# Magnetic helicity evolution in a periodic domain with imposed field

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In helical hydromagnetic turbulence with an imposed magnetic field (which is constant in space and time) the magnetic helicity of the field within a periodic domain is no longer an invariant of the ideal equations. Alternatively, there is a generalized magnetic helicity that is an invariant of the ideal equations. It is shown that this quantity is not gauge invariant and that it can therefore not be used in practice. Instead, the evolution equation of the magnetic helicity of the field describing the deviation from the imposed field is shown to be a useful tool. It is demonstrated that this tool can determine steady state quenching of the alpha-effect. A simple three-scale model is derived to describe the evolution of the magnetic helicity and to predict its sign as a function of the imposed field strength. The results of the model agree favorably with simulations.

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## I. INTRODUCTION

Magnetic helicity has traditionally been used as a diagnostic tool to characterize magnetic field topology. Only in recent years has magnetic helicity also become a useful tool in understanding large scale dynamo action. Magnetic helicity is important because it is conserved in the limit of vanishing resistivity. This is *not* the case with the kinetic helicity, which is also conserved in the inviscid case, but the kinetic helicity dissipation rate diverges in the inviscid limit [1]. In this sense kinetic helicity is not even approximately conserved at large Reynolds numbers, while magnetic helicity is very nearly conserved at large magnetic Reynolds numbers [2].

A key result that has emerged from the concept of magnetic helicity conservation is that, in a periodic domain, a large scale magnetic field generated by the $\alpha$-effect [3,4] saturates on a *resistive* time scale [5]. This time scale can be very long. The helicity concept has also provided us with a simple explanation for the final saturation field strength of helical dynamos in periodic domains; see Ref. [5] for details. In this light, the case with an imposed magnetic field has also been considered [6,7], where it was found that above a certain field strength the dynamo is suppressed.

In the present paper we use similar ideas to obtain a more detailed understanding of the case with an imposed field. We begin with the equation governing the evolution of the magnetic helicity in a periodic domain in the presence of an imposed field. It is well known that in that case the magnetic helicity of the fluctuating magnetic field is no longer conserved in the nonresistive limit [8], but we also point out that a certain generalized (or total) magnetic helicity that has sometimes been used instead is gauge dependent and therefore cannot be used in the present case. We then discuss applications of the magnetic helicity equation to the alpha-effect in mean-field electrodynamics and derive a model equation in order to understand the evolution of the magnetic helicity for different imposed field strengths.

## II. MAGNETIC HELICITY EQUATION

The evolution of the magnetic field $\boldsymbol{B}$ is governed by

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{1}$$

where the electric field is obtained from Ohm's law

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J}, \tag{2}$$

where $\mathbf{u}$ is the velocity, $\boldsymbol{J}$ the current density, and $\eta$ the resistivity. Throughout this paper we adopt SI units, but we set the permeability to unity. We consider all quantities to be triply periodic over a cartesian domain. We consider the case with a finite mean field

$$\langle \boldsymbol{B} \rangle \equiv \boldsymbol{B}_0 = \text{const} \neq \boldsymbol{0}, \tag{3}$$

where angular brackets denote full volume averages. Such averages have no spatial dependence, but they can still depend on time. However, because of periodicity, the volume average of the curl in Eq. (1) vanishes, and hence $\mathrm{d}\langle \boldsymbol{B} \rangle / \mathrm{d}t = 0$. In other words, $\boldsymbol{B}_0$ is not only constant in space, but it is also constant in time.

Next, we split the field into a mean and a fluctuating component $\boldsymbol{B}=\boldsymbol{B}_0+\boldsymbol{b}$ and introduce the magnetic vector potential for the fluctuating component via $\boldsymbol{b}=\boldsymbol{\nabla}\times\boldsymbol{a}$, where $\boldsymbol{a}$ is periodic. The uncurled induction equation reads

$$\frac{\partial \boldsymbol{a}}{\partial t} = -(\boldsymbol{E} + \boldsymbol{\nabla}\phi), \tag{4}$$

where $\phi$ is the scalar potential.

We now consider the magnetic helicity. The imposed field is constant in space and does not therefore contribute to the

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magnetic helicity. We therefore consider only the magnetic helicity of the fluctuating field $H=\langle \boldsymbol{a}\cdot\boldsymbol{b}\rangle$. This quantity is gauge invariant because adding a gradient term to $\boldsymbol{a}$ does not change $H$

$$\langle(\boldsymbol{a}+\boldsymbol{\nabla}\varphi)\cdot\boldsymbol{b}\rangle = H + \langle\boldsymbol{\nabla}\cdot(\varphi\boldsymbol{b})\rangle - \langle\varphi\boldsymbol{\nabla}\cdot\boldsymbol{b}\rangle = H. \tag{5}$$

