaces, rocket nozzles, engines, etc.), in nuclear reactions (such as in the sun, in a fusion reactor, or in nuclear bombs), during atmospheric reentry of space vehicles, etc. As modern technology strives for higher efficiencies, this will require higher and higher temperatures, making thermal radiation ever more important. Other applications that are increasing in importance include solar energy collection and the greenhouse effect (both due to emission from our high-temperature sun). And, finally, one of the most pressing issues for mankind today are the effects of *global warming*, caused by the absorption of solar energy by man-made carbon dioxide released into the Earth's atmosphere.

The same reasons that make thermal radiation important in vacuum and high-temperature applications also make its analysis more difficult, or at least quite different from "conventional" analyses. Under normal conditions, conduction and convection are short-range phenomena: The average distance between molecular collisions *(mean free path for collision)* is generally very small, maybe around 10^{-10} m. If it takes, say, 10 collisions until a high-kinetic-energy molecule has a kinetic energy similar to that of the surrounding molecules, then any external influence is not directly felt over a distance larger than 10^{-9} m. Thus we are able to perform an energy balance on an "infinitesimal volume," i.e., a volume negligibly small in comparison with overall dimensions, but very large in comparison with the mean free path for collision. The principle of *conservation of energy* then leads to a *partial differential equation* to describe the temperature field and heat fluxes for both conduction and convection. This equation may have up to four independent variables (three space coordinates and time) and is *linear in temperature* for the case of constant properties. Thermal radiation, on the other hand, is generally a long-range phenomenon. The *mean free path for a photon* (i.e., the average distance a photon travels before interacting with a molecule) may be as short as 10^{-10} m (e.g., absorption in a metal), but can also be as long as 10^{+10} m or larger (e.g., the sun's rays hitting Earth). Thus, conservation of energy cannot be applied over an infinitesimal volume, but must be applied over the entire volume under consideration. This leads to an *integral equation* in up to seven independent variables (the frequency of radiation, three space coordinates, two coordinates describing the direction of travel of photons, and time).

The analysis of thermal radiation is further complicated by the behavior of the radiative properties of materials. Properties relevant to conduction and convection (thermal conductivity,

---

[1] In this book we shall use the term *heat flux* to denote the flow of energy per unit time and per unit area and the term *heat rate* for the flow of energy per unit time (i.e., not per unit area).

kinematic viscosity, density, etc.) are fairly easily measured and are generally well behaved (isotropic throughout the medium, perhaps with weak temperature dependence). Radiative properties are usually difficult to measure and often display erratic behavior. For liquids and solids the properties normally depend only on a very thin surface layer, which may vary strongly with surface preparation and often even from day to day. All radiative properties (in particular for gases) may vary strongly with wavelength, adding another dimension to the governing equation. Rarely, if ever, may this equation be assumed to be linear.

Because of these difficulties inherent in the analysis of thermal radiation, a good portion of this book has been set aside to discuss radiative properties and different approximate methods to solve the governing energy equation for radiative transport.

1.2 THE NATURE OF THERMAL RADIATION

Thermal radiative energy may be viewed as consisting of *electromagnetic waves* (as predicted by *electromagnetic wave theory*) or as consisting of massless energy parcels, called *photons* (as predicted by *quantum mechanics*). Neither point of view is able to describe completely all radiative phenomena that have been observed. It is, therefore, customary to use both concepts interchangeably. In general, radiative properties of liquids and solids (including tiny particles), and of interfaces (surfaces) are more easily predicted using electromagnetic wave theory, while radiative properties of gases are more conveniently obtained from quantum mechanics.

All electromagnetic waves, or photons, are known to propagate through any medium at a high velocity. Since light is a part of the electromagnetic wave spectrum, this velocity is known as the *speed of light*, c. The speed of light depends on the medium through which it travels, and may be related to the speed of light in vacuum, c_0, by the formula

$$c = \frac{c_0}{n}, \qquad c_0 = 2.998 \times 10^8\,\text{m/s}, \tag{1.4}$$

where n is known as the *refractive index* of the medium. By definition, the refractive index of vacuum is $n \equiv 1$. For most gases the refractive index is very close to unity, for example, air at room temperature has $n = 1.00029$ over the visible spectrum. Therefore, light propagates through gases nearly as fast as through vacuum. Electromagnetic waves travel considerably slower through dielectrics (electric nonconductors), which have refractive indices between approximately 1.4 and 4, and they hardly penetrate at all into electrical conductors (metals). Each wave may be identified either by its

frequency, ν	(measured in cycles/s = s^{-1} = Hz);
wavelength, λ	(measured in μm = 10^{-6} m or nm = 10^{-9} m);
wavenumber, η	(measured in cm^{-1}); or
angular frequency, ω	(measured in radians/s = s^{-1}).

All four quantities are related to one another through the formulae

$$\nu = \frac{\omega}{2\pi} = \frac{c}{\lambda} = c\eta. \tag{1.5}$$

Each wave or photon carries with it an amount of energy, ϵ, determined from quantum mechanics as

$$\epsilon = h\nu, \qquad h = 6.626 \times 10^{-34}\,\text{J s}, \tag{1.6}$$

where h is known as *Planck's constant*. The frequency of light does not change when light penetrates from one medium to another since the energy of the photon must be conserved. On the other hand, wavelength and wavenumber do, depending on the values of the refractive

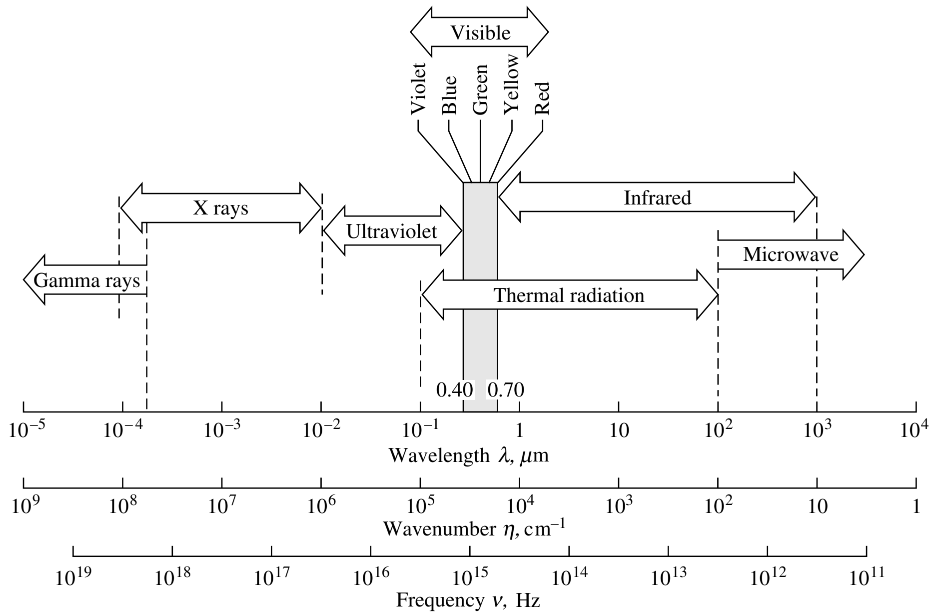

FIGURE 1-1
Electromagnetic wave spectrum (for radiation traveling through vacuum, $n = 1$).

index for the two media. Sometimes electromagnetic waves are characterized in terms of the energy that a photon carries, $h\nu$, using the energy unit *electron volt* ($1\,\text{eV} = 1.6022\times10^{-19}\,\text{J}$). Thus, light with a photon energy (or "frequency") of a eV has a wavelength (in vacuum) of

$$\lambda = \frac{hc}{h\nu} = \frac{6.626\times10^{-34}\,\text{J s}\times 2.998\times10^{8}\,\text{m/s}}{a\,1.6022\times10^{-19}\,\text{J}} = \frac{1.240}{a}\,\mu\text{m}. \tag{1.7}$$

Since electromagnetic waves of vastly different wavelengths carry vastly different amounts of energy, their behavior is often quite different. Depending on their behavior or occurrence, electromagnetic waves have been grouped into a number of different categories, as shown in Fig. 1-1. Thermal radiation may be defined to be those electromagnetic waves which are emitted by a medium due solely to its temperature [1]. As indicated earlier, this definition limits the range of wavelengths of importance for heat transfer considerations to between 0.1 μm (ultraviolet) and 100 μm (midinfrared).

1.3 BASIC LAWS OF THERMAL RADIATION

When an electromagnetic wave traveling through a medium (or vacuum) strikes the surface of another medium (solid or liquid surface, particle or bubble), the wave may be reflected (either partially or totally), and any nonreflected part will penetrate into the medium. While passing through the medium the wave may become continuously attenuated. If attenuation is complete so that no penetrating radiation reemerges, it is known as *opaque*. If a wave passes through a medium without any attenuation, it is termed *transparent*, while a body with partial attenuation is called *semitransparent*.[2] Whether a medium is transparent, semitransparent or opaque depends on the material as well as on its thickness (i.e., the distance the electromagnetic wave must travel through the medium). Metals are nearly always opaque, although it is a

---

[2] A medium that allows a fraction of light to pass through, while scattering the transmitted light into many different directions, for example, milky glass, is called *translucent*.

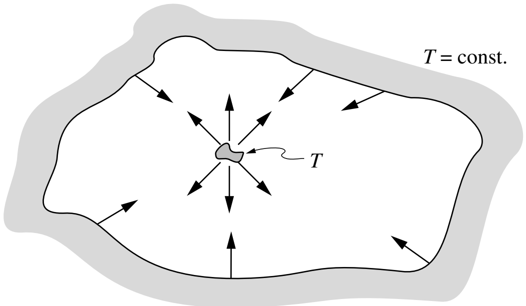

FIGURE 1-2
Kirchhoff's law.

common high school physics experiment to show that light can penetrate through extremely thin layers of gold. Nonmetals generally require much larger thicknesses before they become opaque, and some are quite transparent over part of the spectrum (for example, window glass in the visible part of the spectrum).

An opaque surface that does not reflect *any* radiation is called a *perfect absorber* or a *black surface*: When we "see" an object, our eyes absorb electromagnetic waves from the visible part of the spectrum, which have been emitted by the sun (or artificial light) and have been reflected by the object toward our eyes. We cannot see a surface that does not reflect radiation, and it appears "black" to our eyes.[3] Since black surfaces absorb the maximum possible amount of radiative energy, they serve as a standard for the classification of all other surfaces.

It is easy to show that a black surface also emits a maximum amount of radiative energy, i.e., more than any other body at the same temperature. To show this, we use one of the many variations of *Kirchhoff's law*:[*] Consider two identical black-walled enclosures, thermally insulated on the outside, with each containing a small object—one black and the other one not—as shown in Fig. 1-2. After a long time, in accordance with the Second Law of Thermodynamics, both entire enclosures and the objects within them will be at a single uniform temperature. This characteristic implies that every part of the surface (of the enclosure as well as the objects) emits precisely as much energy as it absorbs. Both objects in the different enclosures receive exactly the same amount of radiative energy. But since the black object absorbs more energy (i.e., the maximum possible), it must also emit more energy than the nonblack object (i.e., also the maximum possible).

By the same reasoning it is easy to show that a black surface is a perfect absorber and emitter at every wavelength and for any direction (of incoming or outgoing electromagnetic waves), and that the radiation field within an isothermal black enclosure is *isotropic* (i.e., the radiative energy density is the same at any point and in any direction within the enclosure).

1.4 EMISSIVE POWER

Every medium continuously emits electromagnetic radiation randomly into all directions at a rate depending on the local temperature and on the properties of the material. This is sometimes referred to as *Prévost's law* (after Pierre Prévost, an early 19th century Swiss philosopher and physicist). The radiative heat flux emitted from a surface is called the *emissive power*, E. We distinguish between *total* and *spectral emissive power* (i.e., heat flux emitted over the entire

---

[3] Note that a surface appearing black to our eyes is by no means a perfect absorber at nonvisible wavelengths and vice versa; indeed, many white paints are actually quite "black" at longer wavelengths.

[*] **Gustav Robert Kirchhoff (1824–1887)**
German physicist. After studying in Berlin, Kirchhoff served as professor of physics at the University of Heidelberg for 21 years before returning to Berlin as professor of mathematical physics. Together with the chemist Robert Bunsen, he was the first to establish the theory of spectrum analysis.

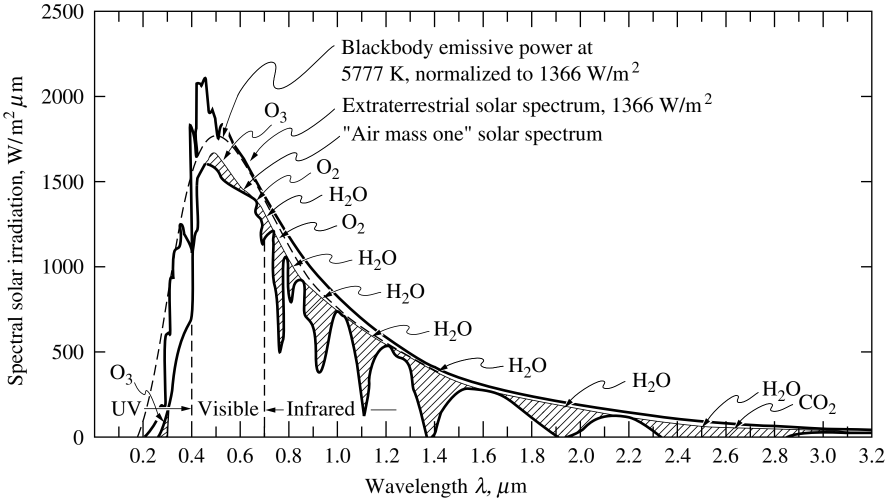

FIGURE 1-3
Solar irradiation onto Earth.

spectrum, or at a given frequency per unit frequency interval), so that

spectral emissive power, $E_\nu \equiv$ emitted energy/time/surface area/frequency,
total emissive power, $E \equiv$ emitted energy/time/surface area.

