Turbulence and Nonlinear Dynamics in MHD Flows
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PARITY SELECTION IN NONLINEAR DYNAMOS

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The stability of different nonlinear α-effect dynamos in spherical geometry is studied. A critical value of the dynamo number is found, above which steady hydromagnetic solutions of even and odd parity are both stable. In $\alpha\omega$-dynamos long-term variations *between* even and odd parity are possible. Comparison with similar variations of the sunspot number is made.

1. INTRODUCTION

The study of stellar stability is a traditional branch in stellar physics. Certain gaps in the Herzsprung Russel diagram and the location of pulsating stars are closely related to the star's stability. A similar issue was proposed in the study of stellar dynamo theory (Krause and Meinel, 1988). Many different dynamo modes are excited, when the dynamo number is highly supercritical. However, Krause and Meinel showed for a simple 1-dimensional model that *only* the solution with the smallest critical dynamo number is stable. *All* other solutions, bifurcating from the trivial solution, are unstable. They suggested, using general results of bifurcation theory, that this holds also for other, more complex dynamos, at least in a finite surrounding of their bifurcation from the trivial solution.

The stability of nonlinear dynamos was already considered by Kleeorin and Ruzmaikin (1984). They found stability for the first bifurcating solution, when it is supercritical, and instability, when it is subcritical. Recently, Brandenburg *et al.* (1988a) investigated the stability of dynamos in 2-D and spherical geometry assuming axisymmetry. They confirmed the stability criterion of Krause and Meinel in the case, where the nonlinear feedback con. s solely through a dependence of α on the magnetic field. But the region of validity was quite small. Already at dynamo numbers exceeding by 1% the critical value at the second bifurcation this second solution was stable. This solution is of even symmetry type. However, investigations by Rädler and Wiedemann (1988) suggest that this solution is unstable against 3-D perturbations. Stability of both even and odd solutions was found in certain examples of hydromagnetic dynamos, where the equations for the mean motions were also solved. In addition, oscillatory $\alpha\omega$-dynamo solutions without any symmetry ('mixed parity') were found to be stable for certain parameters.

The goal of the present report is to examine further the stability of these different dynamo models. We will pay particular attention to the stability behavior in the immediate neighborhood of the second bifurcation for hydromagnetic dynamos. Furthermore, we will study mixed solutions in the $\alpha\omega$-dynamo and discuss the relevance to solar long-term variations.

2. HYDROMAGNETIC MEAN-FIELD DYNAMOS

We consider here the generation of a mean magnetic field $\boldsymbol{B}$ by the α-effect (Steenbeck *et al.* , 1966). The induction equation, supplemented by this additional term is, in dimensionless form:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{B} + \alpha \boldsymbol{B} - \operatorname{curl} \boldsymbol{B}), \tag{1}$$

with $\alpha = C_\alpha \cos\theta$. This equation is solved inside a conducting sphere of constant density together with the momentum equation:

$$\frac{\mathrm{D}\,\boldsymbol{u}}{\mathrm{D}\,t} = -\operatorname{grad}\mathcal{P} - Ta^{1/2}\,\hat{\boldsymbol{z}} \times \boldsymbol{u} - \boldsymbol{B} \times \operatorname{curl}\boldsymbol{B} + P_m \nabla^2 \boldsymbol{u}, \tag{2}$$

where $Ta = (2\Omega R^2/\nu)^2$ is the Taylor number and $P_m = \nu/\eta$ the magnetic Prandtl number (ν and η are kinematic viscosity and magnetic diffusivity, and R is the radius of the conducting sphere). $\mathcal{P}$ is a reduced pressure (which can contain also gravitational and centrifugal potentials), and the other quantities have their usual meaning. The time unit is the *magnetic* diffusive time, as is usual in dynamo theory. We assume $\boldsymbol{u}$ and $\boldsymbol{B}$ to be solenoidal, i.e. $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{B} = 0$.