Here we have used the solenoidality of $\boldsymbol{b}$, and the fact that the volume average over a divergence term vanishes for a periodic domain. The equation for the (gauge-dependent) helicity density of the fluctuating field can be obtained in the form

$$\frac{\partial}{\partial t}(\boldsymbol{a}\cdot\boldsymbol{b}) = -2\boldsymbol{E}\cdot\boldsymbol{b} - \boldsymbol{\nabla}\cdot[(\boldsymbol{E}-\boldsymbol{\nabla}\phi)\times\boldsymbol{a}]. \tag{6}$$

The divergence term vanishes after volume averaging, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle = -2\langle\boldsymbol{E}\cdot\boldsymbol{b}\rangle, \tag{7}$$

where all terms are gauge-independent. Making use of Eq. (2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle = 2\langle(\boldsymbol{u}\times\boldsymbol{B}_0)\cdot\boldsymbol{b}\rangle - 2\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle, \tag{8}$$

where we have used $\langle\boldsymbol{J}\cdot\boldsymbol{b}\rangle=\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle$. Equation (8) can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle = -2\boldsymbol{\mathcal{E}}_0\cdot\boldsymbol{B}_0 - 2\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle, \tag{9}$$

where the electromotive force, $\boldsymbol{\mathcal{E}}_0=\boldsymbol{\mathcal{E}}_0(t)=\langle\boldsymbol{u}\times\boldsymbol{b}\rangle$, has been introduced. If the flow is isotropic and helical, there will be an $\alpha$-effect [3–5] with $\boldsymbol{\mathcal{E}}_0=\alpha\boldsymbol{B}_0$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle = -2\alpha\boldsymbol{B}_0^2 - 2\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle. \tag{10}$$

(In the Appendix, we clarify the implications of a finite $\alpha$ effect when the Faraday displacement current is restored in the Maxwell equations and when it is ignored.) Since $\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle$ is gauge invariant, it is a physically meaningful quantity. If there is a steady state, then $\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle$ must also be steady. In that case we have

$$\alpha = -\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle/\boldsymbol{B}_0^2, \tag{11}$$

which is a relation due to Keinigs [9] for the $\alpha$-effect in the *saturated* (steady) state; see also Ref. [10]. If the field is weak, the $\alpha$-effect will remain finite in the high conductivity limit [4].

The presence of a *finite* $\alpha$-effect means that the structure of Eq. (11) is very different when there is an imposed field. Unlike the case without imposed field ($\boldsymbol{B}_0=0$), the quantity $H=\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle$ is no longer conserved in the limit $\eta\to 0$. This prompted Matthaeus and Goldstein [8] to consider the quantity

$$\hat{H} = H + 2\boldsymbol{A}_0\cdot\boldsymbol{B}_0, \tag{12}$$

where

$$\boldsymbol{A}_0(t) = \int_0^t \boldsymbol{\mathcal{E}}_0(t')\mathrm{d}t'. \tag{13}$$

Note that $\hat{H}$ is constructed such that it satisfies the equation

$$\frac{\mathrm{d}\hat{H}}{\mathrm{d}t} = -2\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle. \tag{14}$$

Indeed, this equation reduces to Eq. (9) after inserting Eqs. (12) and (13) into Eq. (14).

The symbol $\boldsymbol{A}_0$ is chosen to make the extra term in Eq. (12) look like a magnetic helicity, even though $\boldsymbol{\nabla}\times\boldsymbol{A}_0=\boldsymbol{0}\neq\boldsymbol{B}_0$. The reason for $\boldsymbol{\nabla}\times\boldsymbol{A}_0\neq\boldsymbol{B}_0$ is that $\boldsymbol{A}_0$ corresponds to a slowly varying variable, whose curl gives $\boldsymbol{B}_0$ at a *higher* order, and not the order we are working in; see Eq. (A9) of Ref. [11]. Conversely, $\boldsymbol{B}_0$ corresponds to the curl of $\boldsymbol{A}_0$ at a lower order. A rigorous scale expansion is given in the Appendix of Ref. [11].

In the nonresistive limit, the right-hand side of Eq. (14) vanishes, and so $\hat{H}$ is conserved. One may be tempted to conclude that in the steady state, $\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle=0$. This is not generally true, however, and it would be in conflict with Eqs. (10) and (11). Certainly for sufficiently weak fields $\alpha$ is finite [12], so $\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle$ will also remain finite, see Eq. (11). Therefore, $\hat{H}$ cannot be constant in the steady state. The reason for this puzzle is that $\hat{H}$ is not gauge invariant [13], because the definition of $\hat{H}$ involves the quantity $\boldsymbol{A}_0$. At first glance, $\boldsymbol{A}_0$ appears to be gauge invariant, because $\boldsymbol{\mathcal{E}}_0$ is gauge invariant and $\boldsymbol{A}_0$ involves only a time integral over $\boldsymbol{\mathcal{E}}_0$; see Eq. (13). However, the beginning of the time integration is ill-defined, so in general one can replace

$$\boldsymbol{A}_0(t) \to \widetilde{\boldsymbol{A}}_0(t) = \boldsymbol{A}_{00} + \int_0^t \boldsymbol{\mathcal{E}}_0(t')\mathrm{d}t', \tag{15}$$

which would lead to a *different* conserved magnetic helicity, $\hat{H}+\Delta\hat{H}$, where $\Delta\hat{H}=2\boldsymbol{A}_{00}\cdot\boldsymbol{B}_0=$const is undetermined. Therefore, $\hat{H}$ is not a physically meaningful quantity, so it is not surprising that $\hat{H}$ can have a component that grows linearly in time. We emphasize however that $H=\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle$ is still gauge invariant and therefore physically meaningful, even though it is no longer a conserved quantity in the ideal limit.