Here and elsewhere we use the subscripts ν, λ, or η (depending on the choice of spectral variable) to express a spectral quantity whenever necessary for clarification. Thermal radiation of a single frequency or wavelength is sometimes also called *monochromatic radiation* (since, over the visible range, the human eye perceives electromagnetic waves to have the colors of the rainbow). It is clear from their definitions that the total and spectral emissive powers are related by

$$E(T) = \int_0^\infty E_\nu(T, \nu)\, d\nu. \tag{1.8}$$

Blackbody Emissive Power Spectrum

Scientists had tried for many years to theoretically predict the sun's emission spectrum, which we know today to behave very nearly like a blackbody at approximately 5777 K [2]. The spectral solar flux falling onto Earth, or *solar irradiation,* is shown in Fig. 1-3 for extraterrestrial conditions (as measured by high-flying balloons and satellites) and for unity air mass (air mass is defined as the value of $1/\cos\theta_S$, where the *zenith angle* θ_S is the angle between the local vertical and a vector pointing toward the sun) [3,4]. Solar radiation is attenuated significantly as it penetrates through the atmosphere by phenomena that will be discussed in Sections 1.12 and 1.14. Lord Rayleigh (1900) [5][*] and Sir James Jeans (1905) [6][†] independently applied the principles of

---

[*] **John William Strutt, Lord Rayleigh (1842–1919)**
English physical scientist. Rayleigh obtained a mathematics degree from Cambridge, where he later served as professor of experimental physics for five years. He then became secretary, and later president, of the Royal Society. His work resulted in a number of discoveries in the fields of acoustics and optics, and he was the first to explain the blue color of the sky (cf. the Rayleigh scattering laws in Chapter 12). Rayleigh received the 1904 Nobel Prize in Physics for the isolation of argon.

[†] **Sir James Hopwood Jeans (1877–1946)**
English physicist and mathematician, whose work was primarily in the area of astrophysics. He applied mathematics to several problems in thermodynamics and electromagnetic radiation.

classical statistics with its equipartition of energy to predict the spectrum of the sun, with dismal results. Wilhelm Wien (1896) [7]‡ used some thermodynamic arguments together with experimental data to propose a spectral distribution of blackbody emissive power that was very accurate over large parts of the spectrum. Finally, in 1901 Max Planck [8]§ published his work on quantum statistics: Assuming that a molecule can emit photons only at distinct energy levels, he found the spectral *blackbody emissive power* distribution, now commonly known as *Planck's law,* for a black surface bounded by a transparent medium with refractive index n, as

$$E_{b\nu}(T,\nu) = \frac{2\pi h\nu^3 n^2}{c_0^2\left[e^{h\nu/kT}-1\right]}, \tag{1.9}$$

where $k = 1.3807 \times 10^{-23}$ J/K is known as *Boltzmann's constant.*[4] While frequency ν appears to be the most logical spectral variable (since it does not change when light travels from one medium into another), the spectral variables wavelength λ (primarily for surface emission and absorption) and wavenumber η (primarily for radiation in gases) are also frequently (if not more often) employed. Equation (1.9) may be readily expressed in terms of wavelength and wavenumber through the relationships

$$\nu = \frac{c_0}{n\lambda} = \frac{c_0}{n}\eta, \qquad d\nu = -\frac{c_0}{n\lambda^2}\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]d\lambda = \frac{c_0}{n}\left[1-\frac{\eta}{n}\frac{dn}{d\eta}\right]d\eta, \tag{1.10}$$

and

$$E_b(T) = \int_0^\infty E_{b\nu}\,d\nu = \int_0^\infty E_{b\lambda}\,d\lambda = \int_0^\infty E_{b\eta}\,d\eta, \tag{1.11}$$

or

$$E_{b\nu}\,d\nu = -E_{b\lambda}\,d\lambda = E_{b\eta}\,d\eta. \tag{1.12}$$

Here λ and η are wavelength and wavenumber for the electromagnetic waves within the medium of refractive index n (while $\lambda_0 = n\lambda$ and $\eta_0 = \eta/n$ would be wavelength and wavenumber of the same wave traveling through vacuum). Equation (1.10) shows that equation (1.9) gives convenient relations for $E_{b\lambda}$ and $E_{b\eta}$ only if the refractive index is independent of frequency (or wavelength, or wavenumber). This is certainly the case for vacuum ($n = 1$) and ordinary gases ($n \simeq 1$), and may be of acceptable accuracy for some semitransparent media over large parts of the spectrum (for example, for quartz $1.52 < n < 1.68$ between the wavelengths of 0.2 and 2.4 μm). Thus, with the assumption of constant refractive index,

$$E_{b\lambda}(T,\lambda) = \frac{2\pi hc_0^2}{n^2\lambda^5\left[e^{hc_0/n\lambda kT}-1\right]}, \qquad (n = \text{const}), \tag{1.13}$$

---

‡**Wilhelm Wien (1864–1928)**
German physicist, who served as professor of physics at the University of Giessen and later at the University of Munich. Besides his research in the area of electromagnetic waves, his interests included other rays, such as electron beams, X-rays, and α-particles. For the discovery of his displacement law he was awarded the Nobel Prize in Physics in 1911.

§**Max Planck (1858–1947)**
German physicist. Planck studied in Berlin with H. L. F. von Helmholtz and G. R. Kirchhoff, but obtained his doctorate at the University of Munich before returning to Berlin as professor in theoretical physics. He later became head of the Kaiser Wilhelm Society (today the Max Planck Institute). For his development of the quantum theory he was awarded the Nobel Prize in Physics in 1918.

[4] Equation (1.9) is valid for emission into a medium whose absorptive index (to be introduced in Chapter 2) is much less than the refractive index. This includes semitransparent media such as water, glass, quartz, etc., but not opaque materials. Emission into such bodies is immediately absorbed and is of no interest.

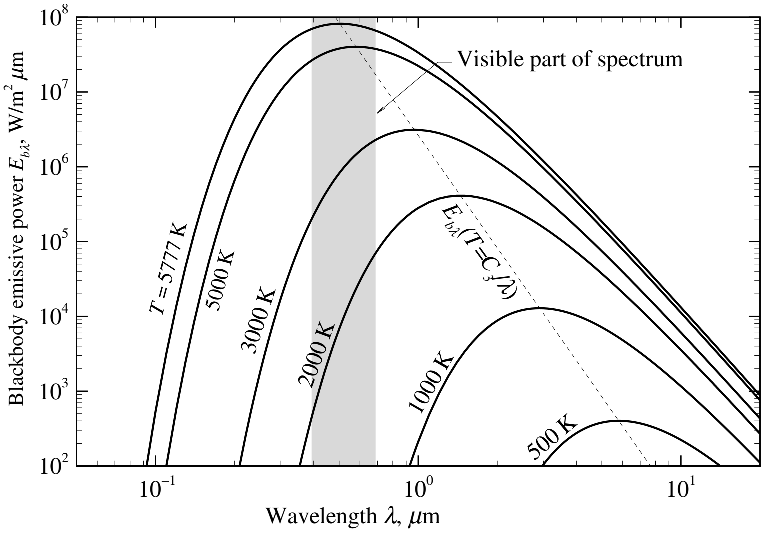

FIGURE 1-4
Blackbody emissive power spectrum.

$$E_{b\eta}(T,\eta) = \frac{2\pi h c_0^2 \eta^3}{n^2\left[e^{hc_0\eta/nkT} - 1\right]}, \qquad (n = \text{const}). \tag{1.14}$$

Figure 1-4 is a graphical representation of equation (1.13) for a number of blackbody temperatures. As one can see, the overall level of emission rises with rising temperature (as dictated by the Second Law of Thermodynamics), while the wavelength of maximum emission shifts toward shorter wavelengths. The blackbody emissive power is also plotted in Fig. 1-3 for an *effective solar temperature* of 5777 K. This plot is in good agreement with extraterrestrial solar irradiation data.

It is customary to introduce the abbreviations

$$C_1 = 2\pi h c_0^2 = 3.7418 \times 10^{-16}\,\text{W m}^2,$$
$$C_2 = hc_0/k = 14{,}388\,\mu\text{m K} = 1.4388\,\text{cm K},$$

so that equation (1.13) may be recast as

$$\frac{E_{b\lambda}}{n^3T^5} = \frac{C_1}{(n\lambda T)^5[e^{C_2/(n\lambda T)} - 1]}, \qquad (n = \text{const}), \tag{1.15}$$

which is seen to be a function of $(n\lambda T)$ only. Thus, it is possible to plot this normalized emissive power as a single line *vs.* the product of wavelength in vacuum $(n\lambda)$ and temperature (T), as shown in Fig. 1-5, and a detailed tabulation is given in Appendix C. The maximum of this curve may be determined by differentiating equation (1.15),

$$\frac{d}{d(n\lambda T)}\left(\frac{E_{b\lambda}}{n^3T^5}\right) = 0,$$

leading to a transcendental equation that may be solved numerically as

$$(n\lambda T)_{\max} = C_3 = 2898\,\mu\text{m K}. \tag{1.16}$$

Equation (1.16) is known as *Wien's displacement law* since it was developed independently by Wilhelm Wien [9] in 1891 (i.e., well before the publication of Planck's emissive power law).

Example 1.1. At what wavelength has the sun its maximum emissive power? At what wavelength Earth?

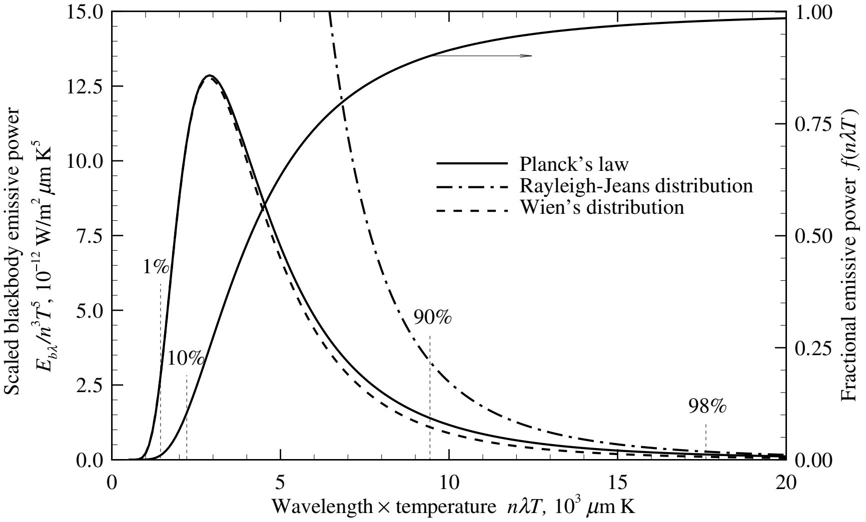

FIGURE 1-5
Normalized blackbody emissive power spectrum.

Solution

From equation (1.16), with the sun's surface at $T_{\text{sun}} \simeq 5777$ K and bounded by vacuum ($n = 1$), it follows that

$$\lambda_{\text{max,sun}} = \frac{C_3}{T_{\text{sun}}} = \frac{2898\,\mu\text{m K}}{5777\,\text{K}} = 0.50\,\mu\text{m},$$

which is near the center of the visible region. Apparently, evolution has caused our eyes to be most sensitive in that section of the electromagnetic spectrum where the maximum daylight is available. In contrast, Earth's average surface temperature may be in the vicinity of $T_{\text{Earth}} = 290$ K, or

$$\lambda_{\text{max,Earth}} \simeq \frac{2898\,\mu\text{m K}}{290\,\text{K}} = 10\,\mu\text{m},$$

that is, Earth's maximum emission occurs in the intermediate infrared, leading to infrared cameras and detectors for night "vision."

It is of interest to look at the asymptotic behavior of Planck's law for small and large wavelengths. For very small values of $hc_0/n\lambda kT$ (large wavelength, or small frequency), the exponent in equation (1.13) may be approximated by a two-term Taylor series, leading to

$$E_{b\lambda} = \frac{2\pi c_0 kT}{n\lambda^4}, \qquad \frac{hc_0}{n\lambda kT} \ll 1. \tag{1.17}$$

The same result is obtained if one lets $h \to 0$, i.e., if one allows photons of arbitrarily small energy content to be emitted, as postulated by classical statistics. Thus, equation (1.17) is identical to the one derived by *Rayleigh* and *Jeans* and bears their names. The *Rayleigh–Jeans distribution* is also included in Fig. 1-5. Obviously, this formula is accurate only for very large values of $(n\lambda T)$, where the energy of the emissive power spectrum is negligible. Thus, this formula is of little significance for engineering purposes.

