New tests for a singularity of ideal MHD

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Analysis using new calculations with 3 times the resolution of the earlier linked magnetic flux tubes confirms the transition from singular to saturated growth rate reported by Grauer and Marliani [2] for the incompressible cases is confirmed. However, all of the secondary tests point to a transition back to stronger growth rate at a different location at late times. Similar problems in ideal hydrodynamics are discussed, pointing out that initial negative results eventually led to better initial conditions that did show evidence for a singularity of Euler. Whether singular or near-singular growth in ideal MHD is eventually shown, this study could have bearing on fast magnetic reconnection, high energy particle production and coronal heating.

The issue currently leading to conflicting conclusions about ideal 3D, incompressible MHD is similar [1,2] to what led to conflicting results on whether there is a singularity of the 3D incompressible Euler. With numerical simulations, it was first concluded that uniform mesh calculations with symmetric initial conditions such as 3D Taylor-Green were not yet singular [3]. Next, a preliminary spectral calculation [4] found weak evidence in favor a singularity in a series of Navier-Stokes simulations at increasing Reynolds numbers, but larger adaptive mesh or refined mesh calculations did not support this result [5,6]. Eventually, numerical evidence in favor of a singularity of Euler was obtained using several independent tests applied to highly resolved, refined mesh calculations of the evolution of two anti-parallel vortex tubes [7]. To date, these calculations have met every analytic test for whether there could be a singularity of Euler.

Several other calculations have also claimed numerical evidence for a singularity of Euler [8–10]. While in all of these cases the evidence is plausible, with the perturbed cylindrical shear flow [10] using the BKM $\|\omega\|_\infty$ test [11], for none has the entire battery of tests used for the anti-parallel case been applied. We have recently repeated one of the orthogonal cases [8] and have applied the BKM test successfully. In all cases using the BKM test, $|\omega\|_\infty \approx A/(T_c - t)$ with $A \approx 19$.

To be able to make a convincing case for the existence of a singularity in higher dimensional partial differential equations, great care must be taken with initial conditions, demonstrating numerical convergence, and comparisons to all known analytic or empirical tests. On the other hand, if no singularity is suspected, some quantity that clearly saturates should be demonstrated, such as the strain causing vorticity growth [5]. It is an even more delicate matter to claim that someone else's calculations or conclusions are incorrect. If it is a matter of suspecting there is inadequate resolution, one must attempt to reproduce the suspicious calculations as nearly as possible and show where inadequate resolution begins to corrupt the calculations and how improved resolution changes the results.

An example of how a detailed search for numerical errors should be conducted can be found in the extensive conference proceeding [12] that appeared prior to the publication of the major results supporting the existence of a singularity of Euler for anti-parallel vortex tubes [7]. The primary difference with earlier work was in the initial conditions. It was found that compact profiles [13] were an improvement, but only if used in conjunction with a high wavenumber filter. Otherwise, the initial unfiltered energy spectrum of the bent anti-parallel vortex tubes went as k^{-2}. Oscillations in the spectrum at high wavenumber in unfiltered initial conditions for linked magnetic flux tubes are shown in Figure 1, showing that the initial MHD spectrum is steep enough that eventually these oscillations are not important.
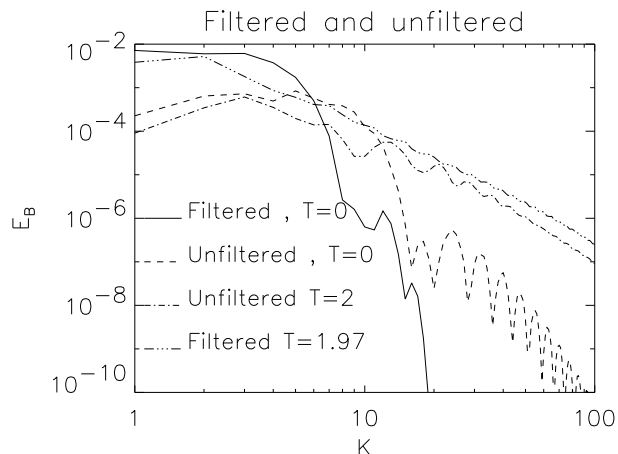

FIG. 1. Filtered and unfiltered initial and final spectrum . The unfiltered spectrum is initialized on a 384^3 mesh.