## III. NONPERIODIC GAUGE POTENTIALS

In this section we want to comment on the related issue that adding a spatially constant vector $\boldsymbol{E}_0(t)$ to the right-hand side of Eq. (4) would not affect the evolution of $\boldsymbol{b}$. The constant vector $\boldsymbol{E}_0(t)$ can readily be absorbed in the definition of the scalar potential $\phi$, because it is specified only up to an additional gauge potential. However, this gauge potential has in general a nonperiodic contribution, even when all other quantities are periodic. More specifically, $\phi$ in Eq. (4) must have an additional component that varies *linearly* in space, i.e.,

$$\phi = \tilde{\phi} - \boldsymbol{E}_0 \cdot \boldsymbol{x}, \tag{16}$$

where $\tilde{\phi}$ is periodic and $\boldsymbol{x}$ is the position vector. We stress that $\phi$ in Eq. (16) is therefore in general not periodic, even if $\boldsymbol{a}$ and $\boldsymbol{b}$ are periodic.

A comment on the helicity flux associated with the gauge field is here in order. This flux is often written as $\phi\boldsymbol{b}$ which would then not be periodic and hence it is not obvious that its surface integral vanishes; see Eqs. (6) and (7). (The importance of this term for magnetic helicity injection has been discussed in Ref. [14].) However, using the identity

$$\boldsymbol{\nabla}\phi \times \boldsymbol{a} = \boldsymbol{\nabla} \times (\phi\boldsymbol{a}) - \phi\boldsymbol{b}, \tag{17}$$

the magnetic helicity flux $\phi\boldsymbol{b}$ can also be written as $\boldsymbol{\nabla}\phi\times\boldsymbol{a}$, which is periodic. Therefore, there is no contribution from the $\phi\boldsymbol{b}$ term in our case. (We note that similar manipulations can be used to turn a nonperiodic vector potential into a periodic one if the velocity is a linear function of coordinates [15].) The term $\boldsymbol{\nabla}\times(\phi\boldsymbol{a})$ has recently been discussed in a formulation of a magnetic helicity conserving dynamo effect [16]. Obviously, such a term does not give a contribution under the divergence and hence cannot be physically meaningful [17].

## IV. APPLICATION TO $\alpha$-QUENCHING

There have been a number of simulations of helically forced periodic flows with an imposed magnetic field. The general objective is to obtain the $\alpha$-effect and its suppression as a function of field strength [5,18,19].

In the steady state, Eq. (11) can be used to determine $\alpha$ by measuring $\langle \boldsymbol{j}\cdot\boldsymbol{b}\rangle$ in a simulation with an applied magnetic field $\boldsymbol{B}_0$. For helical turbulence, $\langle \boldsymbol{j}\cdot\boldsymbol{b}\rangle$ can be approximated by

$$\langle \boldsymbol{j}\cdot\boldsymbol{b}\rangle \approx \epsilon_{\rm f} k_{\rm f} \langle \boldsymbol{b}^2\rangle, \tag{18}$$

where $\epsilon_{\rm f}=\pm 1$ for a fully helical field with positive or negative helicity, and $|\epsilon_{\rm f}|<1$ for fractional helicity. A strongly helical small scale magnetic field is generally expected when the turbulent velocity field is also strongly helical [20]. The steady state $\alpha$ is therefore given by

$$\alpha = -\epsilon_{\rm f} k_{\rm f} \eta \langle \boldsymbol{b}^2\rangle / \boldsymbol{B}_0^2 \quad \text{(steady state value)}. \tag{19}$$

On the other hand, the kinematic value of $\alpha$ can be estimated in terms of the kinetic helicity,

$$\alpha_{\rm K} = -\frac{1}{3}\tau\langle \boldsymbol{\omega}\cdot\boldsymbol{u}\rangle \quad \text{(kinematic value)}, \tag{20}$$

where $\tau$ is the correlation time which, in turn, can be expressed in terms of the turbulent magnetic diffusivity for which we have a similar expression, $\eta_{\rm t}=\frac{1}{3}\tau\langle \boldsymbol{u}^2\rangle$. In analogy to Eq. (18), we write $\langle \boldsymbol{\omega}\cdot\boldsymbol{u}\rangle \approx \epsilon_{\rm f} k_{\rm f}\langle \boldsymbol{u}^2\rangle$, so

$$\frac{\alpha}{\alpha_{\rm K}} \approx \frac{\eta}{\eta_{\rm t}} \frac{\langle \boldsymbol{b}^2\rangle}{\boldsymbol{B}_0^2} \quad \text{(steady state value)}. \tag{21}$$