For large values of $(hc_0/n\lambda kT)$ the -1 in the denominator of equation (1.13) may be neglected, leading to *Wien's distribution* (or *Wien's law*),

$$E_{b\lambda} \simeq \frac{2\pi hc_0^2}{n^2\lambda^5} e^{-hc_0/n\lambda kT} = \frac{C_1}{n^2\lambda^5} e^{-C_2/n\lambda T}, \qquad \frac{hc_0}{n\lambda kT} \gg 1, \tag{1.18}$$

since it is identical to the formula first proposed by Wien, before the advent of quantum mechanics. Examination of Wien's distribution in Fig. 1-5 shows that it is very accurate over most of the spectrum, with a total energy content of the entire spectrum approximately 8% lower than for Planck's law. Thus, Wien's distribution is frequently utilized in theoretical analyses in order to facilitate integration.

Total Blackbody Emissive Power

The total emissive power of a blackbody may be determined from equations (1.11) and (1.13) as

$$\begin{aligned} E_b(T) &= \int_0^\infty E_{b\lambda}(T,\lambda)\,d\lambda = C_1 n^2 T^4 \int_0^\infty \frac{d(n\lambda T)}{(n\lambda T)^5\left[e^{C_2/(n\lambda T)}-1\right]} \\ &= \left[\frac{C_1}{C_2^4}\int_0^\infty \frac{\xi^3\,d\xi}{e^\xi - 1}\right] n^2 T^4, \quad (n = \text{const}). \end{aligned} \tag{1.19}$$

The integral in this expression may be evaluated by complex integration, and is tabulated in many good integral tables:

$$E_b(T) = n^2 \sigma T^4, \quad \sigma = \frac{\pi^4 C_1}{15 C_2^4} = 5.670 \times 10^{-8}\,\frac{\text{W}}{\text{m}^2\,\text{K}^4}, \tag{1.20}$$

where σ is known as the *Stefan–Boltzmann constant*.* If Wien's distribution is to be used then the -1 is absent from the denominator of equation (1.19), and a corrected Stefan–Boltzmann constant should be employed, evaluated as

$$\sigma_W = \frac{6C_1}{C_2^4} = 5.239 \times 10^{-8}\,\frac{\text{W}}{\text{m}^2\,\text{K}^4}, \tag{1.21}$$

indicating that Wien's distribution underpredicts total emissive power by about 7.5%. Historically, the "T^4 radiation law," equation (1.20), predates Planck's law and was found through thermodynamic arguments. A short history may be found in [10].

It is often necessary to calculate the emissive power contained within a finite wavelength band, say between λ_1 and λ_2. Then

$$\int_{\lambda_1}^{\lambda_2} E_{b\lambda}\,d\lambda = \frac{C_1}{C_2^4}\int_{C_2/n\lambda_2 T}^{C_2/n\lambda_1 T} \frac{\xi^3\,d\xi}{e^\xi - 1}\, n^2 T^4. \tag{1.22}$$

It is not possible to evaluate the integral in equation (1.22) in simple analytical form. Therefore, it is customary to express equation (1.22) in terms of the *fraction of blackbody emissive power* contained between 0 and $n\lambda T$,

$$f(n\lambda T) = \frac{\int_0^\lambda E_{b\lambda}\,d\lambda}{\int_0^\infty E_{b\lambda}\,d\lambda} = \int_0^{n\lambda T}\left(\frac{E_{b\lambda}}{n^3 \sigma T^5}\right) d(n\lambda T) = \frac{15}{\pi^4}\int_{C_2/n\lambda T}^\infty \frac{\xi^3\,d\xi}{e^\xi - 1}, \tag{1.23}$$

---

*__Josef Stefan (1835–1893)__
Austrian physicist. Serving as professor at the University of Vienna, Stefan determined in 1879 that, based on his experiments, blackbody emission was proportional to temperature to the fourth power.

Ludwig Erhard Boltzmann (1844–1906)
Austrian physicist. After receiving his doctorate from the University of Vienna he held professorships in Vienna, Graz (both in Austria), Munich, and Leipzig (in Germany). His greatest contributions were in the field of statistical mechanics (Boltzmann statistics). He derived the fourth-power law from thermodynamic considerations in 1889.

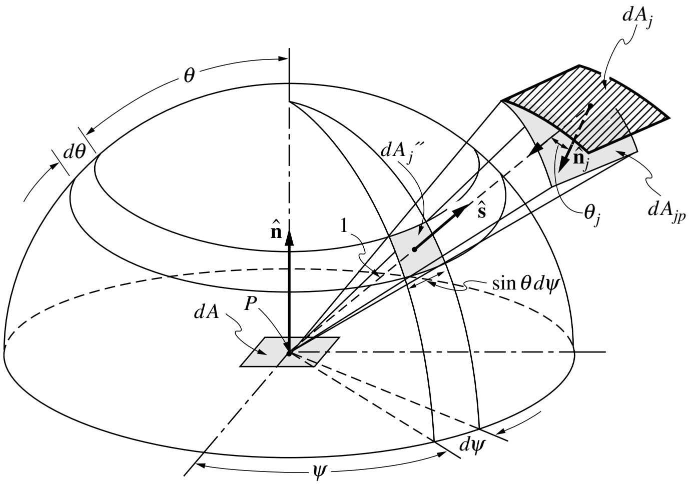

FIGURE 1-6
Emission direction and solid angles as related to a unit hemisphere.

so that

$$\int_{\lambda_1}^{\lambda_2} E_{b\lambda}\, d\lambda = \left[f(n\lambda_2 T) - f(n\lambda_1 T)\right] n^2 \sigma T^4. \tag{1.24}$$

Equation (1.23) can be integrated only after expanding the denominator into an infinite series, resulting in

$$f(n\lambda T) = \frac{15}{\pi^4} \sum_{m=1}^{\infty} \frac{e^{-m\zeta}}{m^4} \left[6 + 6\,(m\zeta) + 3\,(m\zeta)^2 + (m\zeta)^3\right], \quad \zeta = \frac{C_2}{n\lambda T}. \tag{1.25}$$

The fractional emissive power is a function in a single variable, $n\lambda T$, and is therefore easily tabulated, as has been done in Appendix C. For computer calculations a little Fortran routine of equation (1.25), `bbfn`, is given in Appendix F, as well as a stand-alone program, `planck`, which, after inputting wavelength (or wavenumber) and temperature, returns $E_{b\lambda}$, $E_{b\eta}$, and f.

Example 1.2. What fraction of total solar emission falls into the visible spectrum (0.4 to 0.7 μm)?

Solution
With $n = 1$ and a solar temperature of 5777 K it follows that for $\lambda_1 = 0.4\,\mu$m, $n\lambda_1 T_{\text{sun}} = 1 \times 0.4 \times 5777 = 2310.8\,\mu$m K; and for $\lambda_2 = 0.7\,\mu$m, $n\lambda_2 T_{\text{sun}} = 4043.9\,\mu$m K. From Appendix C we find $f(n\lambda_1 T_{\text{sun}}) = 0.12220$ and $f(n\lambda_2 T_{\text{sun}}) = 0.48869$. Thus, from equations (1.20) and (1.24) the visible fraction of sunlight is

$$f(n\lambda_2 T_{\text{sun}}) - f(n\lambda_1 T_{\text{sun}}) = 0.48869 - 0.12220 = 0.36649.$$

(Writing a one-line program `bbfn(4043.9)-bbfn(2310.8)` returns the slightly more accurate value of 0.36661.) Therefore, with a bandwidth of only 0.3 μm the human eye responds to approximately 37% of all emitted sunlight!

1.5 SOLID ANGLES

When radiative energy leaves one medium and enters another (i.e., emission from a surface into another medium), this energy flux usually has different strengths in different directions. Similarly, the electromagnetic wave, or photon, flux passing through any point inside any medium may vary with direction. It is customary to describe the *direction vector* in terms of a *spherical* or *polar coordinate system*. Consider a point P on an opaque surface dA radiating into another medium, say air, as shown in Fig. 1-6. It is apparent that the surface can radiate into infinitely many directions, with every ray penetrating through a *hemisphere* of unit radius as indicated in the figure. The total surface area of this hemisphere, $2\pi\, 1^2 = 2\pi$, is known as the *total solid angle* above the surface. An arbitrary emission direction from the surface is specified by the unit direction vector $\hat{\mathbf{s}}$, which may be expressed in terms of the *polar angle* θ (measured from the *surface normal* $\hat{\mathbf{n}}$) and the *azimuthal angle* ψ (measured between an arbitrary axis on the surface and the projection of $\hat{\mathbf{s}}$ onto the surface). It is seen that, for a hemisphere, $0 \leq \theta \leq \pi/2$ and $0 \leq \psi \leq 2\pi$.

The *solid angle* with which an infinitesimal surface dA_j is seen from a point P is defined as the projection of the surface onto a plane normal to the direction vector, divided by the square of the distance S between dA_j and P, as also shown in Fig. 1-6. If the surface is projected onto the unit hemisphere above the point, the solid angle is equal to the projected area itself, or

$$d\Omega = \frac{dA_{jp}}{S^2} = \frac{\cos\theta_j\, dA_j}{S^2} = dA_j''. \tag{1.26}$$

Thus, an infinitesimal solid angle is simply an infinitesimal area on a unit sphere, or

$$d\Omega = dA_j'' = (1 \times \sin\theta\, d\psi)(1 \times d\theta) = \sin\theta\, d\theta\, d\psi. \tag{1.27}$$

Integrating over all possible directions we obtain

$$\int_{\psi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \sin\theta\, d\theta\, d\psi = 2\pi, \tag{1.28}$$

for the total solid angle above the surface, as already seen earlier. The solid angle, with which a finite surface A_j is seen from point P, follows immediately from equation (1.26) as

$$\Omega = \int_{A_{jp}} \frac{dA_{jp}}{S^2} = \int_A \frac{\cos\theta_j\, dA_j}{S^2} = \int_{A_j} dA_j'' = A_j'', \tag{1.29}$$

i.e., the projection of A_j onto the hemisphere above P.

While a little unfamiliar at first, solid angles are simply two-dimensional angular space: Similar to the way a one-dimensional angle can vary between 0 and π (measured in dimensionless radians, equivalent to length along a semicircular line), the solid angle may vary between 0 and 2π (measured in dimensionless steradians, sr, equivalent to surface area on a hemisphere).

Example 1.3. Determine the solid angle with which the sun is seen from Earth.

Solution

The area of the sun projected onto a plane normal to the vector pointing from Earth to the sun (or, simply, the image of the sun that we see from Earth) is a disk of radius $R_s \simeq 6.96 \times 10^8$ m (i.e., the radius of the sun), at a distance of approximately $S_{ES} \simeq 1.496 \times 10^{11}$ m (averaged over Earth's yearly orbit). Thus the solid angle of the sun is

$$\Omega_S = \frac{(\pi R_S^2)}{S_{ES}^2} = \frac{\pi \times (6.955 \times 10^8)^2}{(1.496 \times 10^{11})^2} = 6.79 \times 10^{-5}\ \text{sr}.$$

This solid angle is so small that we may generally assume that solar radiation comes from a single direction, i.e., that all the light beams are parallel.

Example 1.4. What is the solid angle with which the narrow strip shown in Fig. 1-7 is seen from point "0"?

Solution

Since the strip is narrow we may assume that the projection angle for equation (1.29) varies only in the x-direction as indicated in Fig. 1-7, leading to

$$\Omega = w\int_0^L \frac{\cos\theta_0\, dx}{r^2}, \qquad \cos\theta_0 = \frac{h}{r}, \qquad r^2 = h^2 + x^2,$$

and

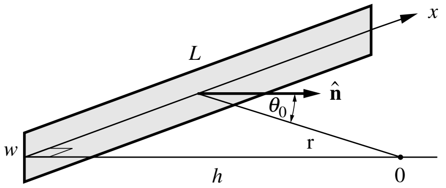

FIGURE 1-7
Solid angle subtended by a narrow strip.

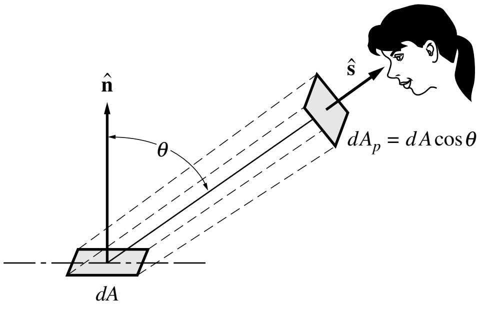

FIGURE 1-8
Relationship between blackbody emissive power and intensity.

$$\Omega = w\int_0^L \frac{h\,dx}{r^3} = wh\int_0^L \frac{dx}{(h^2+x^2)^{3/2}} = \frac{w}{h}\frac{x}{\sqrt{h^2+x^2}}\bigg|_0^L = \frac{wL}{h\sqrt{h^2+L^2}}.$$

1.6 RADIATIVE INTENSITY

While emissive power appears to be the natural choice to describe radiative heat flux leaving a surface, it is inadequate to describe the directional dependence of the radiation field, in particular inside an absorbing/emitting medium, where photons may not have originated from a surface. Therefore, very similar to the emissive power, we define the *radiative intensity I,* as radiative energy flow per unit solid angle and unit area *normal to the rays* (as opposed to surface area). Again, we distinguish between *spectral* and *total intensity*. Thus,

spectral intensity, $I_\lambda \equiv$ radiative energy flow/time/area
normal to rays/solid angle/wavelength,
total intensity, $I \equiv$ radiative energy flow/time/area
normal to rays/solid angle.

