# APPENDIX E

# EXPONENTIAL INTEGRAL FUNCTIONS

The exponential integral functions $E_n(x)$ and their derivatives occur frequently in radiative heat transfer calculations; therefore, a summary of their properties as well as a brief tabulation are given here. More detailed discussions of their properties may be found in the books by Chandrasekhar [1] and Kourganoff [2], or in mathematical handbooks such as [3]. Detailed tabulations are given in [3], and formulae for their numerical evaluation are listed in [3, 4].

The *exponential integral of order n* is defined as

$$E_n(x) = \int_1^\infty e^{-xt}\,\frac{dt}{t^n}, \quad n = 0, 1, 2, \ldots, \tag{E.1}$$

or, setting $\mu = 1/t$,

$$E_n(x) = \int_0^1 e^{-x/\mu}\,\mu^{n-2}\,d\mu, \quad n = 0, 1, 2, \ldots. \tag{E.2}$$

Differentiating equation (E.1), a first recurrence relationship is found as

$$\frac{dE_n}{dx}(x) = -E_{n-1}(x), \quad n = 1, 2, \ldots, \tag{E.3}$$

where

$$E_0(x) = \int_1^\infty e^{-xt}\,dt = \frac{e^{-x}}{x}. \tag{E.4}$$

A second recurrence is found by integrating equation (E.3), or

$$\int_x^\infty E_n(x)\,dx = E_{n+1}(x), \quad n = 0, 1, 2, \ldots. \tag{E.5}$$

An algebraic recurrence between consecutive orders may be obtained by integrating equation (E.1) by parts, or

$$E_{n+1}(x) = \frac{1}{n}\left[e^{-x} - xE_n(x)\right], \quad n = 1, 2, 3, \ldots. \tag{E.6}$$

The integral of equation (E.1) may be solved in a general series expansion as [3]

$$E_n(x) = \frac{(-x)^{n-1}}{(n-1)!}(\psi_n - \ln x) + \sum_{\substack{m=0 \\ m\neq n-1}}^{\infty} \frac{(-x)^m}{m!(n-1-m)}, \quad n = 1, 2, 3, \ldots, \tag{E.7a}$$

where

$$\psi_n = \begin{cases} -\gamma_E, & n = 1, \\ -\gamma_E + \sum_{m=1}^{n-1} \frac{1}{m}, & n \geq 2, \end{cases} \tag{E.7b}$$

and

$$\gamma_E = \int_1^{\infty} \left(1 - e^{-t}\right) \frac{dt}{t} = 0.577216\ldots \tag{E.7c}$$

is known as *Euler's constant.* Substituting values for $n$, one obtains

$$E_1(x) = -(\gamma_E + \ln x) + x - \frac{x^2}{2!2} + \frac{x^3}{3!3} - \frac{x^4}{4!4} + - \ldots, \tag{E.8}$$

$$E_2(x) = 1 + x(\gamma_E - 1 + \ln x) - \frac{x^2}{2!1} + \frac{x^3}{3!2} - \frac{x^4}{4!3} + - \ldots, \tag{E.9}$$

$$E_3(x) = \frac{1}{2} - x + \frac{x^2}{2}\left(-\gamma_E + \frac{3}{2} - \ln x\right) + \frac{x^3}{3!1} - \frac{x^4}{4!2} + - \ldots. \tag{E.10}$$

A function related to $E_1$ that often occurs in radiation calculations is

$$\begin{aligned} \text{Ein}(x) &= \int_0^1 \left(1 - e^{-xt}\right) \frac{dt}{t} = \int_0^{\infty} \left(1 - e^{-xe^{-\xi}}\right) d\xi \\ &= E_1(x) + \ln x + \gamma_E = x - \frac{x^2}{2!2} + \frac{x^3}{3!3} - + \ldots. \end{aligned} \tag{E.11}$$

For vanishing values of $x$ it follows from equation (E.7), or directly integrating equation (E.1), that

$$E_n(0) = \begin{cases} +\infty, & n = 1, \\ \frac{1}{n-1}, & n \geq 2, \end{cases} \tag{E.12a}$$

$$\text{Ein}(0) = 0. \tag{E.12b}$$

For large values of $x$, the asymptotic expansion for the exponential integrals is given by [3]

$$E_n(x) = \frac{e^{-x}}{x}\left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \frac{n(n+1)(n+2)}{x^3} + - \ldots\right], \quad n = 0, 1, 2, \ldots. \tag{E.13}$$

To estimate the relative magnitude of different orders of exponential integrals, the following inequalities are sometimes handy [3]:

$$\frac{n-1}{n}E_n(x) < E_{n+1}(x) < E_n(x), \quad n = 1, 2, 3, \ldots, \tag{E.14}$$

$$\frac{1}{x+n} < e^x E_n(x) \leq \frac{1}{x+n-1}, \quad n = 1, 2, 3, \ldots. \tag{E.15}$$

## References

1. Chandrasekhar, S.: *Radiative Transfer*, Dover Publications, New York, 1960, (originally published by Oxford University Press, London, 1950).
2. Kourganoff, V.: *Basic Methods in Transfer Problems*, Dover Publications, New York, 1963.
3. Abramowitz, M., and I. A. Stegun (eds.): *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.
4. Breig, W. F., and A. L. Crosbie: "Numerical computation of a generalized exponential integral function," *Math Comp.*, vol. 28, no. 126, pp. 575–579, 1974.