The purpose of this letter is to address the claim that a new adaptive mesh refinement (AMR) calculation by Grauer and Marliani [2] supercedes our uniform mesh calculations [1] and that eventually there is a transition to exponential growth. Note that this claim was made without any evidence for whether their numerical method was converged. In all of our earlier calculations, once the calculations become underresolved, we also saw transitions to exponential growth.

Not knowing exactly the initial condition used by the new AMR calculations [2], where and how much grid re-

finement was used, and the short notice we have been given to reply has proven a challenge. Fortunately, we were in the process of new 648^3 calculations in a smaller domain of 4.3^3, yielding effectively 3 times the local resolution of our earlier work [1] in a $(2\pi)^3$ domain on a 384^3 mesh. The case with an initial flux tube diameter of $d = 0.65$, so that the tubes slightly overlap, appears to be closer to their initial condition and so will be the focus of this letter. The importance of our other initial condition, with $d = 0.5$, and no initial overlap of the tubes, is that it is less influenced by an initial current sheet that forms near the origin and is claimed to be the source of the saturation of the nonlinear terms. This was used for the compressible calculations.
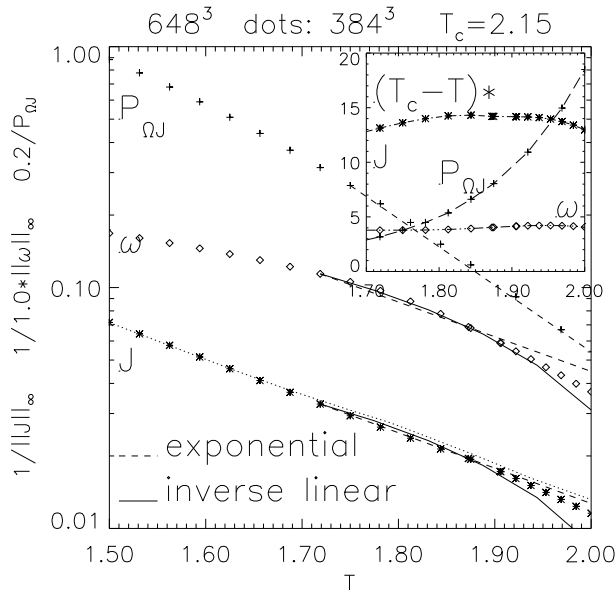
shows that either an exponential or a singular $1/(T_c - t)$ form could fit the data, while the inset shows that taking an estimated singular time of $T_c = 2.15$ and multiplying by $(T_c - T)$ that at least $\|J\|_\infty$ and $\|\omega\|_\infty$ have consistent singular behavior over this time span. The strong growth of $P_{\Omega J} = \int dV(\omega_i e_{ij}\omega_j - \omega_i d_{ij}J_j - 2\varepsilon_{ijk}J_i d_{j\ell}e_{\ell k})$, which is the production of $\int dV(\omega^2 + J^2)$, is discussed below. The 384^3 curve for $1/\|J\|$ demonstrates that lack of resolution tends to exaggerate exponential growth. For the compressible calculations it can be seen that there also is an exponential regime that changes into a regime with $1/\|J\|_\infty \sim (T_c - t)$.

FIG. 2. Replot of $\|J\|_\infty$, $\|\omega\|_\infty$, and $P_{\Omega J}$ for the new incompressible calculations on 4.3^3 domain with initial condition $d = 0.65$ in semi-log coordinates. All plots are from the 648^3 calculation except one 384^3 plot of $1/\|J\|$. Exponential and inverse linear fits are shown for $t = 1.75$ to 1.98. Each works equally well for $\|J\|_\infty$, inverse linear is better for $\|\omega\|_\infty$, and exponential is better for $P_{\Omega J}$. Multiplying $\|J\|_\infty$, $\|\omega\|_\infty$, and $P_{\Omega J}$ by $(T_c - t)$ in the inset emphasizes that $\|J\|_\infty$ and $\|\omega\|_\infty$ might be showing consistent singular behavior. The large growth of $P_{\Omega J}$ is discussed.