Again, spectral and total intensity are related by

$$I(\mathbf{r},\hat{\mathbf{s}}) = \int_0^\infty I_\lambda(\mathbf{r},\hat{\mathbf{s}},\lambda)\,d\lambda. \tag{1.30}$$

Here, **r** is a *position vector* fixing the location of a point in space, and $\hat{\mathbf{s}}$ is a unit direction vector as defined in the previous section. While emissive power depends only on position and wavelength, the radiative intensity depends, in addition, on the direction vector $\hat{\mathbf{s}}$. The emissive power can be related to intensity by integrating over all the directions pointing away from the surface. Considering Fig. 1-8, we find that the emitted energy from dA into the direction $\hat{\mathbf{s}}$, and contained within an infinitesimal solid angle $d\Omega = \sin\theta\,d\theta\,d\psi$ is, from the definition of intensity,

$$I(\mathbf{r},\hat{\mathbf{s}})\,dA_p\,d\Omega = I(\mathbf{r},\hat{\mathbf{s}})\,dA\cos\theta\sin\theta\,d\theta\,d\psi,$$

where dA_p is the projected area of dA normal to the rays (i.e., the way dA is seen when viewed from the $-\hat{\mathbf{s}}$ direction). Thus, integrating this expression over all possible directions gives the

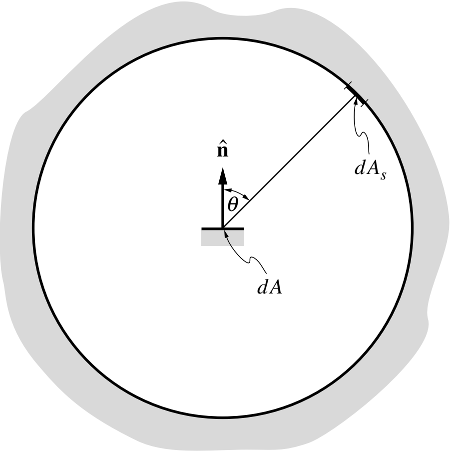

FIGURE 1-9
Kirchhoff's law for the directional behavior of blackbody intensity.

total energy emitted from dA, or, after dividing by dA

$$E(\mathbf{r}) = \int_0^{2\pi} \int_0^{\pi/2} I(\mathbf{r}, \theta, \psi) \cos\theta \sin\theta \, d\theta \, d\psi = \int_{2\pi} I(\mathbf{r}, \hat{\mathbf{s}}) \, \hat{\mathbf{n}} \cdot \hat{\mathbf{s}} \, d\Omega. \tag{1.31}$$

This expression is, of course, also valid on a spectral basis.

The directional behavior of the radiative intensity leaving a blackbody is easily obtained from a variation of Kirchhoff's law: Consider a small, black surface suspended at the center of an isothermal spherical enclosure, as depicted in Fig. 1-9. Let us assume that the enclosure has a (hypothetical) surface coating that reflects all incoming radiation totally and like a mirror everywhere except over a small area dA_s, which also reflects all incoming radiation except for a small wavelength interval between λ and $\lambda + d\lambda$. Over this small range of wavelengths dA_s behaves like a blackbody. Now, all radiation leaving dA, traveling to the sphere (with the exception of light of wavelength λ traveling toward dA_s), will be reflected back toward dA where it will be absorbed (since dA is black). Thus, the net energy flow from dA to the sphere is, recalling the definitions for intensity and solid angle,

$$I_{b\lambda}(T, \theta, \psi, \lambda)(dA \cos\theta) \, d\Omega_s \, d\lambda = I_{b\lambda}(T, \theta, \psi, \lambda)(dA \cos\theta) \left(\frac{dA_s}{R^2}\right) d\lambda,$$

where $d\Omega_s$ is the solid angle with which dA_s is seen from dA. On the other hand, also by Kirchhoff's law, the sphere does not emit any radiation (since it does not absorb anything), except over dA_s at wavelength λ. All energy emitted from dA_s will eventually come back to itself except for the fraction intercepted by dA. Thus, the net energy flow from the sphere to dA is

$$I_{bn\lambda}(T, \lambda) \, dA_s \, d\Omega \, d\lambda = I_{bn\lambda}(T, \lambda) \, dA_s \left(\frac{dA \cos\theta}{R^2}\right) d\lambda,$$

where the subscript n denotes emission into the normal direction ($\theta_s = 0$, ψ_s arbitrary), and $d\Omega$ is the solid angle with which dA is seen from dA_s. Now, from the Second Law of Thermodynamics, these two fluxes must be equal for an isothermal enclosure. Therefore,

$$I_{b\lambda}(T, \theta, \psi, \lambda) = I_{bn\lambda}(T, \lambda).$$

Since the direction (θ, ψ), with which dA_s is oriented, is quite arbitrary we conclude that $I_{b\lambda}$ is *independent of direction*, or

$$I_{b\lambda} = I_{b\lambda}(T, \lambda) \text{ only.} \tag{1.32}$$

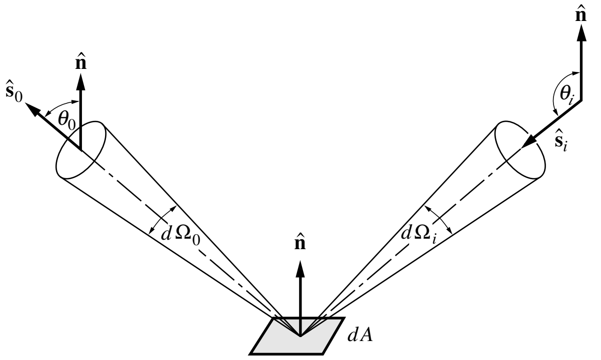

FIGURE 1-10
Radiative heat flux on an arbitrary surface.

Substituting this expression into equation (1.31) we obtain the following relationship between blackbody intensity and emissive power:

$$E_{b\lambda}(\mathbf{r},\lambda) = \pi I_{b\lambda}(\mathbf{r},\lambda). \tag{1.33}$$

This equation implies that the intensity leaving a blackbody (or any surface whose outgoing intensity is independent of direction, or *diffuse*) may be evaluated from the blackbody emissive power (or outgoing heat flux) as

$$I_{b\lambda}(\mathbf{r},\lambda) = E_{b\lambda}(\mathbf{r},\lambda)/\pi. \tag{1.34}$$

In the literature the spectral blackbody intensity is sometimes referred to as the *Planck function.* The directional behavior of the emission from a blackbody is found by comparing the intensity (energy flow per solid angle and *area normal to the rays*) and directional emitted flux (energy flow per solid angle and per unit surface area). The directional heat flux is sometimes called directional emissive power, and

$$E'_{b\lambda}(\mathbf{r},\lambda,\theta,\psi)\,dA = I_{b\lambda}(\mathbf{r},\lambda)\,dA_p,$$

or

$$E'_{b\lambda}(\mathbf{r},\lambda,\theta,\psi) = I_{b\lambda}(\mathbf{r},\lambda)\cos\theta, \tag{1.35}$$

that is, the directional emitted flux of a blackbody varies with the cosine of the polar angle. This is sometimes referred to as *Lambert's law*[*] or the *cosine law.*

1.7 RADIATIVE HEAT FLUX

Consider the surface shown in Fig. 1-10. Let thermal radiation from an infinitesimal solid angle around the direction $\hat{\mathbf{s}}_i$ impinge onto the surface with an intensity of $I_\lambda(\hat{\mathbf{s}}_i)$. Such radiation is often called a "pencil of rays" since the infinitesimal solid angle is usually drawn looking like the tip of a sharpened pencil. Recalling the definition for intensity we see that it imparts an infinitesimal heat flow rate per wavelength on the surface in the amount of

$$dQ_\lambda = I_\lambda(\hat{\mathbf{s}}_i)\,d\Omega_i\,dA_p = I_\lambda(\hat{\mathbf{s}}_i)\,d\Omega_i\,(dA\cos\theta_i),$$

where heat rate is taken as positive in the direction of the outward surface normal (going into the medium), so that the incoming flux going *into the surface* is negative since $\cos\theta_i < 0$. Integrating

---

[*] **Johann Heinrich Lambert (1728–1777)**
German mathematician, astronomer, and physicist. Largely self-educated, Lambert did his work under the patronage of Frederick the Great. He made many discoveries in the areas of mathematics, heat, and light. The lambert, a measurement of diffusely reflected light intensity, is named in his honor (see Section 1.9).

over all 2π incoming directions and dividing by the surface area gives the total incoming heat flux per unit wavelength, i.e.,

$$(q_\lambda)_{\text{in}} = \int_{\cos\theta_i<0} I_\lambda(\hat{\mathbf{s}}_i)\cos\theta_i\,d\Omega_i. \tag{1.36}$$

Heat loss from the surface, along a pencil of rays into the direction $\hat{\mathbf{s}}_o$, and integrated over all outgoing directions, follows as

$$(q_\lambda)_{\text{out}} = \int_{\cos\theta_o>0} I_\lambda(\hat{\mathbf{s}}_o)\cos\theta_o\,d\Omega_o. \tag{1.37}$$

If the surface is black ($\epsilon_\lambda = 1$), there is no energy reflected from the surface and $I_\lambda = I_{b\lambda}$, leading to $(q_\lambda)_{\text{out}} = E_{b\lambda}$. If the surface is not black, the outgoing intensity consists of contributions from emission as well as reflections. The outgoing heat flux is positive since it is going *into the medium.* The net heat flux from the surface may be calculated by adding both contributions, or

$$(q_\lambda)_{\text{net}} = (q_\lambda)_{\text{in}} + (q_\lambda)_{\text{out}} = \int_{4\pi} I_\lambda(\hat{\mathbf{s}})\cos\theta\,d\Omega, \tag{1.38}$$

where a single direction vector $\hat{\mathbf{s}}$ was used to describe the total range of solid angles, 4π. It is readily seen from Fig. 1-10 that $\cos\theta = \hat{\mathbf{n}}\cdot\hat{\mathbf{s}}$ and, since the net heat flux is evaluated as the flux into the positive $\hat{\mathbf{n}}$-direction, one gets

$$(q_\lambda)_{\text{net}} = \mathbf{q}_\lambda\cdot\hat{\mathbf{n}} = \int_{4\pi} I_\lambda(\hat{\mathbf{s}})\,\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}\,d\Omega. \tag{1.39}$$

In order to obtain the *total radiative heat flux* at the surface, equation (1.39) needs to be integrated over the spectrum, and

$$\mathbf{q}\cdot\hat{\mathbf{n}} = \int_0^\infty \mathbf{q}_\lambda\cdot\hat{\mathbf{n}}\,d\lambda = \int_0^\infty\int_{4\pi} I_\lambda(\hat{\mathbf{s}})\,\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}\,d\Omega\,d\lambda. \tag{1.40}$$

Example 1.5. A solar collector mounted on a satellite orbiting Earth is directed at the sun (i.e., normal to the sun's rays). Determine the total solar heat flux incident on the collector per unit area.

Solution
The total heat rate leaving the sun is $\dot{Q}_S = 4\pi R_S^2 E_b(T_S)$, where $R_S \simeq 6.96\times10^8$ m is the radius of the sun. Placing an imaginary spherical shell around the sun of radius $S_{ES} = 1.496\times10^{11}$ m, where S_{ES} is the distance between the sun and Earth, we find the heat flux going through that imaginary sphere (which includes the solar collector) as

$$q_{\text{sol}} = \frac{4\pi R_S^2 E_b(T_S)}{4\pi S_{ES}^2} = I_b(T_S)\frac{\pi R_S^2}{S_{ES}^2} = I_b(T_S)\,\Omega_S,$$

where we have replaced the sun's emissive power by intensity, $E_b = \pi I_b$, and $\Omega_S = 6.79\times10^{-5}$ sr is the solid angle with which the sun is seen from Earth, as determined in Example 1.3. Therefore, with $I_b(T_S) = \sigma T_S^4/\pi$ and $T_S = 5777$ K,

$$\begin{aligned} q_{\text{in}} &= -(\sigma T_S^4/\pi)(\Omega_S) = -\frac{1}{\pi}5.670\times10^{-8}\times5777^4\times6.79\times10^{-5}\,\text{W/m}^2\\ &= -1366\,\text{W/m}^2, \end{aligned}$$

where we have added a minus sign to emphasize that the heat flux is going *into* the collector. The total incoming heat flux may, of course, also be determined from equation (1.36) as

$$q_{\text{in}} = \int_{\cos\theta_i<0} I(\hat{\mathbf{s}}_i)\cos\theta_i\,d\Omega_i.$$

> Since light from the sun arrives only from the extremely small solid angle of Ω_S (over which $\cos\theta_i \simeq -1$), the integral may be written as
>
> $$q_{\text{in}} = I_i \times (-1) \times \Omega_S.$$
>
> Comparing this with the previous expression we see that the incoming solar intensity at the collector, I_i, is identical to the intensity leaving from the sun's surface, $I_b(T_S)$. This invariability property of radiative intensity as it travels through vacuum will be discussed in more detail in Chapter 10.