**TABLE E.1**
**Values of exponential integral functions.**

| $x$ | Ein | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|---|
| 0.00 | 0.000000 | $\infty$ | 1.000000 | 0.500000 | 0.333333 |
| 0.01 | 0.009975 | 4.037929 | 0.949671 | 0.490277 | 0.328382 |
| 0.02 | 0.019900 | 3.354707 | 0.913105 | 0.480968 | 0.323526 |
| 0.03 | 0.029776 | 2.959118 | 0.881672 | 0.471998 | 0.318762 |
| 0.04 | 0.039603 | 2.681263 | 0.853539 | 0.463324 | 0.314085 |
| 0.05 | 0.049382 | 2.467898 | 0.827835 | 0.454919 | 0.309494 |
| 0.06 | 0.059112 | 2.295307 | 0.804046 | 0.446761 | 0.304986 |
| 0.07 | 0.068794 | 2.150838 | 0.781835 | 0.438833 | 0.300559 |
| 0.08 | 0.078428 | 2.026941 | 0.760961 | 0.431120 | 0.296209 |
| 0.09 | 0.088015 | 1.918744 | 0.741244 | 0.423610 | 0.291935 |
| 0.10 | 0.097554 | 1.822924 | 0.722545 | 0.416291 | 0.287736 |
| 0.15 | 0.144557 | 1.464461 | 0.641039 | 0.382276 | 0.267789 |
| 0.20 | 0.190428 | 1.222650 | 0.574201 | 0.351945 | 0.249447 |
| 0.25 | 0.235204 | 1.044283 | 0.517730 | 0.324684 | 0.232543 |
| 0.30 | 0.278920 | 0.905677 | 0.469115 | 0.300042 | 0.216935 |
| 0.35 | 0.321609 | 0.794215 | 0.426713 | 0.277669 | 0.202501 |
| 0.40 | 0.363305 | 0.702380 | 0.389368 | 0.257286 | 0.189135 |
| 0.45 | 0.404039 | 0.625331 | 0.356229 | 0.238663 | 0.176743 |
| 0.50 | 0.443842 | 0.559773 | 0.326644 | 0.221604 | 0.165243 |
| 0.60 | 0.520769 | 0.454379 | 0.276184 | 0.191551 | 0.144627 |
| 0.70 | 0.594310 | 0.373769 | 0.234947 | 0.166061 | 0.126781 |
| 0.80 | 0.664669 | 0.310597 | 0.200852 | 0.144324 | 0.111290 |
| 0.90 | 0.732039 | 0.260184 | 0.172404 | 0.125703 | 0.097812 |
| 1.00 | 0.796600 | 0.219384 | 0.148496 | 0.109692 | 0.086062 |
| 1.10 | 0.858517 | 0.185991 | 0.128281 | 0.095881 | 0.075801 |
| 1.20 | 0.917946 | 0.158408 | 0.111104 | 0.083935 | 0.066824 |
| 1.30 | 0.975031 | 0.135451 | 0.096446 | 0.073576 | 0.058961 |
| 1.40 | 1.029907 | 0.116219 | 0.083890 | 0.064576 | 0.052064 |
| 1.50 | 1.082700 | 0.100020 | 0.073101 | 0.056739 | 0.046007 |
| 1.60 | 1.133528 | 0.086308 | 0.063803 | 0.049906 | 0.040682 |
| 1.70 | 1.182499 | 0.074655 | 0.055771 | 0.043937 | 0.035997 |
| 1.80 | 1.229716 | 0.064713 | 0.048815 | 0.038716 | 0.031870 |
| 1.90 | 1.275274 | 0.056204 | 0.042780 | 0.034143 | 0.028232 |
| 2.00 | 1.319263 | 0.048901 | 0.037534 | 0.030133 | 0.025023 |
| 2.50 | 1.518421 | 0.024915 | 0.019798 | 0.016295 | 0.013782 |
| 3.00 | 1.688876 | 0.013048 | 0.010642 | 0.008931 | 0.007665 |
| 4.00 | 1.967289 | 0.003779 | 0.003198 | 0.002761 | 0.002423 |
| 5.00 | 2.187802 | 0.001148 | 0.000996 | 0.000878 | 0.000783 |