Using semi-log coordinates, Figure 2 plots the growth of $\|\omega\|_\infty$ and $\|J\|_\infty$ for our new high resolution incompressible calculation and Figure 3 plots $\|J\|_\infty$ for a new compressible calculation. By taking the last time all relevant quantities on the 384^3 and 648^3 grids were converged, $\|J\|$ being the worst, then by assuming that the smallest scales are decreasing linearly towards a possible singular time, an estimate of the last time the 648^3 calculation was valid was made. To test exponential versus inverse linear growth, fits were taken between $T = 1.72$ and 1.87, then extrapolated to large T. The large figure
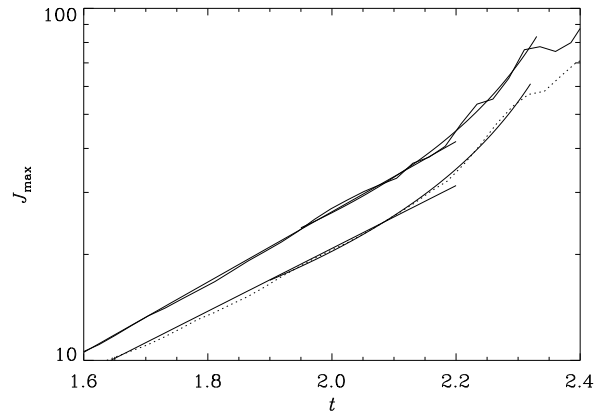

FIG. 3. Semi-logarithmic plot of $\|J\|_\infty$ for a compressible 240^3 calculation in a domain of size 4 (dotted line: filtered, and solid line: unfiltered initial conditions) together with fits to exponential growth and blow-up behavior, respectively. The latter are better fits at later times.

Using the new incompressible calculations and applying the entire battery of tests, based upon Figure 2 we would agree that for the incompressible case there is a transition as reported [2] and signs of saturation at this stage are shown below. Whether the transition is to exponential for all times as claimed [2], or whether there is a still later transition to different singular behavior, will be the focus of this letter. We will look more closely at the structure of the current sheet we all agree exists [1,2] for signs of saturation.

The case against a singularity in early calculations of Euler [5,14,15] was the appearance of vortex sheets, and through analogies with the current in 2D ideal MHD, a suggestion that this leads to a depletion of nonlinearity. The fluid flow most relevant to the linked flux rings is 3D Taylor-Green, due to the initial symmetries [3]. For both TG and linked flux tubes, two sets of anti-parallel vortex pairs form that are skewed with respect to each other and are colliding. In TG, just after the anti-parallel vortex tubes form there is a period of slightly singular development. This is suppressed once the pairs collide with each other, and then vortex sheets dominate for a period. The vortex sheets are very thin, but go across the domain, so fine localized resolution might not be an

advantage at this stage. At late phases in TG, the ends of the colliding pairs begin to interact with each other, so that at 4 corners locally orthogonal vortices begin to form. Due to resolution limitations, an Euler calculation of Taylor-Green has not been continued long enough to determine whether, during this phase, singular behavior might develop. We would draw a similar conclusion for all of MHD cases studied to date [2,16,17], that there might not be enough local resolution to draw any final conclusions even if AMR is applied.

While Taylor-Green has not been continued far enough to rule out singularities, the final arrangement of vortex structures led first to studies of interacting orthogonal vortices [8], and then anti-parallel vortices (see references in [7]). Both of these initial conditions now appear to develop singular behavior. An important piece of evidence for a singularity of Euler was that near the point of a possible singularity, the structure could not be described simply as a vortex sheet. Therefore, there is a precedent to earlier work suggesting sheets, suppression of nonlinearity, and no singularities to later work showing fully three-dimensional structure and singular behavior.

The initial singular growth of $\|J\|_\infty$ and $\|\omega\|_\infty$ for the linked flux rings, then the transition to a saturated growth rate, might be due to the same skewed, anti-parallel vortex pair interaction as in Taylor-Green. Even if this is all that is happening, the strong initial vorticity production and shorter dynamical timescale (order of a few Alfvén times) than earlier magnetic reconnection simulations with anti-parallel flux tubes [17] is a significant success of these simulations. It might be that the vortices that have been generated are strong enough to develop their own Euler singularity. However, the interesting physics is how the magnetic field and current interact with the vorticity. Do they suppress the tendency of the vorticity to become singular, or augment that tendency?
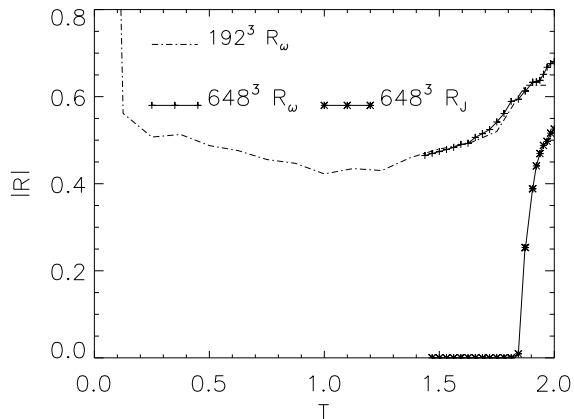