Solar Constant. The absolute value of the heat flux determined in the previous example is known as the *solar constant*. In reality, however, solar irradiation onto the surface of the Earth varies continuously, exhibiting variations on all time scales, from minutes to decades. The most prominent feature is its annual fluctuation due to Earth's varying distance from the Sun (with a maximum of about 1412 W/m^2 in early January, and a minimum of about 1321 W/m^2 in early July; a variation of approximately ±3.4%). Accurate satellite-based observations have also confirmed an 11-year cycle due to cyclical solar activity, with annual averages varying between approximately 1365.6 W/m^2 and 1366.7 W/m^2 (or ±0.04%) [11,12]. The generally accepted annual mean value for the 'solar constant' appears to be

$$q_{\text{sol}} = 1366\,\text{W/m}^2, \tag{1.41}$$

and is qualitatively indicated in Fig. 1-3. The effective solar temperature of $T = 5777$ K has been determined by working Example 1.5 in reverse.

1.8 RADIATION PRESSURE

If we think of radiative energy as photons or (massless) particles carrying an energy of $h\nu$ and traveling at the speed of light c into a certain direction, then these particles should carry momentum in the amount of energy/speed = $h\nu/c = h\eta$ (even though they do not possess mass). Therefore, if photons hit a material surface a momentum transfer takes place, which implies that a stream of photons exerts pressure on the walls of a container, known as *radiation pressure* or *photon pressure*.

Consider a monochromatic beam of intensity $I_\lambda(\hat{\mathbf{s}}_i)$ incident on a surface element dA from a direction $\hat{\mathbf{s}}_i$ and over a solid angle $d\Omega_i$. The energy flow incident on dA due to this beam is $I_\lambda(\hat{\mathbf{s}}_i)\cos\theta_i\,dA\,d\Omega_i$. Therefore, the beam carries with it momentum at a rate of

$$\frac{1}{c} I_\lambda(\hat{\mathbf{s}}_i)\,|\cos\theta_i|\,dA\,d\Omega_i\,\hat{\mathbf{s}}_i,$$

where the unit vector $\hat{\mathbf{s}}_i$ has been added to emphasize that the momentum flow is a vector pointing into the direction of $\hat{\mathbf{s}}_i$. The fraction of momentum falling onto dA in the normal direction is $|\cos\theta_i| = |\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}_i|$. Therefore, the flow of momentum onto dA in the normal direction is, per unit area,

$$\frac{1}{c} I_\lambda(\hat{\mathbf{s}}_i)\,\cos^2\theta_i\,d\Omega_i.$$

According to *Newton's Second Law* the total momentum flux normal to the surface, due to irradiation from all possible directions, must be counteracted by a pressure force $p_\lambda\,dA$ leading to a *spectral radiation pressure*

$$p_\lambda = \frac{1}{c}\int_0^{2\pi}\int_0^{\pi} I_\lambda(\hat{\mathbf{s}})\cos^2\theta\sin\theta\,d\theta\,d\psi. \tag{1.42}$$

As for other spectral properties, a *total radiation pressure* is defined as

$$p = \int_0^\infty p_\lambda\,d\lambda. \tag{1.43}$$

Example 1.6. A very light spacecraft had been entered in the Columbus Competition (which was originally scheduled for 1992, the 500th anniversary of Columbus' discovery of the New World, but which was eventually cancelled for lack of funds), whose goal was to sail the craft to Mars, using solar radiation pressure as the only propellant. Assuming a black sail and a necessary pushing force of 100 N, determine the necessary surface area for the solar sail.

Solution

The solar pressure exerted on the sail may be calculated from equation (1.42). As in the previous example solar radiation is incident over an extremely small solid angle, $\Omega_S = 6.79 \times 10^{-5}$ sr. Thus, the solar pressure may be evaluated as

$$p = \frac{1}{c}\frac{\sigma T_S^4}{\pi}(-1)^2\Omega_S = \frac{q_{\text{sol}}}{c} = \frac{1366\,\text{W/m}^2}{2.998 \times 10^8\,\text{m/s}} = 4.56 \times 10^{-6}\,\text{W s/m}^3 = 4.56 \times 10^{-6}\,\text{N/m}^2.$$

The total force on the sail is $F = pA$, so that a force of 100 N would require a sail area of $A = 100/4.56 \times 10^{-6} = 22 \times 10^6\,\text{m}^2 = 22\,\text{km}^2$!

Note that, for a perfectly reflecting sail, the reflected intensity would be equal to the incoming intensity, but in the opposite direction with $\cos\theta = +1$. The contribution to the radiation pressure would be positive again, doubling the pressure to 9×10^{-6} N and halving the necessary sail area to 11 km^2.

This example demonstrates that radiation pressure, while measurable and nonnegligible for some applications, can certainly not compete with the pressure exerted by a molecular gas.

1.9 VISIBLE RADIATION (LUMINANCE)

Because of the great importance of visible radiation, and since much of the theory developed in this book is directly applicable to lighting design calculations, we shall very briefly discuss the nature of electromagnetic waves as perceived by the human eye. For a more detailed treatment of lighting, the reader should refer to books on the subject, such as the ones by Moon [13] and by Hopkinson and coworkers [14], or the IES (Illuminating Engineering Society) Handbook [15].

When electromagnetic waves of spectral intensity I_λ fall onto the human eye, a certain fraction of the intensity is observed as "light," known as *spectral luminance*,

$$L_\lambda = K_\lambda I_\lambda, \tag{1.44}$$

where the proportionality factor K_λ is called the *luminous efficacy*. Spectral luminance is measured in *lumen* per unit wavelength, per unit solid angle, and per unit area normal to the rays (lm/m^2 μm sr). Therefore, the units of the luminous efficacy are lm/W. Luminance may also be measured using *candelas* (cd), *lux* (lx) or *lamberts* (L), where the different units are related by

$$1\,\text{L} = \frac{1}{\pi}\,\text{cd/cm}^2, \quad 1\,\text{cd} = 1\,\text{lm/sr}, \quad 1\,\text{lx} = 1\,\text{lm/m}^2.$$

The average human eye responds to radiation in the wavelength interval between $\sim 0.4\,\mu$m and $0.7\,\mu$m, and K_λ thus vanishes outside this range. A *standard luminous efficacy* to approximate the average human eye was set by the *CIE* (*Commission Internationale de l'Eclairage* or International Commission on Illumination) in 1924 and is shown in Fig. 1-11. Also shown is the standard *luminous efficiency*, defined as

$$\eta_{\text{lum},\lambda} = K_\lambda/K_{\max}; \quad K_{\max} = K_\lambda(\lambda = 0.555\,\mu\text{m}) = 683\,\text{lm/W}, \tag{1.45}$$

where $K_{\max}$ is the spectral maximum of K_λ, which occurs (for the standard) at $\lambda = 0.555\,\mu$m. The total luminance is calculated from

$$L = \int_0^\infty K_\lambda I_\lambda\, d\lambda, \tag{1.46}$$

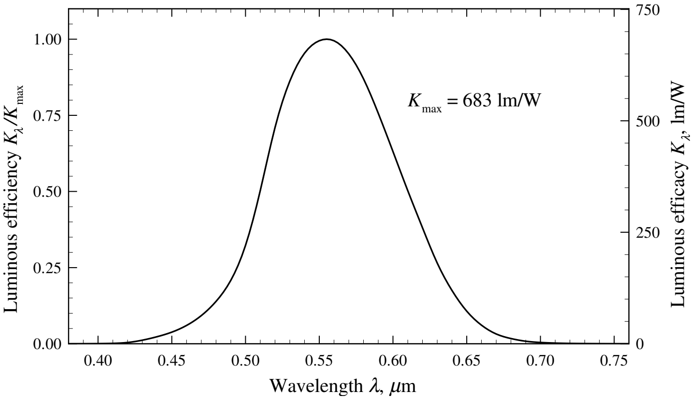

FIGURE 1-11
CIE standards for luminous efficiency and luminous efficacy.

which—if I_λ does not vary too much over the interval $0.4\,\mu\text{m} < \lambda < 0.7\,\mu\text{m}$—may be approximated as

$$\begin{aligned} L &\simeq I_\lambda(\lambda = 0.555\,\mu\text{m})K_{\max}\int_0^\infty \eta_{\text{lum},\lambda}\,d\lambda \\ &\simeq 86\,\frac{\text{lm}\,\mu\text{m}}{\text{W}} \times I_\lambda(\lambda\!=\!0.555\,\mu\text{m}) \simeq 286\,\frac{\text{lm}}{\text{W}}\int_{0.4\mu\text{m}}^{0.7\mu\text{m}} I_\lambda\,d\lambda, \end{aligned} \tag{1.47}$$

where 86 lm μm/W is the area underneath the luminous efficiency in Fig. 1-11, and 286 lm/W is an appropriate average value for K_λ.

Example 1.7. On a clear day the strength of solar radiation has been measured as $q_{\text{sun}} = 800\,\text{W/m}^2$ (normal to the sun's rays), while total sky radiation (from all directions), falling onto a horizontal surface, has been determined as $q_{\text{sky}} = 200\,\text{W/m}^2$. Determine the luminous flux, or *illumination*, onto a horizontal surface if the sun is at a zenith angle of 60°.

Solution
Similar to the radiative heat flux, the luminous flux onto a surface is defined as

$$q_{\text{lum}} = \int_{2\pi} L(\hat{\mathbf{s}})\cos\theta\,d\Omega.$$

Therefore, from equations (1.46) and (1.47)

$$q_{\text{lum}} = \int_{2\pi} 286\,\frac{\text{lm}}{\text{W}}\int_{0.4\mu\text{m}}^{0.7\mu\text{m}} I_\lambda\,d\lambda\,\cos\theta\,d\Omega = 286\,\frac{\text{lm}}{\text{W}}\int_{0.4\mu\text{m}}^{0.7\mu\text{m}}\int_{2\pi} I_\lambda\cos\theta\,d\Omega = 286\,\frac{\text{lm}}{\text{W}}\int_{0.4\mu\text{m}}^{0.7\mu\text{m}} q_\lambda\,d\lambda,$$

where q_λ is the total spectral irradiation from the sun and the sky. Since both contributions are due to solar emission (either by direct travel or after atmospheric scattering), we may write $q_\lambda = (q_{\text{sun}}\cos\theta_{\text{sun}} + q_{\text{sky}})E_{b\lambda}(T_{\text{sun}})/E_b(T_{\text{sun}})$, or

$$q_{\text{lum}} = 286\,\frac{\text{lm}}{\text{W}}(q_{\text{sun}}\cos\theta_{\text{sun}} + q_{\text{sky}})\left[f(0.7\,\mu\text{m}\,T_{\text{sun}}) - f(0.4\,\mu\text{m}\,T_{\text{sun}})\right],$$

or, with $T_{\text{sun}} = 5777\,\text{K}$,

$$q_{\text{lum}} = 286\,\frac{\text{lm}}{\text{W}}\left(800\times\frac{1}{2} + 200\right)\frac{\text{W}}{\text{m}^2}[0.48869 - 0.12220] = 62{,}890\,\text{lm/m}^2 = 62{,}890\,\text{lx}.$$

Actually, for the visible wavelengths a solar temperature of $T_{\text{sun}} = 5777\,\text{K}$ is not very accurate (cf. Fig. 1-3) so usually a solar temperature of 6500 K is employed for luminance calculations. With such a temperature $f(0.7\times 6500\,\mu\text{m}\,\text{K}) - f(0.4\times 6500\,\mu\text{m}\,\text{K}) = 0.57177 - 0.18311$ and

$$q_{\text{lum}} \simeq 66{,}694\,\text{lx}.$$

All relationships for radiative intensity and radiative heat flux that will be developed in this book may also be used to determine luminance and luminous flux simply by using units of *lumens* instead of *watts*.

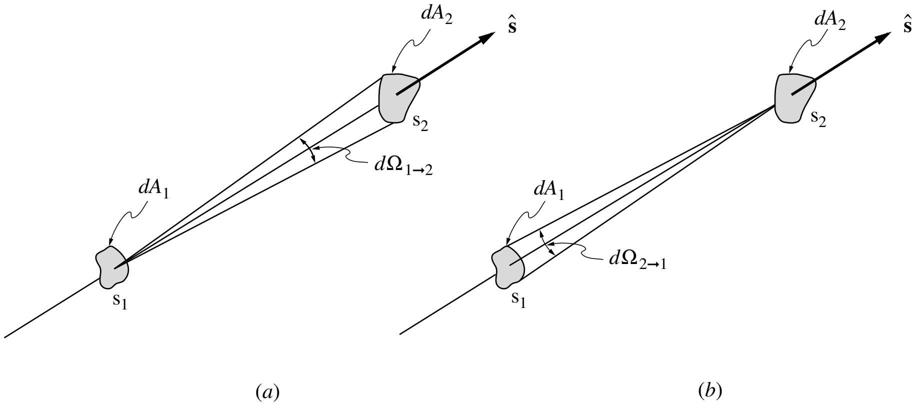

FIGURE 1-12
Radiative intensity in vacuum; *(a)* radiation leaving dA_1 toward dA_2, *(b)* radiation arriving at dA_2 from dA_1.