If we assume that the small scale field is in equipartition, i.e., $\langle \boldsymbol{b}^2\rangle \approx \langle \mu_0\rho\boldsymbol{u}^2\rangle \equiv B_{\rm eq}^2$, and if we define [21] the magnetic Reynolds number as $R_{\rm m}\equiv \eta_{\rm t}/\eta$, then Eq. (21) can be turned into the interpolation formula

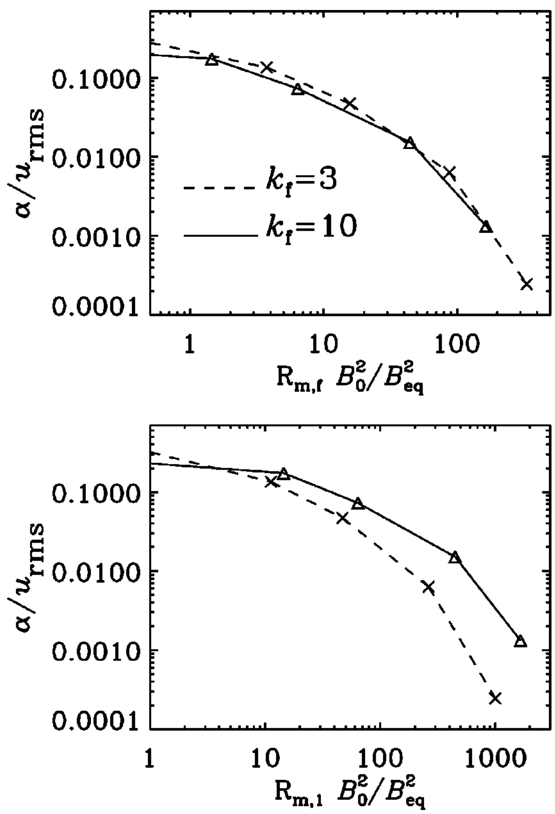


FIG. 1. Normalized $\alpha$-effect versus normalized magnetic energy, scaled with the small scale magnetic Reynolds number (upper panel) and the large scale magnetic Reynolds number (lower panel). Note that the onset of quenching is governed by the small scale magnetic Reynolds number, not the large scale magnetic Reynolds number.

$$\alpha = \frac{\alpha_{\rm K}}{1 + R_m \boldsymbol{B}_0^2/B_{\rm eq}^2}, \tag{22}$$

that recovers Eq. (21) for strong fields and $\alpha=\alpha_{\rm K}$ in the weak field limit, $\boldsymbol{B}_0\to 0$. This equation is known as the catastrophic quenching formula of Vainshtein and Cattaneo [12].

In order to confirm that the onset of steady state quenching depends on the magnetic Reynolds number based on the forcing scale [22], $R_{m,f}\equiv u_{\rm rms}/(\eta k_{\rm f})$, and not on the magnetic Reynolds number based on the scale of the box [5], $R_{m,1}\equiv u_{\rm rms}/(\eta k_1)$, we show two series of simulations obtained for different values of the forcing wave number $k_{\rm f}$, for different values of $B_0$. The forcing of the flow was fully helical; for details on the numerical method see Ref. [5]. The result is shown in Fig. 1.

Next, we consider simulations where we use hyperdiffusivity, i.e., the ordinary magnetic diffusion operator $\eta\nabla^2\boldsymbol{B}$ is replaced by $(-1)^{n-1}\eta_n\nabla^{2n}\boldsymbol{B}$, where $n=1$ corresponds to the standard case. This is a common tool in order to extend the inertial range of the turbulence [23], but it is also clear that this leads to wrong saturation field strengths [24]. In the presence of hyperdiffusivity, the magnetic Reynolds number