Outside the sphere we assume a vacuum, which implies $\operatorname{curl}\boldsymbol{B} = 0$. This leads to a boundary condition for $\boldsymbol{B}$ on $r = R$, which is handled using a method described by Jepps (1975). The boundary is assumed to be stress free, i.e. no angular momentum can leave the sphere. We restrict ourselves to axisymmetric fields so that it is possible to solve this problem in terms of stream functions for $\boldsymbol{u}$ and $\boldsymbol{B}$ together with their azimuthal components. We use a numerical method similar to that of Proctor (1977). The solutions published by Proctor are all antisymmetric with respect to the equatorial plane (dipolar type or odd). The stability of both symmetric and antisymmetric solutions has been investigated for some examples by Brandenburg *et al.* (1988a,b). The test of perturbing the solution with another one of opposite symmetry type was usually applied. The variation of the degree of symmetry was measured by the quantity:

$$P = \frac{E^{(S)} - E^{(A)}}{E^{(S)} + E^{(A)}}, \tag{3}$$

where $E^{(S)}$ and $E^{(A)}$ are the energies of the symmetric and antisymmetric part of the magnetic field. For pure symmetric solutions we have $P = 1$, while for pure antisymmetric solutions $P = -1$. For a dynamo number $C_\alpha = 10$ both symmetric and antisymmetric solutions proved to be stable. This seems to be in contradiction to the general statement of Krause and Meinel (1988) that the second nonlinear solution, bifurcating from C_α^S is in any case unstable. In the case where the only nonlinearity comes through a $\boldsymbol{B}$-dependent α it was shown that in a very small neighborhood of C_α^S (about 1%) the symmetric solution was really unstable. We consider now the same problem for the hydromagnetic dynamo. The result for $Ta = 4$ and $P_m = 1$ is depicted in Figure 1, where we have plotted the variation of P against time t for two different dynamo numbers. For $C_\alpha = 7.88$ (right panel) we find still stability of the symmetric solution, because P always tends back to $+1$ after the disturbance has been applied. However, this solution becomes unstable when the dynamo number is only slightly reduced ($C_\alpha = 7.86$, left panel). We note that the marginal value for the S-type solution is $C_\alpha^S = 7.81$. So we must conclude that the stability criterion of Krause and Meinel holds also for this example of a more complicated dynamo model, but it is perhaps more of academic interest, because the range of validity is so extremely small (less then 1%).

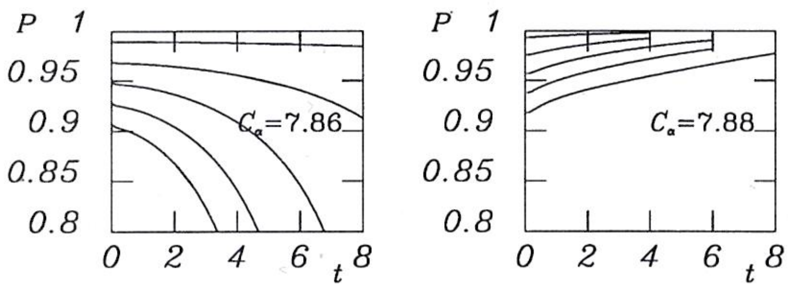

Figure 1: Variation of P for two different dynamo numbers. For $C_\alpha = 7.88$ the quantity P always returns to $+1$ after the perturbation (right panel). This is not the case for $C_\alpha = 7.86$

A snapshot showing the magnetic- and velocity fields of 'mixed parity' type for another dynamo with $Ta = 4 \times 10^4$ (and $P_m = 1$) is given in Figure 2. The time after the perturbation was $t = 0.15$ and $C_\alpha = 10$. The final state is reached at $t = 0.6$ and is of quadrupole type.

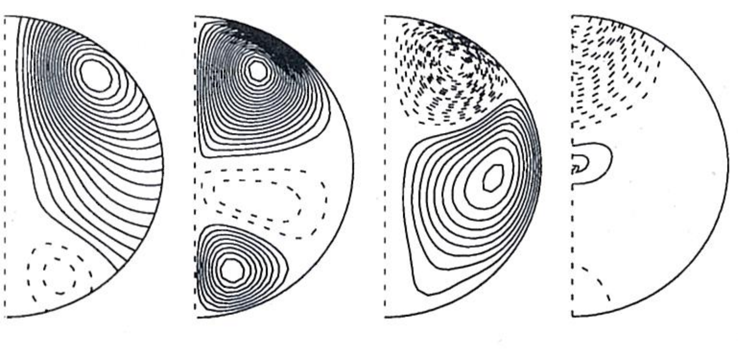

Figure 2: A snapshot showing the poloidal magnetic field, contours of constant toroidal field strength, meridional streamlines and contours of angular velocity for a dynamo with $Ta = 4 \times 10^4$.