1.10 RADIATIVE INTENSITY IN VACUUM

Before we discuss how radiative intensity is affected by absorption, emission, and scattering, it is important to understand how intensity penetrates through a vacuum. When discussing emission from a black surface we noticed that the concept of intensity had one advantage over emissive power, namely, that the emitted intensity did not vary with direction. Within a medium, the definition of an emissive power is not applicable since there is no surface to which to relate it. The intensity, defined as radiative energy transferred per unit time, solid angle, spectral variable, and area normal to the pencil of rays, is the most appropriate variable to describe the radiative transfer within a medium.

Consider radiative intensity penetrating at normal angle through a (fictitious) infinitesimal area dA_1, at location s_1 and time t_1, as shown in the sketch of Fig. 1-12. Based on the definition of intensity, we see that the amount of energy passing through dA_1 over a duration dt and spectral range $d\eta$ that will—a little later—fall onto the infinitesimal surface dA_2, is

$$I_\eta(s_1, t_1)\, dt\, d\Omega_{1\to2}\, d\eta\, dA_1 = I_\eta(s_1, t_1)\, dt \frac{dA_2}{(s_2 - s_1)^2}\, d\eta\, dA_1,$$

where $d\Omega_{1\to2}$ is the solid angle with which dA_2 is seen from an observer on dA_1. Since it takes the radiation until the time $t_2 = t_1 + (s_2 - s_1)/c$ to travel from s_1 to s_2, we can say that the energy going through dA_2 that is coming from dA_1 is

$$I_\eta(s_2, t_2)\, dt\, d\Omega_{2\to1}\, d\eta\, dA_2 = I_\eta(s_2, t_2)\, dt \frac{dA_1}{(s_2 - s_1)^2}\, d\eta\, dA_2.$$

Since both energies must be equal, we conclude that

$$I_\eta(s_2, t_1 + (s_2 - s_1)/c) = I_\eta(s_1, t_1). \tag{1.48}$$

Since the speed of light is so large in comparison with nearly every time scale in engineering problems, we may almost always assume that the radiative energy arrives "instantaneously" everywhere in the medium, i.e.,

$$I_\eta(s_2) = I_\eta(s_1), \tag{1.49}$$

or

$$I_\eta(\hat{\mathbf{s}}) = \text{const.} \tag{1.50}$$

Therefore, within a medium that does not emit, absorb, or scatter radiation (known as a "nonparticipating medium"), the radiative intensity in any given direction is constant along its path.

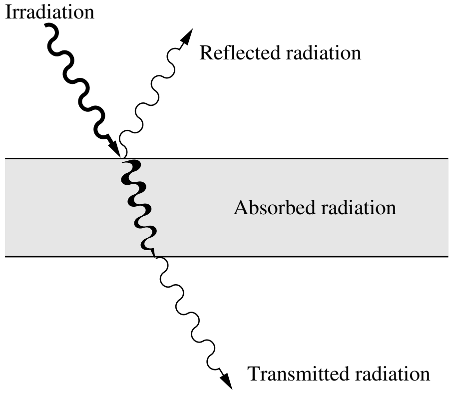

FIGURE 1-13
Absorption, reflection, and transmission by a slab.

Conversely, in a "participating medium" (a medium that does absorb, emit, and/or scatter) any changes in intensity along any given path must be due to one or more of these phenomena.

1.11 INTRODUCTION TO RADIATION CHARACTERISTICS OF OPAQUE SURFACES

We have already noted that thermal radiation, unlike conduction and convection, is a long-range phenomenon. When making an energy balance for a point in space we must account for all photons that may arrive at this point, no matter from how far away. Thus, a *conservation of energy* balance must be performed on an *enclosure* bounded by *opaque walls* (i.e., a medium thick enough that no electromagnetic waves can penetrate through it). Strictly speaking, the surface of an enclosure wall can only reflect radiative energy or allow a part of it to penetrate into the substrate. A surface cannot absorb or emit photons: Attenuation takes place inside the solid, as does emission of radiative energy (and some of the emitted energy escapes through the surface into the enclosure). In practical systems the thickness of the surface layer over which absorption of irradiation from inside the enclosure occurs is very small compared with the overall dimensions of an enclosure—usually a few Å for metals and a few μm for most nonmetals. The same may be said about emission from within the walls that escapes into the enclosure. Thus, in the case of opaque walls it is customary to speak of absorption by and emission from a "surface," although a thin surface layer is implied.

Consider thermal radiation impinging on a medium of finite thickness, as shown in Fig. 1-13. In general, some of the irradiation will be *reflected* away from the medium, a fraction will be *absorbed* inside the layer, and the rest will be *transmitted* through the slab. Based on this observation we define three fundamental radiative properties:[5]

$$\textit{Reflectance,} \quad \rho \equiv \frac{\text{reflected part of incoming radiation}}{\text{total incoming radiation}}, \tag{1.51a}$$

$$\textit{Absorptance,} \quad \alpha \equiv \frac{\text{absorbed part of incoming radiation}}{\text{total incoming radiation}}, \tag{1.51b}$$

$$\textit{Transmittance,} \quad \tau \equiv \frac{\text{transmitted part of incoming radiation}}{\text{total incoming radiation}}. \tag{1.51c}$$

---

[5] The National Institute of Standards and Technology (NIST, formerly NBS) has recommended to reserve the ending "-ivity" for radiative properties of pure, perfectly smooth materials (i.e., reflectivity, absorptivity, transmissivity, and, for emission, emissivity), and "-ance" for rough and contaminated surfaces. Most real surfaces fall into the latter category, discussed in Chapter 3. While we will follow this convention throughout this book, the reader should be aware that many researchers in the field employ endings according to their own personal preference.

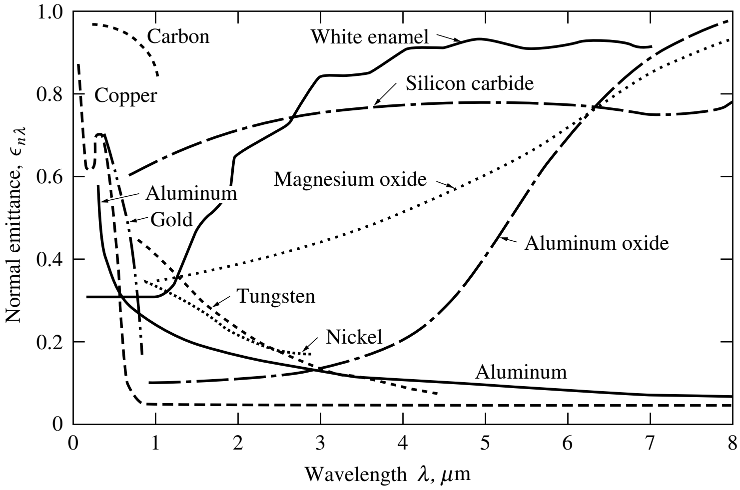

FIGURE 1-14
Normal, spectral emittances for selected materials, from [16].

Since all radiation must be either reflected, absorbed, or transmitted, we conclude

$$\rho + \alpha + \tau = 1. \tag{1.52}$$

If the medium is sufficiently thick to be *opaque*, then $\tau = 0$ and

$$\rho + \alpha = 1. \tag{1.53}$$

We note that all three of these properties are nondimensional and may vary in magnitude between the values 0 and 1. Since a black surface absorbs all incoming radiation it follows for such a surface that $\alpha = 1$ and $\rho = \tau = 0$.

All surfaces also emit thermal radiation (or, rather, radiative energy is emitted within the medium, some of which escapes from the surface). Since we know that, at a given temperature, the maximum possible is emitted by a black surface, we define a fourth nondimensional property:

$$\textit{Emittance,} \quad \epsilon = \frac{\text{energy emitted from a surface}}{\text{energy emitted by a black surface at same temperature}}. \tag{1.54}$$

The emittance can, therefore, also vary between the values of 0 and 1, and for a black surface we have $\epsilon = 1$.

All four properties may be functions of temperature as well as of wavelength (or frequency). The absorptance may be different for different directions of irradiation, while emittance may vary with outgoing directions. Finally, the magnitude of reflectance and transmittance may depend on both incoming and outgoing directions. Thus, we distinguish between spectral and total properties (i.e., an average value over the spectrum), and between directional and hemispherical properties (i.e., an average value over all directions).

Typical spectral behavior of surface emittances is shown in Fig. 1-14 for a few materials, as collected by White [16]. Shown are values for directional emittances in the direction normal to the surface. However, the spectral behavior is similar to *hemispherical emittances* (i.e., emittances averaged over all directions). In general, nonmetals have relatively high emittances, which may vary erratically across the spectrum, and metals behave similarly for short wavelengths but tend to have lower emittances with more regular spectral dependence in the infrared. A more detailed description of the definitions, evaluation, and measurement of radiative surface properties is given in Chapter 3.

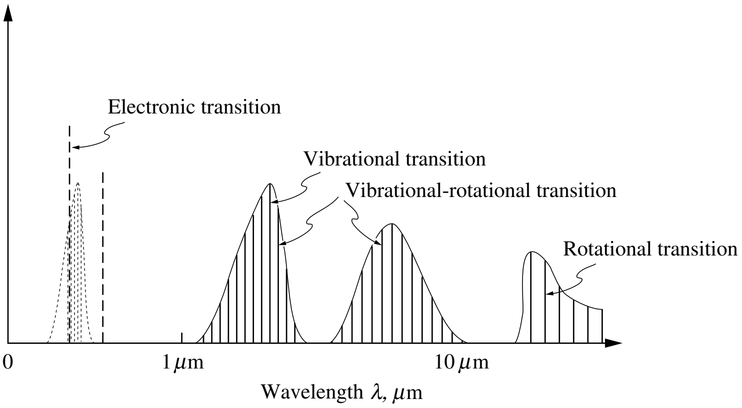

FIGURE 1-15
Spectral lines due to electronic, vibrational, and rotational energy changes in a gas molecule.

1.12 INTRODUCTION TO RADIATION CHARACTERISTICS OF GASES

Like a solid medium (or the thin layer next to its surface), gases can absorb and emit radiative energy. All gas atoms or molecules carry a certain amount of energy, consisting of kinetic energy (translational energy of a molecule) and energy internal to each molecule. The internal molecular energy, in turn, consists of a number of contributions, primarily of the levels of electronic, vibrational, and rotational energy states. Thus, a passing photon may be absorbed by a molecule raising the level of one of the internal energy states. On the other hand, a molecule may spontaneously release (emit) a photon in order to lower one of its internal energy states.

Quantum mechanics postulates that only a finite number of discrete energy levels are possible, i.e., that electrons can circle around the nucleus only on a number of allowed orbits, vibration between nuclei can only occur with a number of distinct frequencies, and the nuclei can rotate around one another only with a number of allowed rotational velocities. Therefore, changing the internal energy of a molecular gas can only destroy or generate photons with very distinct energy levels $h\nu$ and, consequently, only at distinct frequencies or wavelengths. It takes a relatively large amount of energy to change the orbit of an electron, giving rise to absorption–emission lines in the ultraviolet and visible parts of the spectrum, but generally only at extremely high temperatures rarely encountered in engineering. In the case of a monatomic gas only the electronic energy level can be altered by a photon. Changing the vibrational energy level of a molecule requires an intermediate amount of energy, resulting in spectral lines in the near to intermediate infrared ($1\,\mu\text{m} \lesssim \lambda \lesssim 15\,\mu\text{m}$). Finally, rotational energy changes take place with an even smaller amount of energy, so that rotational lines are found in the intermediate to far infrared ($\lambda \gtrsim 10\,\mu\text{m}$). Usually, vibrational energy changes are accompanied by simultaneous changes in rotational energy levels, so that the vibrational lines are surrounded by many rotational lines, as illustrated in Fig. 1-15. Similarly, electronic transitions in molecules are accompanied by many different vibrational and rotational energy changes.

Unless the temperature of a gas is extremely high, the gas will essentially be free of ions and free electrons. In this case, absorption or emission of photons results in *bound–bound transitions* (i.e., there is no ionization before and after the transition), and *discrete spectral lines*, as shown in Fig. 1-15. If absorption of a photon results in ionization and the release of an electron, the transition is termed *bound–free*. Conversely, a free electron may combine with an ion in a *free–bound transition*, producing a photon. Finally, a free electron can absorb or emit photons, resulting in *free–free transitions*. Since an electron can have arbitrary kinetic energy, all transitions involving free electrons are not limited to discrete wavelengths and produce continuous spectra.