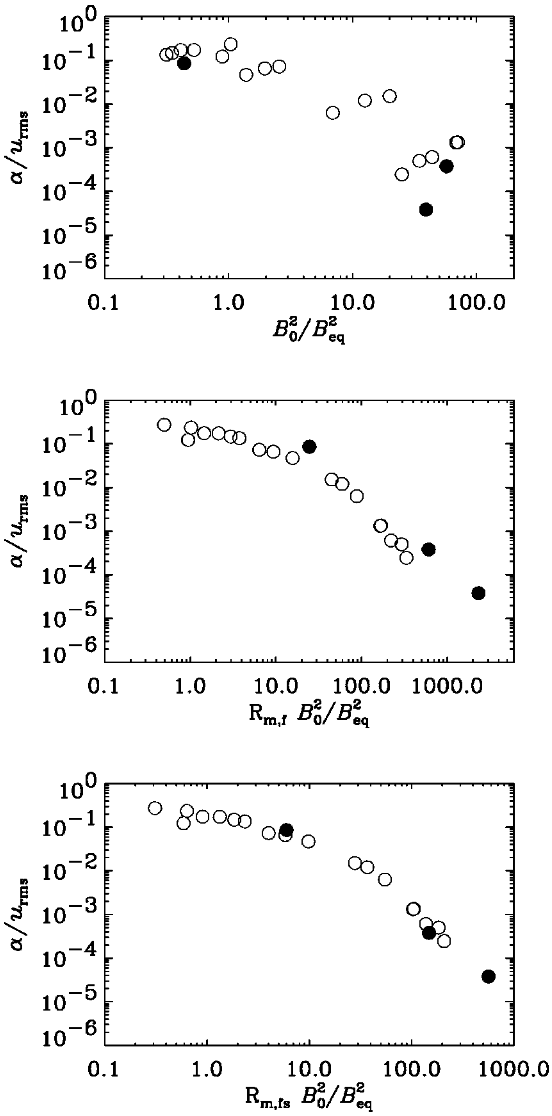


FIG. 2. Normalized $\alpha$-effect versus magnetic field strength. The full dots denote runs where hyperdiffusion has been used. In the second panel, $B_0^2/B_{\rm eq}^2$ is scaled with the magnetic Reynolds number based on the forcing scale $R_{\rm m,f}$. The last panel is similar to the second, but the *effective* forcing wave number has been used which is $k_{\rm f}$ scaled by a factor of 1.6. This brings especially the hyperdiffusive runs (full dots) closer to the rest of the data points.

is defined as $R_{\rm m,f} \equiv u_{\rm rms}/(\eta_n k_f^{2n-1})$. In the following we show, however, that the quenching data are better described by a single quenching curve when $k_{\rm f}$ is rescaled

$$R_{m,\rm fs} \equiv u_{\rm rms}/(\eta_n k_{\rm fs}^{2n-1}), \quad \text{where } k_{\rm fs} = 1.6 k_{\rm f}. \tag{23}$$

The result is shown in Fig. 2. The factor of 1.6 is not expected to be universal but is probably a slowly varying function of magnetic Reynolds number [21,24].

We conclude that Eq. (22) describes the simulations quite well provided the magnetic Reynolds number is defined in a suitable manner. We emphasize however that this equation only applies to the steady state and if there is no mean current. This is generally not the case and therefore the quenching is in practice not automatically catastrophic [21], i.e., the onset of quenching does not depend on $R_m$.

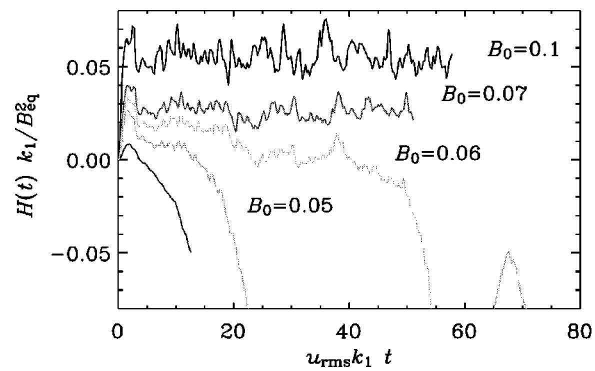


FIG. 3. Evolution of the total magnetic helicity $H = H_1 + H_{\rm f}$ as a function of $t$ for different values of $B_0$, as obtained from the three-dimensional simulation.

## V. EVOLUTION OF LARGE SCALE MAGNETIC HELICITY

Recently, the effect of an imposed field on the inverse cascade has been studied [6,7]. If the imposed magnetic field is weak or absent, there is a strong nonlocal transfer of magnetic helicity and magnetic energy from the forcing scale to larger scales. This leads eventually to the accumulation of magnetic energy at the scale of the box [5,23,25]. As the strength of the imposed field (wave number $k=0$) is increased, the accumulation of magnetic energy at the scale of the box ($k=1$) becomes more and more suppressed [6].

Qualitatively, this can be understood as the result of two competing effects: (i) the inverse cascade that produces magnetic helicity of opposite sign at $k=1$ compared to that at the forcing wave number $k_{\rm f}$, and (ii) the $\alpha$-effect operating on the imposed field producing magnetic helicity of the same sign at $k=1$ than at $k=k_{\rm f}$. This is because the sign of the $\alpha$-effect is opposite to the sign of the magnetic helicity at $k=k_{\rm f}$, and $\alpha$ enters with a minus sign in the evolution Eq. (10) of magnetic helicity. Under the assumption that the turbulence is fully helical, the critical value $B_*$ of the imposed field can be estimated by balancing the two terms on the right-hand side of Eq. (10) and by approximating, as in Sec. IV, $\alpha \approx \eta_{\rm t}\epsilon_{\rm f}k_{\rm f}$ and $\langle \boldsymbol{j}\cdot\boldsymbol{b}\rangle \approx \epsilon_f k_f B_{\rm eq}^2$. This yields

$$B_*^2/B_{\rm eq}^2 \approx \eta/\eta_{\rm t} = R_m^{-1}, \tag{24}$$

where the last equality is again to be understood as a definition of the magnetic Reynolds number, see also Ref. [21]. For $B_0 > B_*$ the sign of the magnetic helicity is the same both at $k=1$ and at $k=k_f$, while for $B_0 < B_*$ the signs are opposite.