FIG. 4. Positions of $\|J\|_\infty$ and $\|\omega\|_\infty$ for $d = 0.65$ in a 4.3^3 domain.

One sign for saturation of the linked flux ring inter-

action would be if the strongest current remains at the origin in this sheet. Figure 4 plots the positions of $\|J\|_\infty$ and $\|\omega\|_\infty$ from the origin as a function of time. During the period where exponential growth is claimed [2], $\|J\|_\infty$ is at the origin, which would support the claims of saturation. However, this situation does not persist.

By analogy to the movement of the L^∞ norms of the components of the stress tensor $u_{i,j}$ in Euler, we expect that the positions of $\|J\|_\infty$ and $\|\omega\|_\infty$ should approach each other and an extrapolated singular point in ideal MHD. Figure 4 supports the prediction that the positions of $\|J\|_\infty$ and $\|\omega\|_\infty$ should approach each other but so far not in a convincingly linear fashion. This is addressed next. We have similar trends for the positions of $\|J\|_\infty$ and $\|\omega\|_\infty$ in the compressible calculations.
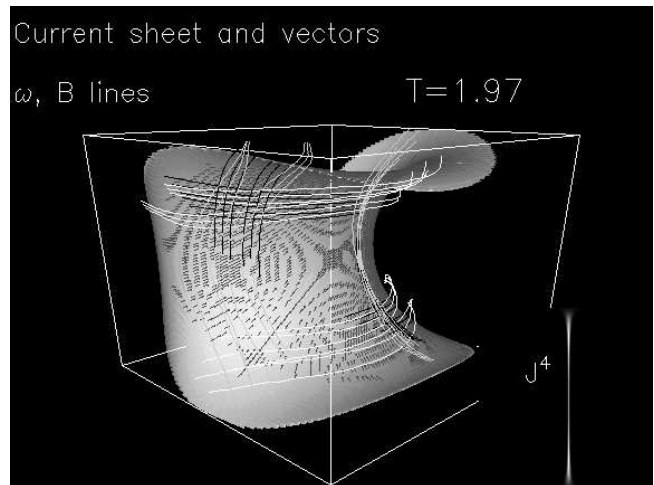

FIG. 5. For $t = 1.97$ on the inner 162^3 grid points, the current sheet is shown with arrows of $\overrightarrow{J}$ overlaid in dark. The current through the ($x/y = z$) plane containing $\|J\|_\infty$ is in lower right. Contours of $|J|^4$ are shown to emphasize where $\|J\|_\infty$ is located. Dark lines are $\overrightarrow{B}$ and light lines are $\overrightarrow{\omega}$ that originated in the vicinity of $\|J\|_\infty$. The vortex lines are predominantly those in the double vortex rings that were originally generated by the Lorenz force, then became responsible for spreading out the current sheet. Where the $\overrightarrow{B}$ lines cross in the upper left and lower right corners are around the locations of $\|J\|_\infty$, which due to symmetries are different views of the same structure. Near $\|J\|_\infty$, $\overrightarrow{B}$ nearly overlies and is parallel to $\overrightarrow{\omega}$ and both $\overrightarrow{B}$ and $\overrightarrow{\omega}$ are nearly orthogonal to their partners across the current sheet, where $\overrightarrow{B}$ and $\overrightarrow{\omega}$ are anti-parallel. Taken from the $d = 0.65$ calculation in a 4.3^3 domain on a 648^3 mesh.