3. $\alpha\omega$-DYNAMOS

The solutions considered so far have all a stationary final state. A richer variety of stability behavior was found for oscillatory nonlinear $\alpha\omega$-dynamos (Brandenburg *et al.* , 1988b). In particular the occurence of stable, but not steady, solutions, which are neither symmetric nor antisymmetric with respect to the equatorial plane, is of interest. The mean magnetic fields on the Sun and several planets are not perfectly antisymmetric and it is tempting to relate this to certain solutions found for the $\alpha\omega$-dynamo.

One of the features of our solutions is that the symmetry type can vary on a very long time scale compared to the magnetic cycle frequency. In some cases this long term variation corresponds to the beat frequency of the two pure solutions. The occurence of a second frequency is typical for solutions whose trajectory describes a torus in some phase space. Poincaré maps showing intersections with a a suitable plane in phase space confirm this (see Brandenburg *et al.* , 1988b for details).

The quantity P, introduced in the previous section, is not so easy to measure, since it involves the knowledge of the magnetic *polarity*. This has been measured on the Sun only over the last sixty years (Hale and Nicholson, 1925). It would be, however, important to have a much longer data base. What has been measured over many centuries is the sunspot number. Counting this number separately on the northern and southern hemisphere might provide a simple tool to infer a variation

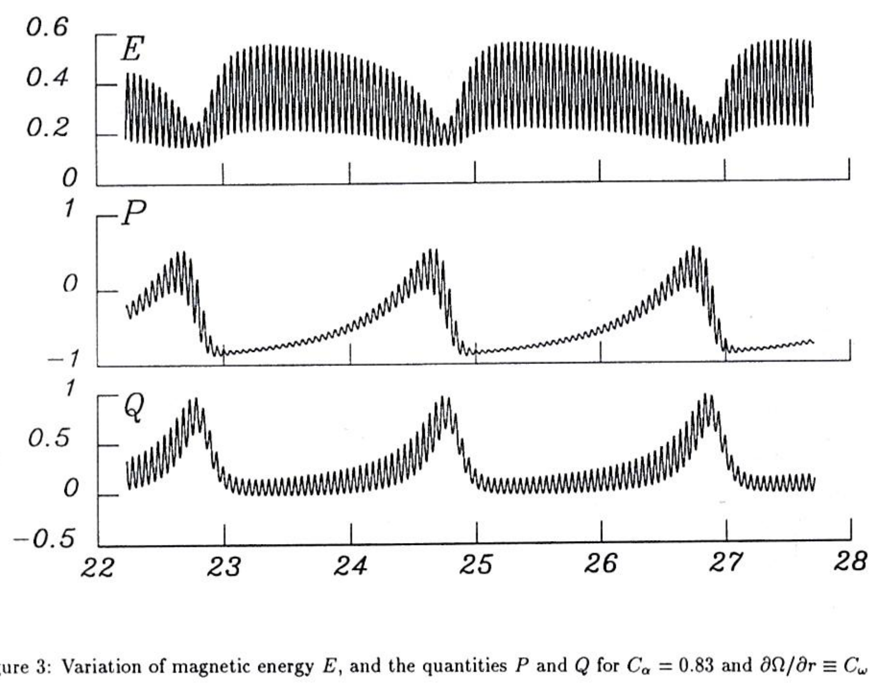

Figure 3: Variation of magnetic energy E, and the quantities P and Q for $C_\alpha = 0.83$ and $\partial\Omega/\partial r \equiv C_\omega = 10^4$.

of the symmetry type. In Figure 3 we have compared the variation of total magnetic energy and the quantity P with another number Q, defined as:

$$Q = \frac{N^{(+)} - N^{(-)}}{N^{(+)} + N^{(-)}}, \qquad (4)$$

where $N^{(+)}$ and $N^{(-)}$ are the mean squared toroidal fields on northern and southern hemisphere, respectively.