A related phenomenological model for saturation of the dynamo effect under influence of $B_0$ has been given [7] that is based upon a Fourier scale separation approach. That approach leads to the conclusion that the critical value $B_* \propto R_m^{-1}$ rather than $R_m^{-1/2}$ as above. Further analysis may be needed to fully reconcile the differences in these approaches, both of which appear to have some support from simulations.

A more quantitative description of the evolution of the magnetic helicity can be obtained by using a modified two-scale model [21,22], where the term $2\boldsymbol{\mathcal{E}}_0\cdot\boldsymbol{B}_0$ from Eq. (9) has been included, so

$$\dot{H}_1 = -2\eta k_1^2 H_1 + 2\langle\boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{B}_1\rangle - 2\boldsymbol{\mathcal{E}}_0\cdot\boldsymbol{B}_0, \qquad (25)$$

$$\dot{H}_{\rm f} = -2\eta k_{\rm f}^2 H_{\rm f} - 2\langle\boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{B}_1\rangle. \qquad (26)$$

Here, $H_1$ and $H_{\rm f}$ are the magnetic helicities at the wave-numbers 1 and $k_f$, respectively, and $\langle\boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{B}_1\rangle$ is the helicity production from $\alpha$-effect and turbulent diffusion operating on the field at $k=1$. We note that the sum of Eqs. (25) and (26) yields Eq. (7). The electromotive force $\boldsymbol{\mathcal{E}}_1$ at wave-number $k_1$ is given by

$$\boldsymbol{\mathcal{E}}_1 = \alpha_{\rm f}\boldsymbol{B}_1 - \eta_{\rm t}\boldsymbol{J}_1. \qquad (27)$$

To calculate $\langle\boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{B}_1\rangle$ in Eqs. (25) and (26) we dot Eq. (27) with $\boldsymbol{B}_1$, volume average, and note that $\langle\boldsymbol{J}_1\cdot\boldsymbol{B}_1\rangle=k_1^2H_1$ and $\langle\boldsymbol{B}_1^2\rangle=k_1|H_1|$. The latter relation assumes that the field at wave-number $k_1$ is fully helical, but that it can have either sign. Thus, we have

$$\langle\boldsymbol{\mathcal{E}}_1\cdot\boldsymbol{B}_1\rangle = \alpha_{\rm f}k_1|H_1| - \eta_{\rm t}k_1^2H_1. \qquad (28)$$

The large scale magnetic helicity production from the $\alpha$-effect operating on the imposed field is $\boldsymbol{\mathcal{E}}_0\times\boldsymbol{B}_0=\alpha_1\boldsymbol{B}_0^2$. The $\alpha$-effect is proportional to the residual magnetic helicity of Pouquet, Frisch, and Léorat [20], with

$$\alpha = -\frac{1}{3}\tau(\langle\boldsymbol{\omega}\cdot\boldsymbol{u}\rangle - \langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle/\rho_0), \qquad (29)$$

where $\tau$ is the correlation time and $\rho_0$ the average density. In terms of $H_1$ and $H_{\rm f}$ we write

$$\alpha_1 = \alpha_{\rm K} + \frac{1}{3}\tau k_1^2H_1, \qquad (30)$$

for the $\alpha$-effect with feedback from $H_1$ and $H_{\rm f}$, respectively. Here, $\alpha_{\rm K}$ is the contribution to the $\alpha$-effect from the kinematic helicity, as defined in Eq. (20).

The above set of equations for the case of an imposed magnetic field is similar to a recently proposed four-scale model [26], where two smaller scales were added relative to the two-scale model. In the present case, on the other hand, instead of including scales smaller than the forcing scale, the imposed field at the infinite scale is included, albeit fixed in time.

For finite values of $B_0$, the final value of $H_1$ is particularly sensitive to the value of $\alpha_{\rm K}$ and turns out to be too large compared with the simulations. This disagreement with simulations is readily removed by taking into account that $\alpha_{\rm K}=\frac{1}{3}\tau\langle\boldsymbol{\omega}\cdot\boldsymbol{u}\rangle$ should itself be quenched when $B_0$ becomes comparable to $B_{\rm eq}$. Thus, we write

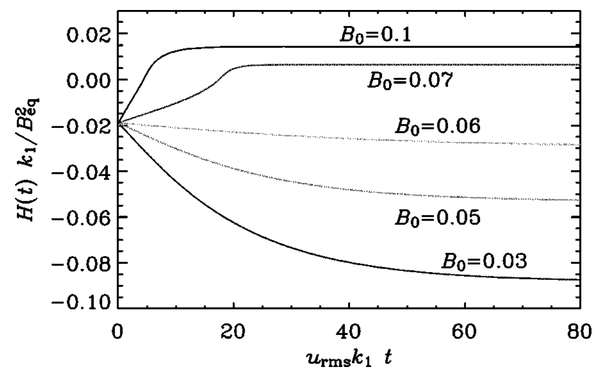


FIG. 4. Evolution of magnetic helicity as a function of $t$ for different values of $B_0$, as obtained from the two-scale model.

$$\alpha_{\rm K} = \alpha_{\rm K0}/(1+B_0^2/B_{\rm eq}^2), \qquad (32)$$

which is a good approximation to more elaborate expressions [27]. We emphasize that this equation only applies to $\alpha_{\rm K}$ and is therefore distinct from Eq. (22).