In reality, not even the lines from bound–bound transitions are truly discrete, but are *broad-*

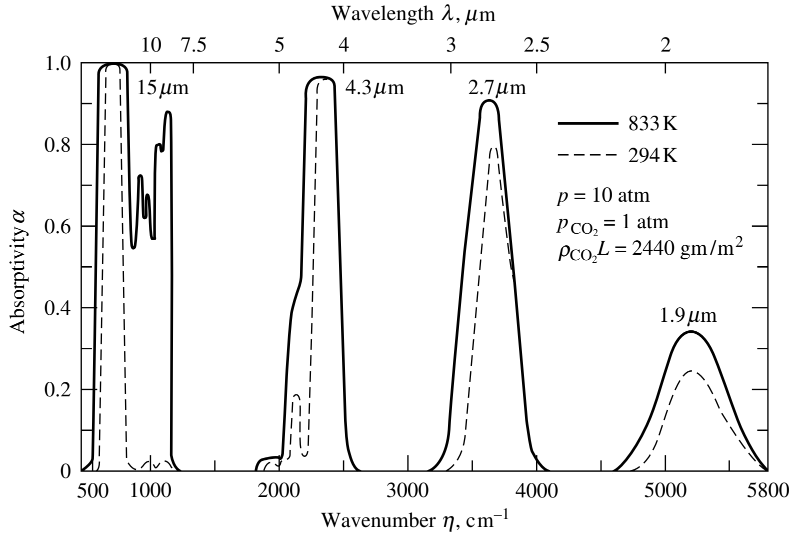

FIGURE 1-16
Spectral absorptivity of an isothermal mixture of nitrogen and carbon dioxide, from Edwards [17].

ened slightly. The rotational lines accompanying a vibrational transition usually overlap, forming what is termed *vibration–rotation bands,* as also indicated in Fig. 1-15.

As radiative energy penetrates through a gas layer it gradually becomes attenuated by absorption. Experience (and theoretical development) shows that this absorption leads to an exponential decay of incident radiation, so that the *transmissivity* of a homogeneous isothermal gas layer may be written as

$$\tau_\eta = e^{-\kappa_\eta s}, \tag{1.55}$$

where s is the thickness of the gas layer and the proportionality constant κ_η is known as the *absorption coefficient*. The exponential decay of incident radiation is also commonly called *Beer's law*. In the expression for transmissivity we have used the wavenumber η as the spectral variable, since the wavenumber is commonly chosen by researchers working in the field of gas radiation. Since, in the case of a gas layer, incident radiation is either transmitted or absorbed, we define the spectral absorptivity of a gas layer as

$$\alpha_\eta = 1 - \tau_\eta = 1 - e^{-\kappa_\eta s}. \tag{1.56}$$

Figure 1-16 shows a typical absorption spectrum for a nitrogen–carbon dioxide mixture, taken from the early work of Edwards [17]. The formation of vibration–rotation bands in the infrared due to bound–bound transitions, separated by spectral windows, is clearly visible. The line structure is lost in this figure, partly because of the aforementioned line broadening, which tends to be strong for high pressures, and partly because of poor spectral resolution in these old measurements.

Not all gases have vibration–rotation bands. In particular, dry air (i.e., nitrogen and oxygen) at moderate temperature does not absorb or emit radiation in the infrared. Radiative properties of gases are described in much more detail in Chapter 11.

1.13 INTRODUCTION TO RADIATION CHARACTERISTICS OF SOLIDS AND LIQUIDS

There are a number of liquid and solid substances that absorb radiative energy only gradually, such that these materials cannot be approximated as "opaque surfaces." They are known

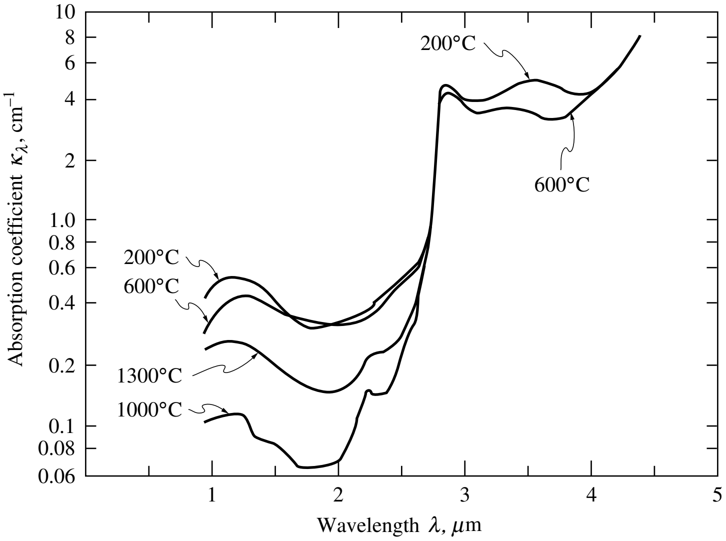

FIGURE 1-17
Spectral absorption coefficient of window glass, from Neuroth [18].

as *semitransparent.* Typical examples for the visible part of the spectrum are water, glass, quartz, etc. Absorption and emission of photons in liquids are due to interactions with free electrons, and within solids due to interactions with free electrons as well as due to excitation of lattice vibrations (*phonons*). Consequently, semitransparent materials are always poor electric conductors (few free electrons). The absorption behavior of these materials is qualitatively similar to that of dissociated gases (free–free transitions for the electrons), with the possibility of strong absorption bands due to *photon–phonon* interactions. As an example the absorption coefficient for window glass is shown in Fig. 1-17, taken from measurements by Neuroth [18]. A slightly more detailed discussion of the radiative properties of semitransparent media is given in Chapter 13.

1.14 INTRODUCTION TO RADIATION CHARACTERISTICS OF PARTICLES

The interaction between photons, or electromagnetic waves, and small particles is somewhat different from that with a homogeneous gas, liquid, or solid. As for a homogeneous medium, radiation traveling through a particle cloud may be transmitted, reflected, or absorbed. In addition, interaction with a particle may change the direction in which a photon travels, as schematically shown in Fig. 1-18. This can occur by one of three different mechanisms: *(i)* The path of a photon may be altered, without ever colliding with the particle, by *diffraction, (ii)* a photon may change its direction by *reflection* from the particle, and *(iii)* the photon may penetrate into the particle, changing its direction because of *refraction.* All three phenomena together are known as the *scattering* of radiation. Absorption takes place as the electromagnetic wave penetrates into the particle. Therefore, in the presence of scattering, the equation for transmissivity of a material layer, equation (1.55), must be augmented to

$$\tau_\eta = e^{-(\kappa_\eta+\sigma_{s\eta})s} = e^{-\beta_\eta s}, \tag{1.57}$$

where $\sigma_{s\eta}$ is known as the *scattering coefficient* and $\beta_\eta = \kappa_\eta + \sigma_{s\eta}$ as the *extinction coefficient.* Scattered rays travel on unattenuated, and the probability that a ray from one direction will be scattered into a certain other direction by an angle θ is described by the *scattering phase function,*

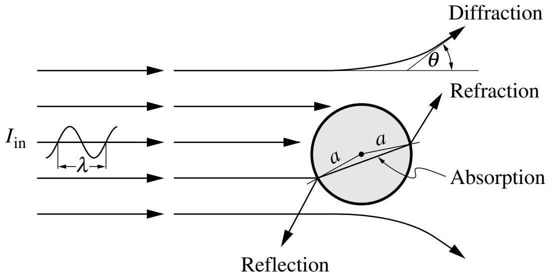

FIGURE 1-18
Interaction of electromagnetic waves with small particles.

Φ_η. It is also common practice to introduce the *single scattering albedo*, defined as

$$\omega_\eta \equiv \frac{\sigma_{s\eta}}{\kappa_\eta + \sigma_{s\eta}} = \frac{\sigma_{s\eta}}{\beta_\eta}, \tag{1.58}$$

which gives the relative importance of scattering: a medium that does not scatter at all has $\omega = 0$, while $\omega = 1$ indicates a pure scatterer (nonabsorbing).

The nature of the interaction between electromagnetic waves and particles is determined by the relative size of the particles compared with the wavelength of the radiation. Defining a *size parameter*

$$x = \frac{2\pi a}{\lambda}, \tag{1.59}$$

where a is the effective radius of the particle, we distinguish among three different regimes:

1. $x \ll 1$, or *Rayleigh scattering*, named after Lord Rayleigh, who studied the interaction of atmospheric air (whose molecules are, in fact, very small particles) with sunlight [19]. He observed that, for very small particles, scattering is proportional to ν^4 or $1/\lambda^4$. Thus, within the visible part of the spectrum, blue light is scattered the most (accounting for blue skies) and red the least (resulting in red sunsets).

2. $x = \mathcal{O}(1)$, or *Mie scattering*, named after Gustav Mie, who developed a comprehensive (and difficult) theory for the interaction between electromagnetic waves and particles [20].

3. $x \gg 1$. In this case the surface of the particle may be treated as a normal surface, and properties may be found through *geometric optics.*

A more detailed discussion of the radiative properties of particles is given in Chapter 12.

1.15 THE RADIATIVE TRANSFER EQUATION

To determine the radiative flux onto a surface (or at any point inside the medium) requires knowledge of the radiative intensity at that point, for all directions and for all wavelengths or wavenumbers. The intensity field inside a participating medium is governed by the *radiative transfer equation*, which is a radiative energy balance along a thin pencil of rays, as was done in Section 1.10, but now for a medium that emits, absorbs, and/or scatters radiation. This equation, which will be rigorously developed in Chapter 10, states that, along a line of sight into the direction $\hat{\mathbf{s}}$, the spectral radiative intensity changes according to

$$\frac{dI_\eta}{ds} = \kappa_\eta I_{b\eta} - \kappa_\eta I_\eta - \sigma_{s\eta} I_\eta + \frac{\sigma_{s\eta}}{4\pi}\int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\,\Phi_\eta(\hat{\mathbf{s}}_i,\hat{\mathbf{s}})\,d\Omega_i. \tag{1.60}$$

The terms on the right-hand-side indicate how, along a path s, the intensity is augmented by emission ($\kappa_\eta I_{b\eta}$) and decreased by absorption ($\kappa_\eta I_\eta$) and by scattering away from $\hat{\mathbf{s}}$ ($\sigma_{s\eta} I_\eta$).

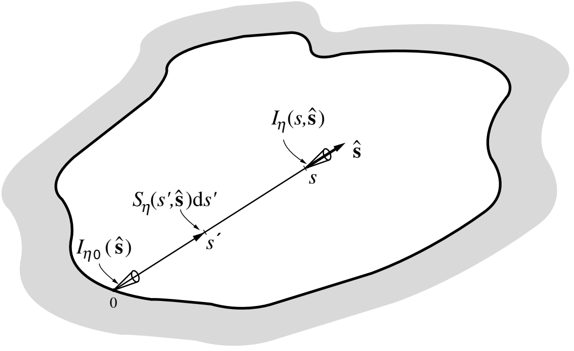

FIGURE 1-19
Intensity variation inside a participating medium.

Scattered radiation is simply channeled into other directions and, thus, the last term in equation (1.60) is the increase of intensity due to radiation from other directions (the scattering phase function $\Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})$ gives the probability that a ray coming from $\hat{\mathbf{s}}_i$ is scattered toward $\hat{\mathbf{s}}$). We note that if $\kappa_\eta = \sigma_{s\eta} = 0$ (nonparticipating medium) equation (1.60) reduces to equation (1.50):

$$\frac{dI_\eta}{ds} = 0 \quad \text{or} \quad I_\eta(\hat{\mathbf{s}}) = \text{const.} \tag{1.61}$$

To better distinguish between attenuation and augmentation of radiation, we may introduce the *source function* for radiative intensity,

$$S_\eta(s, \hat{\mathbf{s}}) = (1 - \omega_\eta)I_{b\eta} + \frac{\omega_\eta}{4\pi} \int_{4\pi} I_\eta(\hat{\mathbf{s}}_i)\, \Phi_\eta(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\, d\Omega_i, \tag{1.62}$$

where we have used the single scattering albedo ω, defined by equation (1.58). This simplifies the radiative transfer equation, commonly abbreviated as RTE, to

$$\frac{dI_\eta}{ds} = \beta_\eta[S_\eta(s, \hat{\mathbf{s}}) - I_\eta]. \tag{1.63}$$

Its formal solution is readily found, and for the special case of a spatially invariant extinction coefficient, $\beta_\eta = \text{const}$, follows as

$$I_\eta(s) = I_\eta(0)\, e^{-\beta_\eta s} + \int_0^s S_\eta(s', \hat{\mathbf{s}})\, e^{-\beta_\eta(s-s')}\, \beta_\eta\, ds'. \tag{1.64}$$

The local intensity at s consists of two parts (cf. Fig. 1-19): *(i)* intensity entering the medium at $s = 0$ (say, from a bounding wall) attenuated by absorption and out-scattering from 0 to s, and *(ii)* increase of intensity along the path at s' (by emission and/or in-scattering) attenuated along the remaining distance between s' and s. It is important to remember that, if the medium scatters ($\omega > 0$), the source function in equation (1.64) contains the radiative intensity at every point along the path, for all possible directions (not just $\hat{\mathbf{s}}$): the radiative transfer equation, equation (1.60), is an integro-differential equation (intensity appears, both, as a derivative and also inside the integral on the right-hand side) in five dimensions (three space dimensions and two directional coordinates). This makes the RTE extremely difficult to solve, and much of this book will be devoted to describing the various methods of solution that have been devised over the years (Chapters 14–21).