Figure 3 makes evident that Q is, indeed, a suitable indicator for the variation of the degree of symmetry as well. Qualitatively the same information can be read from the quantities P and Q. During phases of little total magnetic energy (upper panel) a strong asymmetry between fields in the northern and southern hemisphere appears. Q becomes nearly $+1$, that is the field on the northern hemisphere exceeds that of the southern hemisphere. This can also be seen from a butterfly diagram (Figure 4) showing the latitudinal distribution of toroidal field versus time.

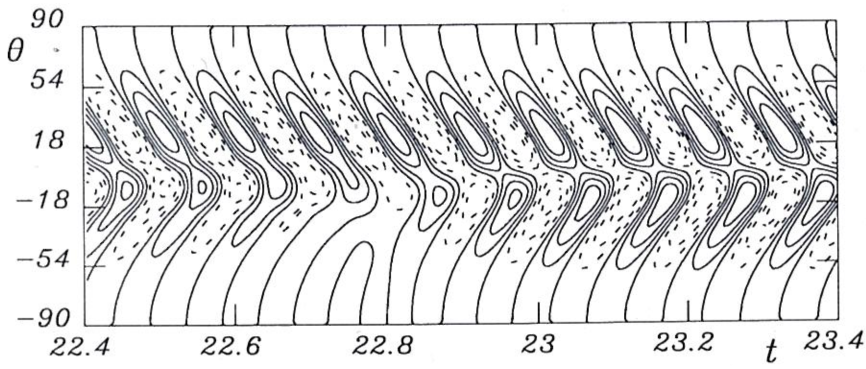

Figure 4: Butterfly diagram of the toroidal magnetic field for the same model as in Figure 3. The dashed lines indicate a couterclockwise directed field.

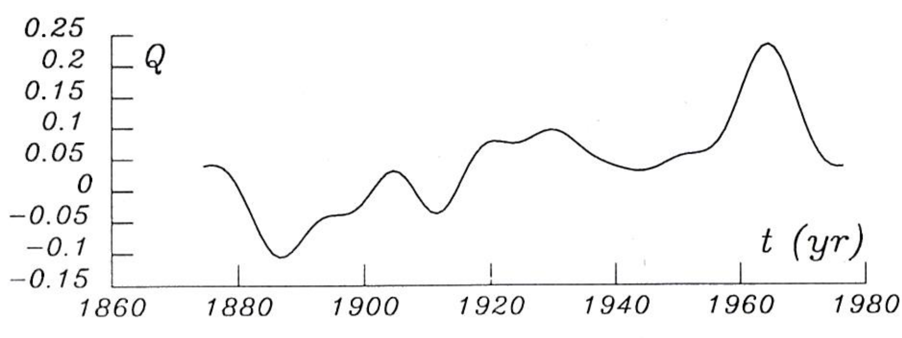

Figure 5: Variation of Q obtained from historical sunspot observations.

Finally, we compare our results with historical measurements of sunspots. We applied Eq.(4) using for $N^{(+)}$ and $N^{(-)}$ the total sunspot area (from Greenwich Photographic Results) on the northern and southern hemispheres, respectively. The time series of Q, obtained from the yearly values of the total areas, was filtered to reduce the 11-year signal of the solar cycle. The result may suggest that the value of Q changes sign from negative to positive in the beginning of this century.

4. CONCLUSIONS

The general statement by Krause and Meinel saying that only the solution with the lowest marginal dynamo number is stable is confirmed also for an example of a more complicated dynamo model including the dynamics of the mean flow. The regime of validity is finite, but it can be quite small. We have also demonstrated some more of the richness that even a simple nonlinearity (α quenching) can introduce into oscillating $\alpha\omega$-dynamos. In particular a long-term variation of the type of symmetry, found in these models, may be of relevance for understanding solar "grand minima". We note, however, that the models we consider here are not really of solar type. Firstly, the (turbulent) conductivity is not constant, but varies considerably through the Sun. Secondly, the gradient of angular velocity is in these models prescribed and constant. More realistic and self-consistent models should therefore be investigated. But is it hoped that some of the features found in these simplified models may still be relevant.

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