In Fig. 4 we show the result of a numerical integration of Eqs. (25) and (26). Both the three-dimensional simulation and the two-scale model show a similar value of $B_0\approx 0.06,\ldots,0.07$, above which $H_1$ changes sign. This confirms the validity of our estimate of the critical value $B_*$ obtained from Eq. (24). Second, the time evolution is slow when $B_0 < B_*$ and faster when $B_0 > B_*$. In the simulation, however, the field attains its final level for $B_0 > B_*$ almost instantaneously, which is not the case in the model. It is possible [7] that the almost instantaneous adjustment in the simulations is a consequence of the Alfvén effect, which is not included in the present model. This, and other shortcomings of the present model may also be responsible for the mismatch between the magnetic helicity amplitudes seen in the simulations and the model. Most characteristic in the simulations is the fact that $H_1\to 0$ while $H_f\neq 0$ in the limit of strong imposed field strength.

## VI. CONCLUSIONS

We have shown that (i) in the presence of an imposed field and (ii) using triple-periodic boundary conditions, the generalized magnetic helicity [8] in Eq. (12) is not gauge-invariant and therefore cannot be used for practical purposes. This quantity has frequently been used in the solar wind community as an alternative to the ordinary magnetic helicity which is known not to be conserved in the limit of vanishing resistivity. We have argued, however, that even though the ordinary magnetic helicity is not conserved in the presence of an imposed field and in the limit of low resistivity, it remains an extremely useful quantity that has predictive power–similar to the case without imposed field [5].

Based on analytic considerations and confirmed by the simulations, we have shown that the *sign* of the magnetic helicity depends on the strength of the imposed magnetic field. If the field is weak enough, the situation is similar to the case without the imposed magnetic field and the sign of the magnetic helicity is opposite to the sign of the helicity of the turbulence. If the field exceeds a certain threshold, which is $R_m^{-1/2}$ times the equipartition field strength, where $R_m$ is the magnetic Reynolds number based on the forcing wave number, the sign of magnetic helicity changes and becomes equal to the sign of the helicity of the turbulence. This can be understood as a consequence of the $\alpha$-effect operating on the imposed field. In finite systems, this $\alpha$-effect would cause the large scale field to have opposite helicity compared to the small scale field. In an infinite (or periodic) system, this is not possible, and the entire field in the computational domain plays the role of a small scale field which must then have the same sign of helicity as the turbulence.

The two-scale model used to describe the nonlinear evolution of helical dynamos [21,22] can be generalized to take account of the large scale field. The formalism is similar to a recently proposed four-scale model [26]. The nonlinear two-scale and multiscale models play important roles in modern mean-field dynamo theory. Given that we are still lacking a proper understanding of solar and stellar dynamos in the nonlinear regime, an independent confirmation of the nonlinear multiscale model must therefore be regarded as a crucial step toward understanding the origin and maintenance of magnetic fields in turbulent astrophysical bodies.

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## APPENDIX A: THE ROLE OF THE DISPLACEMENT CURRENT IN TRIPLY PERIODIC SYSTEMS WITH IMPOSED FIELD

In this appendix we discuss a paradoxical situation [28] that arises when comparing volume averages of the present equations (where periodic boundary conditions are used and a uniform field is imposed) with the full Maxwell equations (where the displacement current is included). The displacement current is given by the fourth Maxwell equation

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{J}, \tag{A1}$$

where $c$ is the speed of light. We recall that the permeability has been put to unity. Applying volume averages, and noting that the volume average of the curl vanishes, we have

$$\frac{1}{c^2}\frac{\partial \langle \boldsymbol{E}\rangle}{\partial t} = -\langle \boldsymbol{J}\rangle. \tag{A2}$$

On the other hand, the volume average of Ohm's law (2) yields

TABLE I. Comparison of various volume averages in the pre-Maxwellian approximation and the full Maxwell equations. The asterisk denotes that this result is strictly valid only when also a time average is considered.

| | pre-Maxwell | Maxwell |
|---|---|---|
| $\langle \boldsymbol{u}\times\boldsymbol{b}\rangle$ | $\neq\boldsymbol{0}$ | $\neq\boldsymbol{0}$ |
| $\langle \boldsymbol{E}\rangle$ | $\neq\boldsymbol{0}$ | $\neq\boldsymbol{0}$ |
| $\langle \boldsymbol{J}\rangle$ | $=\boldsymbol{0}$ | $\neq\boldsymbol{0}$ |
| $\langle \boldsymbol{J}\rangle\times\langle \boldsymbol{B}\rangle$ | $=\boldsymbol{0}$ | $=\boldsymbol{0}^*$ |

$$\langle \boldsymbol{E}\rangle = -\langle \boldsymbol{u}\times\boldsymbol{B}\rangle + \eta\langle \boldsymbol{J}\rangle. \tag{A3}$$

We assume that there is no net flow, i.e. $\langle \boldsymbol{u}\rangle=\boldsymbol{0}$. Therefore, $\langle \boldsymbol{u}\times\boldsymbol{B}\rangle=\langle \boldsymbol{u}\times\boldsymbol{b}\rangle$.