1.16 OUTLINE OF RADIATIVE TRANSPORT THEORY

When considering heat transfer by conduction and/or convection within a medium, we require knowledge of a number of material properties, such as thermal conductivity k, thermal diffusivity α, kinematic viscosity ν, and so on. This knowledge, together with the *law of conservation* *of energy*, allows us to calculate the energy field within the medium in the form of the basic variable, temperature T. Once the temperature field is determined, the local heat flux vector may be found from Fourier's law. The evaluation of radiative energy transport follows a similar pattern: Knowledge of radiative properties is required (emittance ϵ, absorptance α, and reflectance ρ, in the case of surfaces, as well as absorption coefficient κ and scattering coefficient σ_s for semitransparent media), and the law of conservation of energy is applied to determine the energy field. Two major differences exist between conduction/convection and thermal radiation that make the analysis of radiative transport somewhat more complex: *(i)* Unlike their thermophysical counterparts, radiative properties may be functions of direction as well as of wavelength, and *(ii)* the basic variable appearing in the law of conservation of radiative energy, the *radiative transfer equation* introduced in the previous section, is not temperature but radiative intensity, which is a function not only of location in space (as is temperature), but also of direction. Only after the intensity field has been determined can the local temperatures (as well as the radiative heat flux vector) be calculated.

Thermal radiation calculations are always performed by making an energy balance for an enclosure bounded by opaque walls (some of which may be artificial to account for radiation penetrating through openings in the enclosure). If the enclosure is evacuated or filled with a nonabsorbing, nonscattering medium (such as air at low to moderate temperatures), we speak of *surface radiation transport.* If the enclosure is filled with an absorbing gas or a semitransparent solid or liquid, or with absorbing and scattering particles (or bubbles), we refer to it as *radiative transport in a participating medium.* Of course, radiation in a participating medium is always accompanied by surface radiation transport.

References

1. Sparrow, E. M., and R. D. Cess: *Radiation Heat Transfer*, Hemisphere, New York, 1978.
2. *NASA web page on the solar system and the sun* http://solarsystem.nasa.gov/planets, 2011.
3. Thekaekara, M. P.: "The solar constant and spectral distribution of solar radiant flux," *Solar Energy*, vol. 9, no. 1, pp. 7–20, 1965.
4. Thekaekara, M. P.: "Solar energy outside the earth's atmosphere," *Solar Energy*, vol. 14, pp. 109–127, 1973.
5. Rayleigh, L.: "The law of complete radiation," *Phil. Mag.*, vol. 49, pp. 539–540, 1900.
6. Jeans, J. H.: "On the partition of energy between matter and the ether," *Phil. Mag.*, vol. 10, pp. 91–97, 1905.
7. Wien, W.: "Über die Energieverteilung im Emissionsspektrum eines schwarzen Körpers," *Annalen der Physik*, vol. 58, pp. 662–669, 1896.
8. Planck, M.: "Distribution of energy in the spectrum," *Annalen der Physik*, vol. 4, no. 3, pp. 553–563, 1901.
9. Wien, W.: "Temperatur und Entropie der Strahlung," *Annalen der Physik*, vol. 52, pp. 132–165, 1894.
10. Crepeau, J.: "A brief history of the T^4 radiation law," ASME paper no. HT2009-88060, 2009.
11. Fröhlich, C., and J. Lean: "Solar radiative output and its variability: evidence and mechanisms," *Astron. Astrophys. Rev*, no. 3, pp. 273–320, 2004.
12. *World Radiation Center solar constant web page* http://www.pmodwrc.ch/pmod.php?topic=tsi/composite/SolarConstant, 2010.
13. Moon, P.: *Scientific Basis of Illuminating Engineering*, Dover Publications, New York, 1961, (originally published by McGraw-Hill, New York, 1936).
14. Hopkinson, R. B., P. Petherbridge, and J. Longmore: *Daylighting*, Pitman Press, London, 1966.
15. Kaufman, J. E. (ed.): *IES Lighting Handbook*, Illuminating Engineering Society of North America, New York, 1981.
16. White, F. M.: *Heat Transfer*, Addison-Wesley, Reading, MA, 1984.
17. Edwards, D. K.: "Radiation interchange in a nongray enclosure containing an isothermal CO_2–N_2 gas mixture," *ASME Journal of Heat Transfer*, vol. 84C, pp. 1–11, 1962.
18. Neuroth, N.: "Der Einfluss der Temperatur auf die spektrale Absorption von Gläsern im Ultraroten, I (Effect of temperature on spectral absorption of glasses in the infrared, I)," *Glastechnische Berichte*, vol. 25, pp. 242–249, 1952.
19. Rayleigh, L.: *Phil. Mag.*, vol. 12, 1881.
20. Mie, G. A.: "Beiträge zur Optik trüber Medien, speziell kolloidaler Metallösungen," *Annalen der Physik*, vol. 25, pp. 377–445, 1908.

Problems

1.1 Solar energy impinging on the outer layer of Earth's atmosphere (usually called the "solar constant") has been measured as 1366 W/m^2. What is the solar constant on Mars? (Distance from Earth to sun $= 1.496 \times 10^{11}$ m, Mars to sun $= 2.28 \times 10^{11}$ m.)

1.2 Assuming Earth to be a blackbody, what would be its average temperature if there was no internal heating from the core of Earth?

1.3 Assuming Earth to be a black sphere with a surface temperature of 300 K, what must Earth's internal heat generation be in order to maintain that temperature (neglect radiation from the stars, but not the sun) (radius of the Earth $R_E = 6.37 \times 10^6$ m).

1.4 To estimate the diameter of the sun, one may use solar radiation data. The solar energy impinging onto the Earth's atmosphere (called the "solar constant") has been measured as 1366 W/m^2. Assuming that the sun may be approximated to have a black surface with an effective temperature of 5777 K, estimate the diameter of the sun (distance from sun to Earth $S_{ES} \simeq 1.496 \times 10^{11}$ m).

1.5 Solar energy impinging on the outer layer of Earth's atmosphere (usually called the "solar constant") has been measured as 1366 W/m^2. Assuming the sun may be approximated as having a surface that behaves like a blackbody, estimate its effective surface temperature (distance from sun to Earth $S_{ES} \simeq 1.496 \times 10^{11}$ m, radius of sun $R_S \simeq 6.96 \times 10^8$ m).

1.6 A rocket in space may be approximated as a black cylinder of length $L = 20$ m and diameter $D = 2$ m. It flies past the sun at a distance of 140 million km such that the cylinder axis is perpendicular to the sun's rays. Assuming that *(i)* the sun is a blackbody at 5777 K and *(ii)* the cylinder has a high conductivity (i.e., is essentially isothermal), what is the temperature of the rocket? (Radius of sun $R_S = 696{,}000$ km; neglect radiation from Earth and the stars.)

1.7 A black sphere of very high conductivity (i.e., isothermal) is orbiting Earth. What is its temperature? (Consider the sun but neglect radiation from the Earth and the stars.) What would be the temperature of the sphere if it were coated with a material that behaves like a blackbody for wavelengths between 0.4 μm and 3 μm, but does not absorb and emit at other wavelengths?

1.8 A 100 W lightbulb may be considered to be an isothermal black sphere at a certain temperature. If the light flux (i.e., visible light, $0.4\,\mu\text{m} < \lambda < 0.7\,\mu\text{m}$) impinging on the floor directly (2.5 m) below the bulb is 42.6 mW/m^2, and assuming conduction/convection losses to be negligible, what is the lightbulb's effective temperature? What is its efficiency?

1.9 When a metallic surface is irradiated with a highly concentrated laser beam, a plume of plasma (i.e., a gas consisting of ions and free electrons) is formed above the surface that absorbs the laser's energy, often blocking it from reaching the surface. Assume that a plasma of 1 cm diameter is located 1 cm above the surface, and that the plasma behaves like a blackbody at 20,000 K. Based on these assumptions calculate the radiative heat flux and the total radiation pressure on the metal directly under the center of the plasma.

1.10 Solar energy incident on the surface of the Earth may be broken into two parts: a direct component (traveling unimpeded through the atmosphere) and a sky component (reaching the surface after being scattered by the atmosphere). On a clear day the direct solar heat flux has been determined as $q_{\text{sun}} = 1000$ W/m^2 (per unit area normal to the rays), while the intensity of the sky component has been found to be diffuse (i.e., the intensity of the sky radiation hitting the surface is the same for all directions) and $I_{\text{sky}} = 70$ W/m^2 sr. Determine the total solar irradiation onto Earth's surface if the sun is located 60° above the horizon (i.e., 30° from the normal).

1.11 A window (consisting of a vertical sheet of glass) is exposed to direct sunshine at a strength of 1000 W/m^2. The window is pointing due south, while the sun is in the southwest, 30° above the horizon. Estimate the amount of solar energy that *(i)* penetrates into the building, *(ii)* is absorbed by the window, and *(iii)* is reflected by the window. The window is made of *(a)* plain glass, *(b)* tinted glass, whose radiative properties may be approximated by

$$\rho_\lambda = 0.08 \qquad \text{for all wavelengths (both glasses),}$$

$$\tau_\lambda = \begin{cases} 0.90 & \text{for } 0.35\,\mu\text{m} < \lambda < 2.7\,\mu\text{m} \\ 0 & \text{for all other wavelengths} \end{cases} \quad \text{(plain glass),}$$

$$\tau_\lambda = \begin{cases} 0.90 & \text{for } 0.5\,\mu\text{m} < \lambda < 1.4\,\mu\text{m} \\ 0 & \text{for all other wavelengths} \end{cases} \quad \text{(tinted glass).}$$

(c) By what fraction is the amount of visible light ($0.4\,\mu\text{m} < \lambda < 0.7\,\mu\text{m}$) reduced, if tinted rather than plain glass is used? How would you modify this statement in the light of Fig. 1-11?

1.12 On an overcast day the directional behavior of the intensity of solar radiation reaching the surface of the Earth after being scattered by the atmosphere may be approximated as $I_{\text{sky}}(\theta) = I_{\text{sky}}(\theta = 0)\cos\theta$, where θ is measured from the surface normal. For a day with $I_{\text{sky}}(0) = 100\,\text{W/m}^2\,\text{sr}$ determine the solar irradiation hitting a solar collector, if the collector is *(a)* horizontal, *(b)* tilted from the horizontal by 30°. Neglect radiation from the Earth's surface hitting the collector (by emission or reflection).

1.13 A 100 W lightbulb is rated to have a total light output of 1750 lm. Assuming the lightbulb to consist of a small, black, radiating body (the light filament) enclosed in a glass envelope (with a transmittance $\tau_g = 0.9$ throughout the visible wavelengths), estimate the filament's temperature. If the filament has an emittance of $\epsilon_f = 0.7$ (constant for all wavelengths and directions), how does it affect its temperature?

1.14 A *pyrometer* is a device with which the temperature of a surface may be determined remotely by measuring the radiative energy falling onto a detector. Consider a black detector of 1 mm × 1 mm area that is exposed to a $1\,\text{cm}^2$ hole in a furnace located a distance of 1 m away. The inside of the furnace is at 1500 K and the intensity escaping from the hole is essentially blackbody intensity at that temperature. *(a)* What is the radiative heat rate hitting the detector? *(b)* Assuming that the pyrometer has been calibrated for the situation in *(a)*, what temperature would the pyrometer indicate if the nonabsorbing gas between furnace and detector were replaced by one with an (average) absorption coefficient of $\kappa = 0.1\,\text{m}^{-1}$?

1.15 Consider a pyrometer, which also has a detector area of 1 mm × 1 mm, which is black in the wavelength range $1.0\,\mu\text{m} \le \lambda \le 1.2\,\mu\text{m}$, and perfectly reflecting elsewhere. In front of the detector is a focusing lens ($f = 10\,\text{cm}$) of diameter $D = 2\,\text{cm}$, and transmissivity of $\tau_l = 0.9$ (around $1\,\mu\text{m}$). In order to measure the temperature inside a furnace, the pyrometer is focused onto a hot black surface inside the furnace, a distance of 1 m away from the lens.

(a) How large a spot on the furnace wall does the detector see? (Remember that geometric optics dictates

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}; \qquad M = \frac{h\,(\text{detector size})}{H\,(\text{spot size})} = \frac{v}{u},$$

where $u = 1\,\text{m}$ is the distance from lens to furnace wall, and v is the distance from lens to detector.)

(b) If the temperature of the furnace wall is 1200 K, how much energy is absorbed by the detector per unit time?

(c) It turns out the furnace wall is not really black, but has an emittance of $\epsilon = 0.7$ (around $1\,\mu\text{m}$). Assuming there is no radiation reflected from the furnace surface reaching the detector, what is the true surface temperature for the pyrometer reading of case *(b)*?

(d) To measure higher temperatures pyrometers are outfitted with filters. If a $\tau_f = 0.7$ filter is placed in front of the lens, what furnace temperature would provide the same pyrometer reading as case *(b)*?