Figure 5 gives an overall view of the current, vorticity and magnetic field around the inner current sheet. The vortex pattern has developed out of the four initial vortices, two sets of orthogonal, anti-parallel pairs that are responsible for the initial compression and stretching of the current sheet. By this late time, the ends of the those vortices have begun to interact as new sets of orthogonal vortex pairs. The lower right inset in Figure 5 is a 2D $x/(y = z)$ slice through this domain that goes through

3

$\|J\|_\infty$ at $t = 1.97$ to show that while $\|J\|_\infty$ is large at the origin $(0,0,0)$, $\|J\|_\infty$ is larger where it is being squeezed between the new orthogonal vortices. Along one of the new vortices $\overrightarrow{B}$ is parallel to and overlying $\overrightarrow{\omega}$ and on the orthogonal partner they are anti-parallel and overlying.

The location of $\|\omega\|_\infty$ is not in the vortex lines shown, but is on the outer edges of the current sheet. Therefore, the exact position of $\|\omega\|_\infty$ in Figure 4 is an artifact of the initial development and does not accurately reflect the position of $\overrightarrow{\omega}$ most directly involved in amplifying $\|J\|_\infty$, which is probably why the positions of $\|J\|_\infty$ and $\|\omega\|_\infty$ are not approaching each other faster. The continuing effects of the initial current sheet is probably also behind the strong exponential growth of $P_{\Omega J}$ in Figure 2, stronger even than the the possible singular growth of $\|J\|_\infty$ and $\|\omega\|_\infty$ in the inset. More detailed analysis in progress should show that near the position of $\|J\|_\infty$, the growth of $P_{\Omega J}$ and the position of $\|\omega\|_\infty$ are more consistent with our expectations for singular growth and has already shown that some of the components of $P_{\Omega J}$ have consistent singular growth.

As noted, for Euler all available calculations find $\|\omega\|_\infty \approx A/(T_c - t)$ with $A \approx 19$. A represents how much smaller the strain along $\|\omega\|_\infty$ is than $\|\omega\|_\infty$. Here, $A \approx 4$, indicating stronger growth in $\|\omega\|_\infty$ for ideal MHD than Euler. Another Euler result was that the asymptotic energy spectrum as the possible singularity was approached was k^{-3}, whereas purely sheet-like structures in vorticity should yield k^{-4} spectrum. k^{-3} indicates a more complicated 3D structure than sheets. In Figure 1 the late time spectra are again k^{-3}.

The next the initial condition we will investigate will be magnetic flux and vortex tubes that nearly overlay each other and are orthogonal to their partners. Our new calculations of orthogonal vortex tubes for Euler show that they start becoming singular as filaments are pulled off of the original tubes and these filaments become anti-parallel, suggesting that the fundamental singular interaction in Euler is between anti-parallel vortices. Whether the next step for ideal MHD is to become anti-parallel or something else can only be determined by new calculations. AMR might be useful, but great care must be taken with the placement of the inner domains and a large mesh will still be necessary. The complicated structures in the domain in Figure 5 are not fully contained in this innermost 162^3 mesh points and the innermost domain should go out to the order of 300^3 points. There are examples of how to use AMR when there are strong shears on the boundaries of sharp structures [18]. This uncertainty of where to place the mesh is why we believe in using uniform mesh calculations as an unbiased first look at the problem.

These final results are hardly robust and their usefulness is primarily to suggest a new more localized initial condition and to show that none of the calculations to date is the full story. For J and ω to show singular behavior as long as they have has been surprising. Recall that for Euler, velocity, vorticity and strain are all man-ifestations of the same vector field, but for ideal MHD there are two independent vector fields even though the only analytic result in 3D is a condition on the combination, $\int dV \left[\|\boldsymbol{\omega}\|_\infty(t) + \|\mathbf{J}\|_\infty(t) \right] dt \rightarrow \infty$ [19]. Eventually, one piece of evidence for singular growth must be a demonstration of strong coupling between the current and vorticity so that they are acting as one vector field. It could be that our strong growth is due to the strongly helical initial conditions and there are no singularities. This would still be physically interesting since helical conditions could be set up by footpoint motion in the corona.

Could the magnetic and electric fields blow up too? There are signs this might be developing around the final position of $\|J\|_\infty$, in which case there might exist a mechanism for the direct acceleration of high energy particles. This has been considered on larger scale [20], but to our knowledge a mechanism for small-scale production of super-Dreicer electric fields has not been proposed before. A singular rise in electric fields could explain the sharp rise times in X-ray production in solar coronal measurements [21], which could be a consequence of particle acceleration coming from reconnection. This would also have implications for the heating of the solar corona by nanoflares [22] and the production of cosmic rays.

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