In helical hydromagnetic turbulence there is an $\alpha$-effect [3–5], so $\boldsymbol{\mathcal{E}}_0\equiv\langle \boldsymbol{u}\times\boldsymbol{b}\rangle=\alpha\boldsymbol{B}_0\neq\boldsymbol{0}$, and therefore $\langle \boldsymbol{E}\rangle\neq 0$ and, because of Eq. (A2), $\langle \boldsymbol{J}\rangle\neq\boldsymbol{0}$. The latter condition is, of course, inconsistent with our assumption that $\langle \boldsymbol{J}\rangle=\langle\boldsymbol{\nabla}\times\boldsymbol{B}\rangle=\boldsymbol{0}$. This discrepancy could be particularly important when considering the contribution of the volume averaged field to the Lorentz force $\langle \boldsymbol{J}\rangle\times\langle \boldsymbol{B}\rangle$, which vanishes in the pre-Maxwellian magnetohydrodynamics (MHD) approximation, but not when the displacement current is retained [28].

Eliminating $\langle \boldsymbol{E}\rangle$ from Eqs. (A2) and (A3) we have

$$\left(c^2+\eta\frac{\mathrm{d}}{\mathrm{d}t}\right)\langle \boldsymbol{J}\rangle = \dot{\boldsymbol{\mathcal{E}}}_0, \tag{A4}$$

where the dot on $\boldsymbol{\mathcal{E}}_0$ denotes time differentiation. Equation (A4) could be solved for $\langle \boldsymbol{J}\rangle$ either in terms of the Green's function $\exp[-(t-t')c^2/\eta]$ or via series expansion [28], confirming that $\langle \boldsymbol{J}\rangle\neq\boldsymbol{0}$. However, here we are interested in the Lorentz force, so we write

$$\left(c^2+\eta\frac{\mathrm{d}}{\mathrm{d}t}\right)(\langle \boldsymbol{J}\rangle\times\langle \boldsymbol{B}\rangle) = \dot{\boldsymbol{\mathcal{E}}}_0\times\boldsymbol{B}_0=\boldsymbol{0}. \tag{A5}$$

The right-hand side of Eq. (A5) vanishes, because $\langle \boldsymbol{B}\rangle=\boldsymbol{B}_0$ is independent of time and $\dot{\boldsymbol{\mathcal{E}}}_0=\dot{\alpha}\boldsymbol{B}_0$ is parallel to $\boldsymbol{B}_0$. The use of the equation $\boldsymbol{\mathcal{E}}_0=\alpha\boldsymbol{B}_0$ ignores random fluctuations in time about zero. In that sense, Eq. (A5) is strictly valid only when the averages are also taken over time.

The solution to Eq. (A5) shows that, if the Lorentz force from the mean field was vanishing initially, it must vanish at all times. Table I summarizes which of the different volume averages discussed in this section vanish in the pre-Maxwellian MHD approximation and which quantities remain finite when the full Maxwell equations are used.

The apparent inconsistency is removed by noting that Eq. (A1) does not simply exist in the pre-Maxwellian MHD formulation and hence cannot be invoked in the discussion. [The situation is similar to the incompressibility assumption $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$, or the anelastic approximation $\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})=0$, both of which do not imply $\partial\rho/\partial t=0$. Indeed, the original continuity equation is no longer used and has instead been *replaced* by $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$ or $\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})=0$, respectively.] Nevertheless, as far as the Lorentz force is concerned, the neglect of the displacement current is inconsequential, because it vanishes in either of the two cases; see Table I.

The mismatch between $\langle \boldsymbol{J} \rangle = \mathbf{0}$ in the pre-Maxwellian approximation and the exact result, $\langle \boldsymbol{J} \rangle \neq 0$, is negligible, but can be quantified using a rigorous expansion in terms of slowly and rapidly varying variables [11]. Such an approach also demonstrates quite nicely that the difficulties introduced into the periodic model by the presence of a nonzero uniform mean field are due to imposing periodic boundary conditions on the entire (infinite volume) system. If instead (see the Appendix of Ref. [11]) the turbulence is assumed to be modeled as *locally* homogeneous, in the statistical sense, and periodicity is employed in a two scale expansion as a local leading order model, no such problems emerge.

Paradoxical situations arising from the assumption of triple periodicity are commonly resolved using scale expansion. Another such example is the famous Jeans swindle [29], where the assumed zero order equilibrium state does not obey triple periodicity; see Refs. [30,31] for a stability analysis using a proper equilibrium solution. We emphasize, however, that the problem with the Jeans swindle is *distinct* from the problem with the displacement current discussed here. The latter is completely resolved by staying fully within the pre-Maxwellian formulation, while the former is a true mathematical swindle.


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